Self-force in a radiation gauge

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I. Intro

A single complex Weyl scalar, either

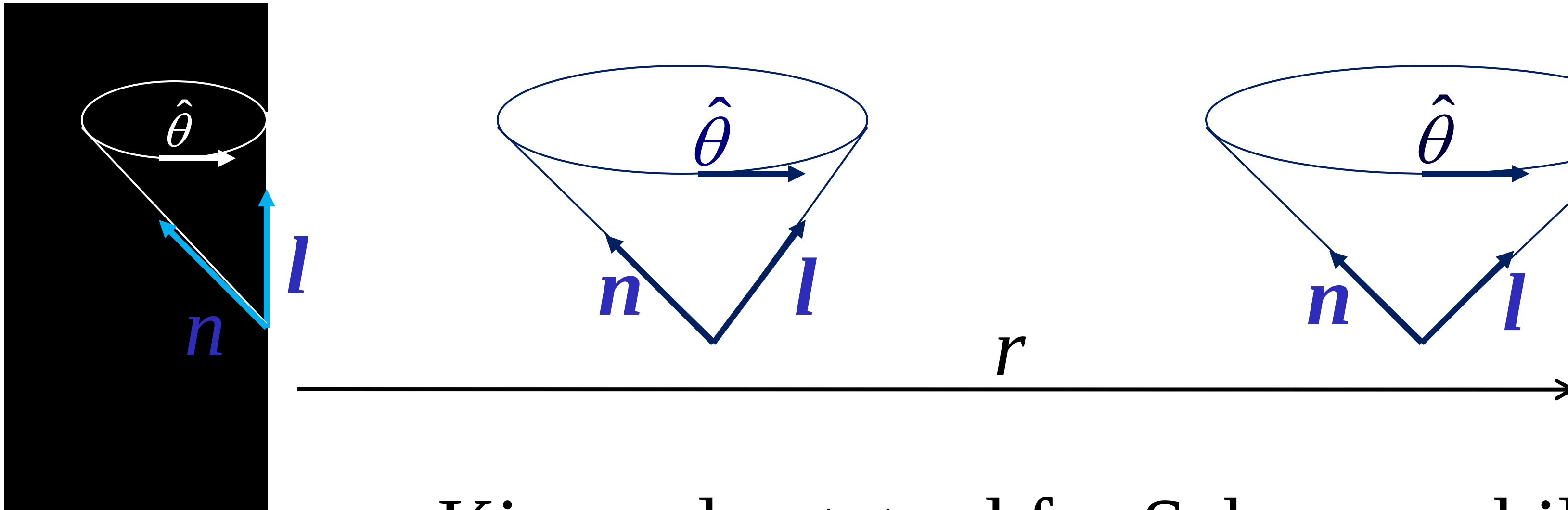
$$\psi_0 \text{ or } \psi_4 ,$$

determines gravitational perturbations of a Kerr geometry (outside perturbative matter sources) up to changes in mass, angular momentum, and change in the center of mass.

ψ_0 and ψ_4 are each a component of the perturbed Weyl tensor along a tetrad associated with the two principal null directions of the spacetime.
Each satisfies a separable wave equation, the Teukolsky equation for that component.

Newman-Penrose Formalism

Null tetrad $\quad l^{\alpha}, n^{\alpha}, m^{\alpha}, \overline{m}^{\alpha}$



e.g.,Kinnersley tetrad for Schwarzschild

$$l^{\alpha} = \frac{r^2}{\Delta} t^{\alpha} + r^{\alpha} \qquad n^{\alpha} = \frac{1}{2}\left(t^{\alpha} - \frac{\Delta}{r^2} r^{\alpha} \right)$$

$$m^{\alpha} = \frac{1}{\sqrt{2}}\left(\hat{\theta}^{\alpha} + i\hat{\phi}^{\alpha} \right) \qquad \overline{m}^{\alpha} = \frac{1}{\sqrt{2}}\left(\hat{\theta}^{\alpha} - i\hat{\phi}^{\alpha} \right)$$

$$\Delta = r^2 - 2Mr$$

$$\psi_0 = -\delta C_{\alpha\beta\gamma\delta} l^\alpha m^\beta l^\gamma m^\delta \qquad \psi_4 = -\delta C_{\alpha\beta\gamma\delta} n^\alpha \overline{m}^\beta n^\gamma \overline{m}^\delta$$

