Self-force and Averaging

(also self-force differences)

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Outline

1. Self-force differences
   with Theo Drivas, arXiv:1009.0504

   • How much does the central object type affect the self-force?

2. Self-force and averaging
   arXiv:1104.5635v1

   • Self-force as angle-average of the bare force
   • Gauge-dependence, mode sum regularization
   • Mass renormalization and the "4/3 problem"

Self-force Differences

The self-force on a scalar charge moving through a vacuum region of spacetime is given by (Quinn 2001)

$$F^\mu = q^2 \left[\underbrace{\frac{1}{3}\left(\dot{a}^\mu - a^2 u^\mu\right)}_{\text{Local ALD-type force}} + \underbrace{\lim_{\epsilon \to 0^+} \int_{-\infty}^{\tau-\epsilon} \left(g^{\mu\nu} + u^\mu u^\nu\right) \nabla_\nu G(z(\tau), z(\tau')) d\tau'}_{\text{Non-local tail integral}} \right]$$

A self-force is hard to compute, requiring regularization. However, consider the *difference* in self-force between charges moving through spacetimes that agree locally, but not globally. (The orbit is kept the same; for example, a circular orbit around a star versus that around a black hole.) This self-force difference …

- …is easy to compute (no regularization).
- …probes the (non)locality of the force.
- …gives the self-force for the new spacetime once the result is known for the old spacetime.

Computing a Self-force Difference

The Green's function satisfies

$$\left[g^{\mu\nu}\nabla_\mu\nabla_\nu - \xi R\right] G(x,x') = -4\pi\delta^4(x,x')$$

The "Green's function difference" ΔG satisfies the source-free equation and is ~~smooth at the particle~~ zero in a neighborhood of the particle.

regularization

self-force: $$\lim_{\epsilon\to 0^+}\int_{-\infty}^{\tau-\epsilon}\left(g^{\mu\nu}+u^\mu u^\nu\right)\nabla_\nu G(z(\tau),z(\tau'))d\tau'$$

self-force difference: $$\Delta F^\mu = q^2\int\left(g^{\mu\nu}+u^\mu u^\nu\right)\nabla_\nu \Delta G(z(\tau),z(\tau'))d\tau'.$$

straightforward

After computing the Green's function (or field), you still must do a delicate regularization to compute the self-force. After computing the Green's function difference (or field difference), no regularization is required to get the self-force difference.

For Static Spherically Symmetric Central Bodies

Suppose our charge orbits a star instead of the black hole.
How does the self-force change?

The Green's function for each spacetime (star and black hole) can be computed in a mode expansion. Then subtract mode-by-mode,

r0: radius of star   r: field point   rq: source point

Green's function difference: $$\Delta G_{\ell\omega} = N_{\ell\omega} K_{\ell\omega}(r_0) R^{\infty}_{\ell\omega}(r) R^{\infty}_{\ell\omega}(r_q),$$

$$K_{\ell\omega} = \left. \frac{W(I_{\ell\omega}, R^H_{\ell\omega}) - \xi f^{-1} \chi I_{\ell\omega} R^H_{\ell\omega}}{W(I_{\ell\omega}, R^{\infty}_{\ell\omega}) - \xi f^{-1} \chi I_{\ell\omega} R^{\infty}_{\ell\omega}} \right|_{r=r_0}$$

$N_{\ell\omega}$ A normalization factor

$I_{\ell\omega}$: The solution of the star radiation equation regular at the origin

$R^H_{\ell\omega}$ The solution of the Schwarzschild radial equation ingoing at the horizon

$R^{\infty}_{\ell\omega}$ The solution of the Schwarzschild radial equation outgoing at infinity