Spin weight: Under a rotation of the $\hat{\theta}, \hat{\phi}$ basis

$$\hat{\theta} \to \hat{\theta}\cos\psi - \hat{\phi}\sin\psi$$

$$\hat{\phi} \to \hat{\phi}\cos\psi + \hat{\theta}\sin\psi,$$

m and $\overline{m}$ change by $m \to m e^{i\psi}, \quad \overline{m} \to \overline{m} e^{-i\psi}$

An object η is said to have spin-weight s if it changes under this basis rotation by

$$\eta \to \eta e^{is\psi}$$

Then m and $\overline{m}$ have spin weights +1 and -1, and ψ_0 and ψ_4 have spin-weights +2 and -2.

They are gauge invariant, because the Weyl tensor of the unperturbed Kerr geometry has only Ψ_2 = nonzero.

A gauge transformation changes the Weyl tensor by

$$\delta C_{\alpha\beta\gamma\delta} = \mathcal{L}_\xi C_{\alpha\beta\gamma\delta}$$

Components of ξ are
$$\begin{array}{ll} \xi\cdot l, \quad \xi\cdot n & \text{spin } 0 \\ \xi\cdot m, \xi\cdot\overline{m} & \text{spin } \pm 1 \end{array}$$

$C_{\alpha\beta\chi\delta}$ has only a spin 0 component

$$\Rightarrow \mathcal{L}_\xi C_{\alpha\beta\gamma\delta} \text{ has only spin } 0, \pm 1$$

Teukolsky equation: $\mathcal{O}_s\,\psi = S$

$$\mathcal{O}_s = \left[\frac{(r^2+a^2)^2}{\Delta} - a^2\sin^2\theta\right]\frac{\partial^2}{\partial t^2} - 2s\left[\frac{M(r^2-a^2)}{\Delta} - r - ia\cos\theta\right]\frac{\partial}{\partial t} + \frac{4Mar}{\Delta}\frac{\partial^2}{\partial t\partial\phi} - \Delta^{-s}\frac{\partial}{\partial r}\left(\Delta^{s+1}\frac{\partial}{\partial r}\right)$$
$$-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) - 2s\left[\frac{a(r-M)}{\Delta} + \frac{i\cos\theta}{\sin^2\theta}\right]\frac{\partial}{\partial\phi}$$
$$+\left[\frac{a^2}{\Delta} - \frac{1}{\sin^2\theta}\right]\frac{\partial^2}{\partial\phi^2} + (s^2\cot^2\theta - s)$$

Source function $S = \mathcal{T}^{\alpha\beta}T_{\alpha\beta}$,

$T_{\alpha\beta}$ = energy-momentum tensor

$\mathcal{T}^{\alpha\beta}$ a 2nd-order differential operator

•Outgoing Radiation Gauge (ORG)

$$h_{\alpha\beta} n^{\beta} = 0 \qquad\qquad h = 0$$

5 constraints, similar to those for *ingoing* waves in flat space with a transverse-tracefree gauge. The metric perturbation satisfying these conditions is given by

$$h_{\alpha\beta} = L_{\alpha\beta}\Psi$$

where $L_{\alpha\beta}$ is a 2nd-order differential operator involving only $\eth$ and ∂_t

Two theorems:

1. Let h be given by $h = L_{ORG}\ \Psi$, with Ψ an ORG potential obtained from $\delta\psi_0$ $(\delta\psi_4)$. If ψ satisfies the sourcefree Teukolsky equation then h satisfies the vacuum Einstein equation.

   (Chzrzanowski; Cohen, Kegeles; Wald)

2a. Given a solution $\psi_0\ (\psi_4)$ to the sourcefree Teukolsky equation on a globally hyperbolic type D vacuum spacetime, one can find a local ORG gauge:

There is a perturbed metric $h_{\alpha\beta}$ in an ORG gauge that satisfies the linearized vacuum Einstein equation, and for which $\delta\psi_0\ (\delta\psi_4)$ is the perturbed Weyl tensor component.