A Prettier Expression

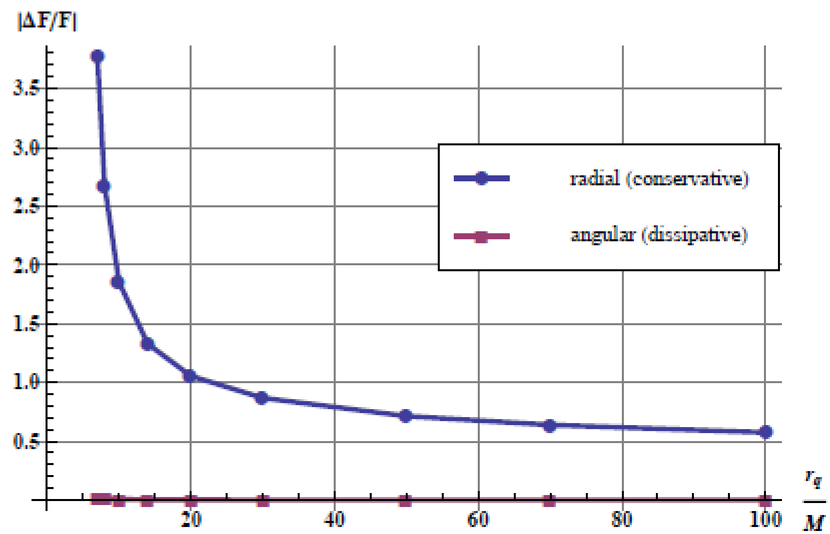
For body A/B, define radial function A/B to be the interior regular solution continued into the Schwarzschild exterior by matching at the boundary.

Then the self-force difference B minus A is given by

$$\Delta G^{AB}_{\ell\omega} = -N^{AB}_{\ell\omega} R^{\infty}_{\ell\omega}(r) R^{\infty}_{\ell\omega}(r_q), \qquad N^{AB}_{\ell\omega} = 4\pi \frac{W(B_{\ell\omega}, A_{\ell\omega})}{r^2 f W(A_{\ell\omega}, R^{\infty}_{\ell\omega}) W(B_{\ell\omega}, R^{\infty}_{\ell\omega})},$$

The Wronskian of the origin-regular solutions of the respective spacetimes controls the self-force difference.

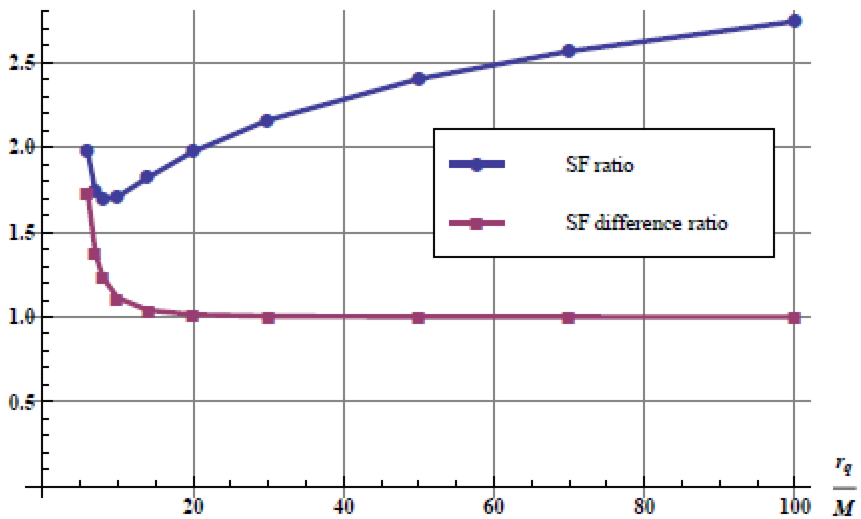
Circular orbit about a thin-shell spacetime



r_q:	6	7	8	10	14	20	30	50	70	100
radial	7.248	3.772	2.679	1.856	1.329	1.055	.8681	.7136	.6386	.5742
angular	4.229e-2	2.385e-2	1.489e-2	6.984e-3	2.233e-3	7.546e-4	2.141e-4	4.471e-5	1.611e-5	5.447e-6

FIG. 4. The fractional self-force difference $|\Delta F/F^{\text{black hole}}|$, for radial (conservative) and angular (dissipative) components. (Black hole self-force results are taken from [41].) Here we have used units where $M = 1$ and taken $\xi = 0$, $r_0 = 5$. The change in central object is seen to have a much larger effect on the radial force than on the angular force. The increase in the radial component near the shell is associated with the divergence of the radial shell self-force when $r_q = r_0$.

Comparing radial and circular orbit SF's and SFD's



r_q:	6	7	8	10	14	20	30	50	70	100
$F_r^{\mathrm{circ}}/F_r^{\mathrm{static}}$	1.975	1.746	1.693	1.710	1.822	1.975	2.161	2.404	2.567	2.742
$\Delta F_r^{\mathrm{circ}}/\Delta F_r^{\mathrm{static}}$	1.735	1.380	1.232	1.111	1.039	1.014	1.004	1.001	1.000	1.000

FIG. 5. A plot of the ratio of the circular orbit radial self-force to the static case radial self-force, along with the same quantity for the self-force difference (which agrees with the self-force in the static case, $\Delta F_r^{\mathrm{static}} = F_r^{\mathrm{static}}$). We see that there is disagreement in the self-force even for large r_q, while there is very good agreement for the self-force difference. The results of [41] have been used in the calculation of these quantities. We have used units where $M = 1$, and taken $r_0 = 5$, $\xi = 0$.

Self-force Differences Results

1. We did a complete analysis of the force on charge held static at a large distance from an arbitrary central body.

2. For the self-force difference (from Schwarzschild) for circular orbits a thin-shell spacetime, we found

    1. The change in central body has a much bigger effect on the radial (conservative) self-force than on the angular (dissipative) self-force

    2. The self-force difference for a static charge is a good estimator of the radial self-force difference for a circular orbit. That is:

       Shell circular orbit SF = BH circular orbit SF + static case SF Diff

Part II. Self-force and Averaging

1. Motivation

2. Review of the formalism in Gralla&Wald 2008

3. Hamiltonian Center of Mass (CM) and parity condition

4. Simple derivation of equations of motion

$$\ddot{Z}^a = \langle F^a[h] \rangle_{r\to 0} = \langle \left(g^{ab} + u^a u^b\right) \left(\nabla_d h_{cd} - \frac{1}{2}\nabla_b h_{cd}\right) u^c u^d \rangle_{r\to 0}$$

5. Gauge-invariance of mode-sum regularization

6. Mass renormalization and the 4/3 problem

7. Ideology: averaging vs. regular/singular decomposition

The MiSaTaQuWa equation gives the Lorenz gauge motion,

Lorenz gauge
Green's Function

$$\frac{Du^\mu}{d\tau} = -\frac{1}{2}\left(g^{\mu\nu} + u^\mu u^\nu\right)\left(2h^{\rm tail}_{\nu\lambda\rho} - h^{\rm tail}_{\lambda\rho\nu}\right)u^\lambda u^\rho.$$

$$h^{\rm tail}_{\mu\nu\lambda} = 4m\int_{-\infty}^{\tau^-} \nabla_\lambda\left(G_{+\mu\nu\mu'\nu'} - \frac{1}{2}g_{\mu\nu}G_{+\;\;\rho\mu'\nu'}^{\;\;\rho}\right)\left(z(\tau), z(\tau')\right)u^{\mu'}u^{\nu'}\,d\tau'.$$

Lorenz Gauge force

To get the force in another gauge, need to use a transformation law.

- At a theoretical level, a particular gauge choice is playing a preferred role, even though Einstein's equation (which determines the motion) is covariant.

- At a practical level, it appears that the computation of a self-force in an alternative gauge must proceed through Lorenz gauge, eliminating much of the practical appeal of working in the Lorenz gauge in the first place.