(Whiting, L. Price)

2b. Under the conditions of (a) there is a local solution Ψ to the sourcefree Teukolsky equation for which $h = L\Psi$ is a perturbed vacuum metric associated with ψ_0 (ψ_4). (Not yet proved)

Explicitly,

$$\frac{1}{8}\left[\left(\eth - ia\sin\theta\,\partial_t\right)^4 \overline{\Psi} + 12M\partial_t \Psi\right] = \psi_0$$

Integrate 4 times with respect to θ

Algebraic solution

for each frequency and angular harmonic

$$\psi_{0\ell m\omega} = {}_2R_{\ell m\omega}(r)\ {}_2S_{\ell m\omega}(\theta)\, e^{i(m\phi-\omega t)}$$

$$\Psi_{\ell m\omega} = 8\frac{(-1)^m D\bar{\psi}_{0\ell -m-\omega} + 12iM\omega\psi_{0\ell m\omega}}{D^2 + 144M^2\omega^2}$$

Resulting metric has only the $\ell \geq 2$ part of the source. The $\ell = 0,1$ part has support on a spherical shell at $r=r_0$, and the metric is discontinuous at $r=r_{0.}$

Equivalent alternative involves radial derivatives along principal null geodesics:

$$\psi_0 = (l^\mu \partial_\mu)^4 \Psi = \partial_r{}^4 \Psi(u, r, \theta, \widetilde{\phi})$$

For each angular harmonic of ψ_0, this gives a unique solution satisfying the Teukolsky equation: e.g., for $r > r_0$,

$$\psi_0 = (l^\mu \partial_\mu)^4 \Psi = \partial_r{}^4 \Psi(u, r, \theta, \widetilde{\phi})$$

$$\Psi = \int_r^\infty dr_1 \int_{r_1}^\infty dr_2 \int_{r_2}^\infty dr_3 \int_{r_3}^\infty dr_4 \psi_0(u, r_4, \theta, \widetilde{\phi}).$$

$$(\text{Kerr coordinates}: \tilde{\phi} = \phi + \int_r^\infty \frac{dr}{\Delta})$$

Because h^{ret} and h^{sing} have the same source, $h^{ren} = h^{ret} - h^{sing}$ is a solution to the vacuum perturbation equations, and $\psi^{ren}{}_0 = \psi^{ret}{}_0 - \psi^{sing}{}_0$ is a solution to the vacuum Teukolsky equation, allowing one to compute a Hertz potential.

Although the argument used the order

$$\begin{array}{ccc} \psi^{ret}_{rad} & \longrightarrow & \psi^{R}_{rad} \\ & & \downarrow \\ & & h^{R}_{rad} \end{array}$$

The diagram commutes

$$\begin{array}{ccc} \psi^{ret}_{rad} & \longrightarrow & \psi^{R}_{rad} \\ \downarrow & & \downarrow \\ h^{ret}_{rad} & \longrightarrow & h^{R}_{rad} \end{array}$$

because of the algebraic character of the operations

Outline of method

1. Compute ψ_0^{ret} from the Teukolsky equation as a mode sum over l,m,ω.
2. Find the Hertz potential Ψ^{ret} from ψ_0^{ret} or ψ_4^{ret} algebraically from angular equation or as a 4 radial integrals from the radial equation.
   The angular harmonics of Ψ^{ret} are defined for $r>r_0$ or $r<r_0$, with r_0 the radial coordinate of the particle.
3. Find, in a radiation gauge, the components of $h^{\text{ret}}_{\alpha\beta}$ and its derivatives that occur in the expression for a^α.

3. Compute $a^{\text{ret}\,\alpha}$ from the perturbed geodesic equation as a mode sum truncated at $\ell_{\max}$. Compute the renormalization vectors A^α and B^α (and C^α ?), numerically matching a power series in $L := \ell + 1/2$ to the values of $a_\ell^{\text{ret}\,\alpha}$, and subtract from $a_\ell^{\text{ret}\,\alpha}$ the truncated sum

$$\sum_{\ell=0}^{\ell_{\max}} A^\alpha L + B^\alpha + \frac{C^\alpha}{L}$$

to obtain $a_\ell^{\text{ren}\,\alpha}$.

4. Determine the contribution to $h^{\text{ret}}_{\alpha\beta}$ of the perturbations in mass, angular momentum, and change in center of mass.

ψ_0^{ret} –> Hertz potential Ψ –> $h^{ret}[\psi_0]$ -> $a^{ret}[\psi_0]$ –> $a^{ren}[\psi_0]$

But $h^{ret}[\psi_0]$ is not the full metric.