I will identify a class of gauges based on the requirement that the Hamiltonian center of mass of the particle is well-defined, and show that the force is given by the angle-average of the bare force. Interpetation: self-force = net gravitational force.

Formalism and Mass Dipole CM

To do perturbation theory of small bodies in a mathematically rigorous manner, we consider a one-parameter-family of metrics containing a body that scales to zero size and mass with the perturbation parameter, λ.

The worldline, γ, that the body ``disappears to'' characterizes its lowest-order motion. We characterize the corrected motion by a deviation vector, Z, on γ.

Our definition of Z is essentially to compute the center of mass of the body in the ``near-zone'', where the metric is stationary and asymptotically flat. We used a "mass dipole" notion,

$$D^i(t_0) \equiv \frac{3}{8\pi} \lim_{\bar{r}\to\infty} \int \bar{g}^{(0)}_{00} n^i \bar{r}^2 d\Omega \qquad Z^i(t_0) \equiv D^i(t_0)/M(t_0).$$

near-zone background metric

Hamiltonian CM and Parity Condition

Regge and Teitelboim (1974) gave a definition of center of mass as the conserved quantity associated with the asymptotic boost symmetry in the Hamiltonian formulation of asymptotically flat general relativity.

The definition is **more general** than the mass dipole definition in that it applies to non-stationary spacetimes, but also **less general** in that it requires a ``parity condition'' on the coordinates in order to be defined.

Near zone background: $$\bar{g}^{(0)}_{\bar{\mu}\bar{\nu}} = \eta_{\mu\nu} + \frac{C_{\mu\nu}(t_0, \vec{n})}{\bar{r}} + \frac{F_{\mu\nu}(t_0, \vec{n})}{\bar{r}^2} + O\left(\frac{1}{\bar{r}^3}\right)$$

Parity condition: $$C_{ij}(\vec{n}) = C_{ij}(-\vec{n})$$ (i,j label spatial coordinates)

In terms of the far-zone picture this restricts the divergent part of the first order perturbation,

$$h_{\mu\nu} = g^{(1)}_{\mu\nu} = \frac{C_{\mu\nu}(t, \vec{n})}{r} + D_{\mu\nu}(t, \vec{n}) + rE_{\mu\nu}(t, \vec{n}) + O(r^2)$$

To Parity or not to Parity?

The CM definitions are equivalent in the region of shared validity. The question of which to use is simply the question of whether to impose the parity condition.

On the **one hand**, one could view Regge-Teitelboim analysis as indicating that parity-condition-violating coordinates are “too irregular” to define the center of mass, even if the mass dipole formula is still finite.

On the **other hand**, one could view the mass dipole formula as providing an extension of the of the RT center of mass to a larger class of coordinates within the stationary case.

In any case, the parity condition offers a number of simplifications in the present context, and <u>I will adopt it</u>.

Equation of Motion

Under gauge transformations allowed by our assumptions,

$$\xi^\mu = \alpha^\mu(t, \vec{n}) + O(r),$$

The change in the spatial part of the first-order far-zone perturbation is

$$\delta h_{ij} = \underset{1/r}{-2\partial_{(i}\alpha_{j)}} + O(1) \equiv \frac{\delta C_{ij}(t, \vec{n})}{r} + O(1)$$

In order to preserve the parity condition that C is even parity, α must be of the form of a constant plus an odd parity function,

$$\xi^i = c^i(t) + \Sigma^i(t, \vec{n}) + O(r), \qquad \Sigma^i(t, -\vec{n}) \; = \; -\Sigma^i(t, \vec{n}).$$

In this case the center of mass (“deviation vector”) changes by the constant part c. We can write this as an angle-average,

$$\boxed{\delta Z^i = \langle \xi^i \rangle_{r\to 0} \equiv \frac{1}{4\pi} \lim_{r\to 0} \int \xi^i d\Omega.}$$