The missing pieces

ψ_0 and ψ_4 do not determine the full perturbation:
Spin-weight 0 and 1 pieces undetermined.

There are algebraically special perturbations of Kerr, perturbations for which ψ_0 and ψ_4 vanish:
changing mass δm

changing angular momentum δJ
(and singular perturbations –
to C-metric and to Kerr-NUT).

And gauge transformations

$h^{\mathrm{ret}}_{\alpha\beta}[\psi_0]$ via CCK procedure

$h^{\mathrm{ret}}_{\alpha\beta}[\delta m]$ from the conserved currents associated with the background Killing vector t^α.

$h^{\mathrm{ret}}_{\alpha\beta}[\delta J]$ from the conserved currents associated with the background Killing vector ϕ^α, for the part of $\delta \boldsymbol{J}$ along background $\boldsymbol{J}$.

There are two remaining pieces:

$h^{\mathrm{ret}}_{\alpha\beta}[\delta J_{\perp}]$ the part of $\delta\boldsymbol{J}$ orthogonal to the background $\boldsymbol{J}$

$h^{\mathrm{ret}}_{\alpha\beta}[CM]$ the change in the center of mass

Each is pure gauge outside the source, but the gauge transformation is discontinuous across the source.

1° $\quad h_{\alpha\beta}[\delta m], \quad h_{\alpha\beta}[\delta J]$

$$j_{(t)}{}^{\alpha} = \delta(2T^{\alpha}{}_{\beta} - \delta^{\alpha}_{\beta} T)t^{\beta} \qquad j_{(\phi)}{}^{\alpha} = -\delta T^{\alpha}{}_{\beta}\phi^{\beta}$$

Background $T^{\alpha}{}_{\beta} = 0 \quad \Rightarrow$

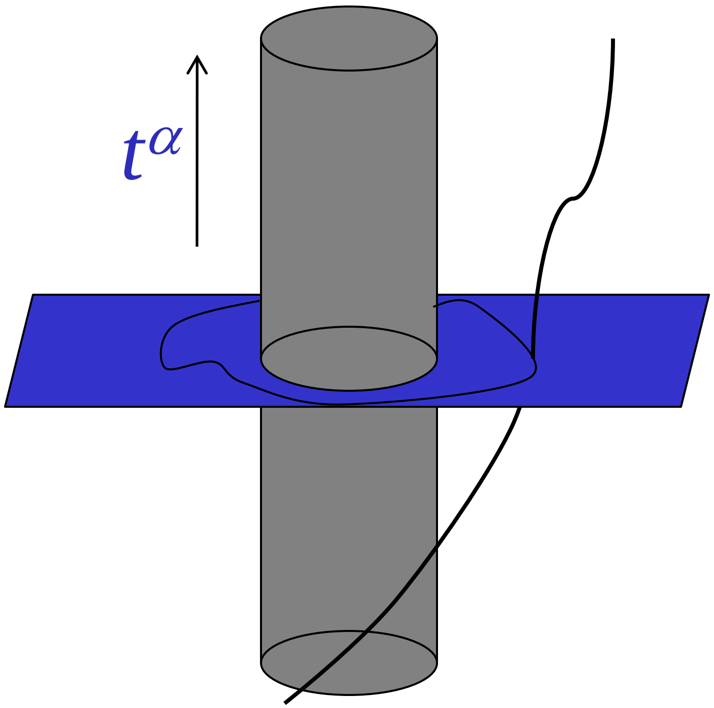
$$\nabla_{\alpha} j_{(t)}{}^{\alpha} = 0, \qquad \nabla_{\alpha} j_{(\phi)}{}^{\alpha} = 0$$

$$\delta m = \int j_{(t)}{}^{\alpha}\, dS_{\alpha} \qquad \delta J = -\int j_{(\phi)}{}^{\alpha}\, dS_{\alpha}$$

$$= m\left(2u^{\alpha}\nabla_{\alpha} t - \frac{1}{u_{\alpha} t^{\alpha}}\right) \qquad = -m u_{\alpha}\phi^{\alpha}$$

Then $h_{\alpha\beta}[\delta m]$, $h_{\alpha\beta}[\delta J]$ are the perturbed metrics in any desired gauge associated with the infinitesimal changes δm, δJ outside the particle.

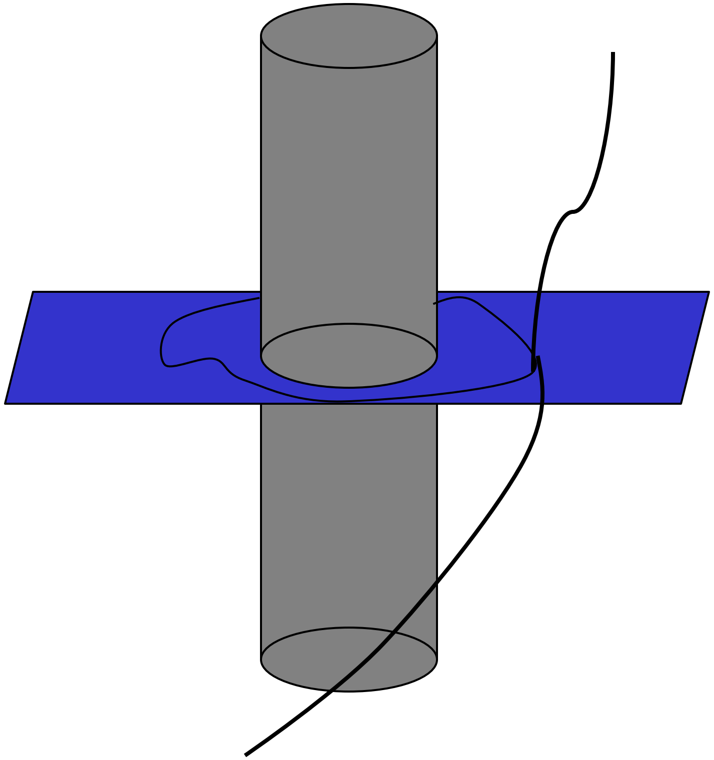
What does *outside the particle* mean?



Schwarzschild background:
Decompose the source into spherical harmonics on symmetry spheres:

$h_{\alpha\beta}[\delta m]$, $h_{\alpha\beta}[\delta J]$ are nonzero outside the symmetry sphere $r = r_0$ through the particle, zero or pure gauge inside.

They each have as source a spherical shell at $r = r_0$ but the sum of all angular harmonics of $h_{\alpha\beta}$ is a perturbed metric whose source is the point particle.



Kerr background:
No natural symmetry spheres.
No separation of different
θ harmonics in field
equations for $h_{\alpha\beta}$.

Free to chose any stationary axisymmetric radial coordinate, e.g. Boyer-Lindquist r, with
$h_{\alpha\beta}[\delta m]$, $h_{\alpha\beta}[\delta J]$ nonzero outside $r = r_0$

$2^{o} \qquad h_{\alpha\beta}[\delta J_{\perp}], \quad h_{\alpha\beta}[CM]$

Two questions:

If they are pure gauge, how can they have a source?

$h^{g}_{\alpha\beta} = \pounds_{\xi}\, g_{\alpha\beta}\, \Theta(r - r_0)$ is not pure gauge at $r=r_0$

($h^{g}_{\alpha\beta} = \pounds_{\xi\Theta(r-r_0)}\, g_{\alpha\beta}$ is pure gauge)

For Schwarzschild these are *l=1* perturbations, with axial and polar parity, respectively.

How do we identify them in Kerr?

The idea is to find the part of the source that has not contributed to $h^{\mathrm{ret}}_{\alpha\beta}[\psi] + h_{\alpha\beta}[\delta m] + h_{\alpha\beta}[\delta J]$

One could in principle simply subtract from $\delta T^{\alpha\beta}$ the contribution from these three terms. Writing

$$\mathcal{E} h_{\alpha\beta} := \delta G_{\alpha\beta}$$

we have

$$\mathscr{E} h^{\text{ret}}{}_{\alpha\beta} = 8\pi\delta T_{\alpha\beta},$$

$$8\pi\delta T^{\text{remaining}}_{\alpha\beta} = 8\pi\delta T_{\alpha\beta} - \mathscr{E}\left(h^{\text{ret}}[\psi] + h[\delta m] + h[\delta J]\right)_{\alpha\beta}$$

Find ξ at r_0 from the jump condition

$$\int_{r_0-\varepsilon}^{r_0+\varepsilon} (\mathscr{E}\, h^{\text{gauge}})_{\alpha\beta} = \int_{r_0-\varepsilon}^{r_0+\varepsilon} 8\pi\delta T^{\text{remaining}}_{\alpha\beta}$$

For h^{gauge} continuous, the jump in $\mathscr{E}\, h^{\text{gauge}}$ involves only the few terms in $\mathscr{E}$ with second derivatives in the radial direction orthogonal to u^α .