Key Manipulation

Consider the change in *acceleration* due to a change of gauge,

Christoffel term; vanishes by parity

$$
\begin{aligned}
\delta \ddot{Z}_i &= \langle \partial_0 \partial_0 \xi_i \rangle_{r\to 0} \\
&= \langle \nabla_0 \nabla_0 \xi_i + R_{0j0k} x^k \partial_j \xi_i + \overbrace{\nabla_0 \nabla_i \xi_0 - \nabla_i \nabla_0 \xi_0 + R_{i00}{}^{j} \xi_j}^{\text{Ricci Identity}} \rangle_{r\to 0} \\
&= \langle -(\nabla_0 \delta h_{0i} - \frac{1}{2} \nabla_i \delta h_{00}) \rangle_{r\to 0} - R_{0i0j} \langle \xi^j \rangle_{r\to 0} \\
&= \delta \langle F_i \rangle_{r\to 0} - R_{0i0j} \delta Z^j
\end{aligned}
$$

recall $-\delta h_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\nu$

Now we can "drop the deltas" to obtain

$$
\ddot{Z}^i = \langle F^i \rangle_{r\to 0} - R_{0j0}{}^{i} Z^j + \mathcal{A}^i,
$$

Where A is the constant of integration—some unknown gauge-invariant piece. We may now work in any convenient (parity-regular) gauge to determine A. (It works out to be the Papapetrou spin force.)

What if you don't impose the parity condition?

As far as I can tell, it's a big mess:

$$\ddot{Z}^i - \langle F^i \rangle_{r\to 0} + R_{0j0}{}^i Z^j - M^{-1} S^{kl} R_{kl0i} =$$
$$\langle (-\partial_0 \partial_0 \xi_j (\delta^i{}_j - 3n^i n_j) + R_{0j0k} x^k \partial^j \xi^i - R_{0i0}{}^j \xi_k (\delta^k{}_j - 3n^k n_j)) \rangle_{r\to 0},$$

The equation of motion in a parity-irregular gauge contains a gauge vector to some reference (parity-regular) gauge. **The acceleration is not given by a local expression involving just the spacetime metric!**

“Parity-Regular” gauges

I *believe* that my results apply to all gauges in which the metric perturbation goes like 1/r and satisfies the parity condition.

However, I have only *shown* that my results hold for gauges related to a gauge I used (equivalently related to Lorenz gauge) by transformations of the form

$$\xi^\mu = \alpha^\mu(t, \vec{n}) + O(r),$$

where certain restricted log terms are also allowed. I name these gauges “parity-regular”.

Bob and I were unable to find a complete proof of our belief that these two classes are equivalent, but it seems obvious. See arXiv:1104.5205.

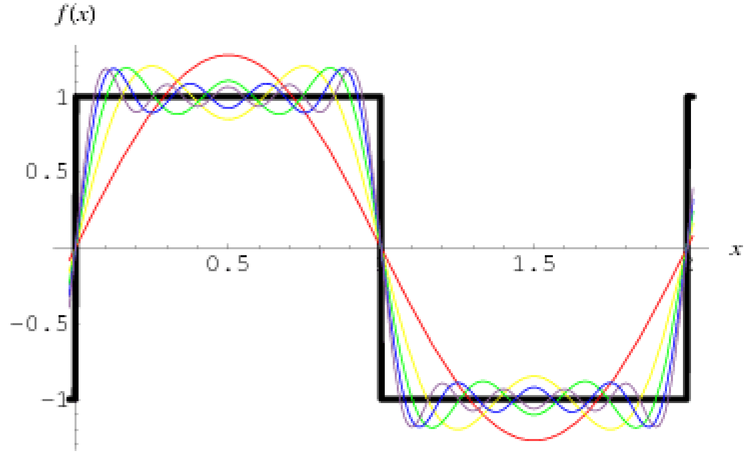
Mode-sum Regularization

To compute the self-force in a black hole background, Barack and Ori told us to numerically determine the spherical harmonic modes of the metric perturbation, and to perform a subtraction from each mode.