But

Now we're back to the old difficulty of handling terms that are singular at the particle.

Instead of trying directly to evaluate

$$8\pi\delta T_{\alpha\beta} - \mathscr{E}\left(h^{\text{ret}}[\psi] + h[\delta m] + h[\delta J]\right)_{\alpha\beta}$$

use the fact that h^{sing} has source $\delta T_{\alpha\beta}$:

$$\mathscr{E} h^{\text{gauge}}{}_{\alpha\beta} = \mathscr{E}(h^{\text{sing}})_{\alpha\beta} - \mathscr{E}\left(h^{\text{ret}}[\psi] + h[\delta m] + h[\delta J]\right)_{\alpha\beta}$$

$$\int_{r_0-\varepsilon}^{r_0+\varepsilon} \mathscr{E} h^{\text{gauge}}{}_{\alpha\beta} = -\int_{r_0-\varepsilon}^{r_0+\varepsilon} \mathscr{E}\left(h^{\text{ren}}[\psi] + h[\delta m] + h[\delta J]\right)_{\alpha\beta}$$

$$\int_{r_0-\varepsilon}^{r_0+\varepsilon} \mathscr{E}\, h^{\text{gauge}}{}_{\alpha\beta} = -\int_{r_0-\varepsilon}^{r_0+\varepsilon} \mathscr{E}\left(h^{\text{ren}}[\psi] + h[\delta m] + h[\delta J]\right)_{\alpha\beta}$$

The test:
First calculation involving renormalization

Computation of ΔU^{ren} and a^{ren} for a mass in circular orbit in a Schwarzschild spacetime. (Shah et al. 2010)
($\Delta U(\Omega) = \delta u^t |_{\text{fixed }\Omega}$ is the quantity that is gauge-invariant under helically symmetric gauge transformations.)

- ❑ Obtain agreement to the six-place accuracy of the comparison of U^{ren} computed in Regge-Wheeler (Detweiler) and Lorenz (Barack-Sago) gauges.

- ❑ Renormalization coefficients A, B, C(=0) agree with those of Lorenz with fractional error < 10^{-10}

We find the singular part of the self-force by matching a power series to its numerically computed large-L behavior. Explicitly, we match the sequence of values to successive terms in a series of the form

$$AL + B + \sum_{k=1}^{k_{\max}} \frac{\tilde{E}_{2k}}{P_{2k}(\ell)}$$

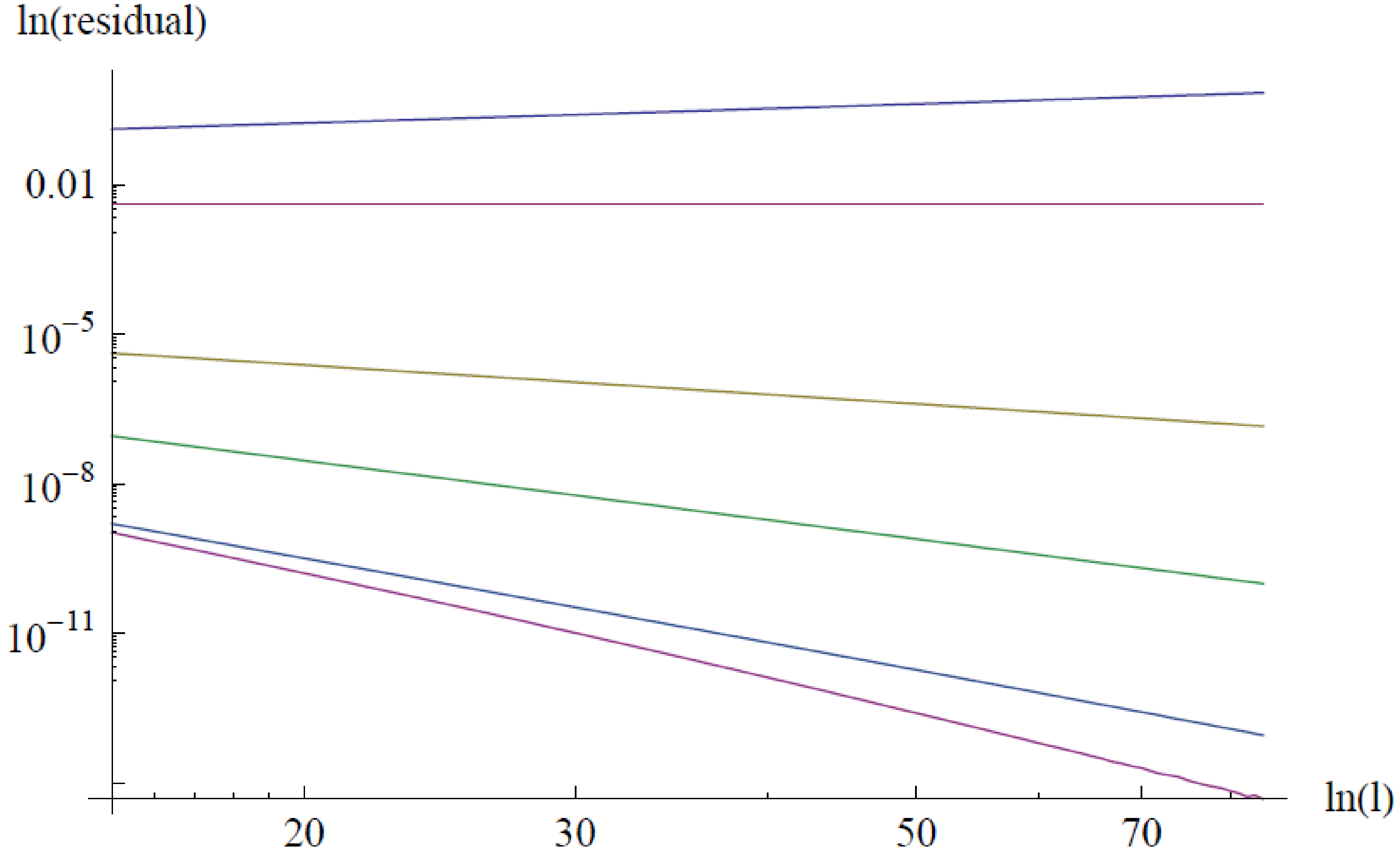
$$P_{2k}(\ell) = \prod_{i=1}^{k}\left(\ell - k - \frac{1}{2} + i\right)\prod_{j=1}^{k}\left(\ell + k + \frac{3}{2} - j\right)$$