In terms of the angle-average result, we must find an S such that

$$\langle F^i \rangle_{r\to 0} = \sum_{\ell=0}^{\infty} \left(F^i_\ell - S^i_\ell \right) .$$

Mode sum regularization relates a mode decomposition to a local average.
This type of relationship is familiar…

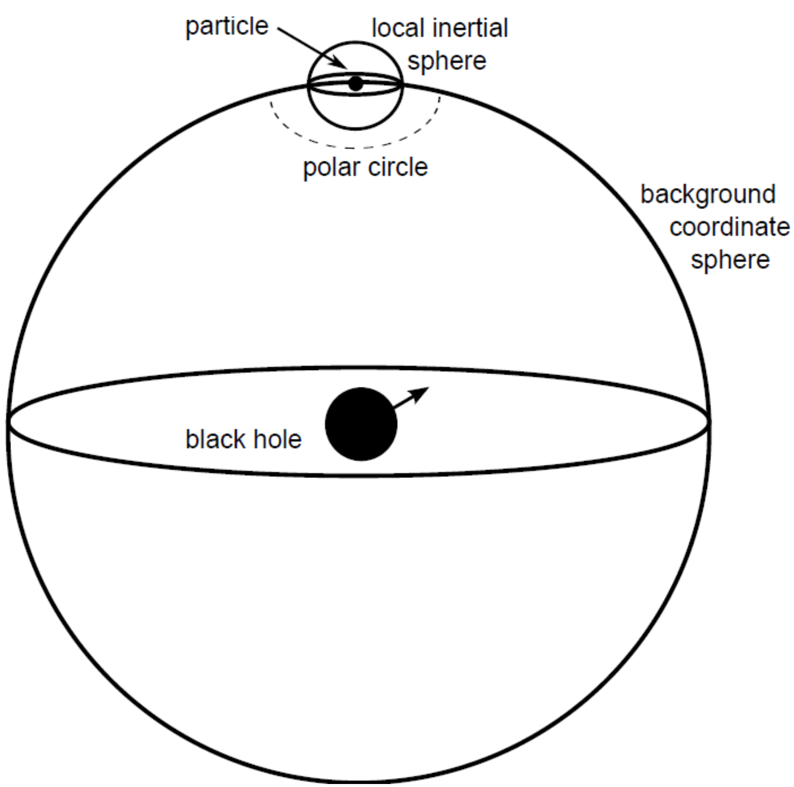


The Fourier series of a discontinuous function converges to the average at discontinuity.

For spherical harmonics, the average is over a small circle surrounding the pole.
For the change in force under a change of parity-regular gauge, we have

$$\sum_{\ell} (\delta F^i)_\ell = \underbrace{\lim_{\tilde{\theta}\to 0} \frac{1}{2\pi} \int \delta F^i|_{\tilde{t}=0,\tilde{r}=r_0}(\tilde{\theta},\tilde{\phi})d\tilde{\phi}}_{\text{polar circle average}} = \langle \delta F^i \rangle_{r\to 0},$$

local inertial sphere average
(i.e., self-force)



The first equality is a general theorem.

The second equality follows from the parity condition: δF has the form of "constant plus odd-parity", and pretty much any old average will pick out the constant part.

The process of decomposing into modes and summing automatically averages the change in bare force, computing its contribution to the self-force!

$$\sum_\ell (\delta F^i)_\ell = \langle \delta F^i \rangle_{r\to 0},$$

This means that the same subtraction S may be used in any parity-regular gauge.

$$\begin{aligned}\sum_\ell (F^i_\ell - S^i_\ell) &= \sum_\ell \left[(F^i_{\rm old})_\ell + (\delta F^i)_\ell - S^i_\ell \right] \\ &= \sum_\ell \left[(F^i_{\rm old})_\ell - S^i_\ell \right] + \langle \delta F^i \rangle_{r\to 0} \\ &= \langle F^i_{\rm old} + \delta F^i \rangle_{r\to 0}.\end{aligned}$$

modesum formula using new F with the same S...