$$\sum_{\ell=0}^{\infty} \frac{1}{P_{2k}(\ell)} = 0$$

r_0/M	A^+	B	$\left\lvert \frac{A^+}{A^+_{\text{analytic}}} \right\rvert - 1$	$\left\lvert \frac{B}{B_{\text{analytic}}} \right\rvert - 1$
6	$-1.964185503296099 \times 10^{-2}$	$-9.719920769918032 \times 10^{-3}$	6.8×10^{-14}	-5.0×10^{-11}
7	$-1.542712134731597 \times 10^{-2}$	$-7.595781032643107 \times 10^{-3}$	3.7×10^{-14}	-3.0×10^{-11}
8	$-1.235264711003273 \times 10^{-2}$	$-6.072295959309139 \times 10^{-3}$	$< 10^{-16}$	-5.7×10^{-12}
9	$-1.008020470281125 \times 10^{-2}$	$-4.954081856693618 \times 10^{-3}$	1.8×10^{-14}	-3.3×10^{-12}
10	$-8.366600265340854 \times 10^{-3}$	$-4.113353788131433 \times 10^{-3}$	1.2×10^{-14}	-7.3×10^{-12}
11	$-7.047957565474786 \times 10^{-3}$	$-3.467126055149815 \times 10^{-3}$	8.9×10^{-15}	-3.7×10^{-12}
12	$-6.014065304058753 \times 10^{-3}$	$-2.960554843842139 \times 10^{-3}$	2.5×10^{-14}	-1.1×10^{-11}
13	$-5.189692421934956 \times 10^{-3}$	$-2.556541533529994 \times 10^{-3}$	-2.4×10^{-14}	6.5×10^{-12}
14	$-4.522475818510165 \times 10^{-3}$	$-2.229391187912286 \times 10^{-3}$	8.4×10^{-14}	-2.9×10^{-11}
15	$-3.975231959999661 \times 10^{-3}$	$-1.960906506358998 \times 10^{-3}$	8.9×10^{-15}	-3.6×10^{-12}
16	$-3.521046167445508 \times 10^{-3}$	$-1.737934146698723 \times 10^{-3}$	5.8×10^{-14}	-1.7×10^{-11}
17	$-3.140087242121197 \times 10^{-3}$	$-1.550788700414580 \times 10^{-3}$	-4.6×10^{-15}	-6.7×10^{-15}
18	$-2.817502867825028 \times 10^{-3}$	$-1.392217662603554 \times 10^{-3}$	3.5×10^{-14}	-1.1×10^{-11}
19	$-2.542002591363557 \times 10^{-3}$	$-1.256707170227638 \times 10^{-3}$	-3.1×10^{-15}	9.4×10^{-13}
20	$-2.304886114323209 \times 10^{-3}$	$-1.140007036978271 \times 10^{-3}$	-5.7×10^{-15}	1.8×10^{-12}
25	$-1.500933043143495 \times 10^{-3}$	$-7.437542878990537 \times 10^{-4}$	-1.6×10^{-15}	-3.0×10^{-14}
30	$-1.054092553389464 \times 10^{-3}$	$-5.230319186714355 \times 10^{-4}$	4.2×10^{-15}	-1.1×10^{-12}
35	$-7.805574591571754 \times 10^{-4}$	$-3.876932093890911 \times 10^{-4}$	2.6×10^{-14}	-6.5×10^{-12}
40	$-6.011057519272318 \times 10^{-4}$	$-2.987922074634742 \times 10^{-4}$	3.8×10^{-15}	-8.5×10^{-13}
45	$-4.770823620144716 \times 10^{-4}$	$-2.372891353438443 \times 10^{-4}$	9.3×10^{-15}	-2.5×10^{-12}
50	$-3.878143885933070 \times 10^{-4}$	$-1.929854912525034 \times 10^{-4}$	1.8×10^{-15}	-2.3×10^{-13}
55	$-3.214363205318354 \times 10^{-4}$	$-1.600201924066196 \times 10^{-4}$	6.9×10^{-15}	-1.7×10^{-12}
60	$-2.707442873558057 \times 10^{-4}$	$-1.348310298624353 \times 10^{-4}$	4.2×10^{-15}	-9.6×10^{-13}
65	$-2.311598761671936 \times 10^{-4}$	$-1.151520162987173 \times 10^{-4}$	2.0×10^{-16}	-8.0×10^{-14}
70	$-1.996605675599292 \times 10^{-4}$	$-9.948607781191319 \times 10^{-5}$	-2.0×10^{-16}	-5.0×10^{-14}

Comparison of ΔU

r_0/M	ΔU	ΔU(from SD)	ΔU(from BS)
6	-0.29602751	-0.2960275	-0.296040244
7	-0.22084753	-0.2208475	-0.220852781
8	-0.17771974	-0.1777197	-0.177722443
9	-0.14936061	-0.1493606	-0.149362192
10	-0.12912227	-0.1291222	-0.129123253
11	-0.11387465	-0.1138747	-0.113875315
12	-0.10193557	-0.1019355	-0.101936046
13	-0.092313311	-0.09231331	-0.092313661
14	-0.084381953	-0.08438195	-0.084382221
15	-0.077725319	-0.07772532	-0.077725527
16	-0.072055057	-0.07205505	-0.072055223
18	-0.062901899	-0.06290189	-0.062902026
20	-0.055827719	-0.05582771	-0.055827795
25	-0.043599843	-0.04359984	-0.043599881
30	-0.035778314	-0.03577831	-0.035778334
40	-0.026339677	-0.02633967	-0.026339690
50	-0.020844656	-0.02084465	-0.020844661
60	-0.017247593	-0.01724759	-0.017247596
70	-0.014709646	-0.01470964	-0.014709648
80	-0.012822961	-0.01282296	-0.012822962
90	-0.011365316	-0.01136531	-0.011365317



Verification of power-law residuals after subtraction of successive terms in L^k expansion of $a^{\mathrm{ret},\,r}$