...does indeed give the correct new force

Both Lorenz gauge and the modified radiation gauge of the Milwaukee group satisfy the parity conditi HOPEFULLY... and Ori 's Lorenz gauge results may be used in the radia

Self-force and Averaging Results

1. The self-force in any parity-regular gauge is given by the angle-average of the bare force in that gauge.

2. The perturbed mass is constant in time

3. The mode sum regularization scheme is gauge-invariant under the parity condition.

Supertranslation Dependence of Mass Dipole Center of Mass

Consider a supertranslation,

$$\delta\bar{x}^i = \alpha^i(\vec{n}) + O(1/\bar{r})$$

The old g00 is given by

$$\bar{g}_{00} = -1 + 2M/\bar{r} + O(1/\bar{r}^2)$$

So, just by plugging in we have

$$\delta\bar{g}_{00} = 2M\alpha^i n_i/\bar{r}^2 + O(1/\bar{r}^3)$$

and the center of mass changes by

$$\delta Z^i = \frac{3}{4\pi}\int \alpha^j n_j n^i d\Omega. \longrightarrow = \frac{1}{4\pi}\int \alpha^i d\Omega$$

with parity condition:

$$\underbrace{\alpha^i = c^i}_{\text{constant}} + \underbrace{\Sigma^i(\vec{n})}_{\text{odd parity}}$$

(so that the 1/r part of hij is even parity)

If I had used the left form in my derivation I would have naturally been led to the expression <(F.n)n> instead of <F> for the self-force. An aesthetic choice?

Does averaging work for electromagnetism?

At first glance, the answer is no: the average gives

$$e\langle u^\nu F_{\mu\nu}\rangle_{r\to 0} = -(\delta m)a_\mu + e^2(g_{\mu\nu}+u_\mu u_\nu)\left(\frac{2}{3}\dot{a}^\nu + \frac{1}{3}R^\nu{}_\lambda u^\lambda\right) + eF^{\text{tail}}_{\mu\nu}u^\nu$$

Mass renormalization! AHH!!! $\delta m := \lim_{s_0\to 0}\frac{2e^2}{3s_0}$

In my opinion this kind of mass renormalization is a complete physical disaster (any mathematical issues aside): it requires the body to be made of **negative energy matter**. Another worry is numerical coefficient, which doesn't match the energy of a shell: "the 4/3 problem".

However, remember the (F.n)n average? It turns out that works!

$$\boxed{e\langle u^\nu (F_{\gamma\nu}n^\gamma)n_\mu\rangle_{r\to 0} = e^2(g_{\mu\nu}+u_\mu u_\nu)\left(\frac{2}{3}\dot{a}^\nu + \frac{1}{3}R^\nu{}_\lambda u^\lambda\right) + eF^{\text{tail}}_{\mu\nu}u^\nu}$$

Ideology Slide

(Given knowledge of the correct self-force equations,
what is the best way to think about them?)

Averaging	Reg./Sing. Decomposition
Works in a big class of gauges	Works only in Lorenz
Simple to write down (requires only notion of local inertial coordinates)	Requires Hadamard decomposition to define
Physical interpretation: self-force is net force on the body	Physical interpretation: ??? Why doesn't the singular field affect the motion?
Mathematical properties: ???	Mathematical properties: smooth solution of source-free field equation.
Works for EM too	Works for EM too

Final score: 4-2. Same as Mexico > USA.

SF Differences Results

1. The change in central body has a much bigger effect on the radial (conservative) self-force than on the angular (dissipative) self-force

2. The self-force difference for a static charge is a good estimator of the radial self-force difference for a circular orbit.

Gauge and Averaging Results

1. The self-force in any parity-regular gauge is given by the angle-average of the bare force in that gauge.

2. The perturbed mass is constant in time

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