Foundational aspects of the self-force

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The self-force problem

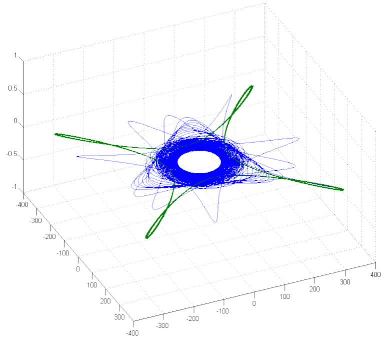
Compact objects coupled to long-range fields carry some of these fields with them as they move. *How does this self-field affect their motion?*

- *Fields can irreversibly radiate energy and momentum. This implies a recoil.*
- *Fields can also reversibly transfer energy and momentum to (and from) matter. There are therefore "conservative" effects too.*
  Self-force is not just radiation reaction!

Typical consequences of the self-force

For a sufficiently small object near internal equilibrium in orbit around a large central attractor:

- Trajectories usually precess, circularize, and decay.
- Orbital frequencies change.
- Linear and angular momentum are shifted (along with higher multipole moments of the stress-energy tensor)



Small charged particle orbiting a large spinning charge in flat spacetime with and without self-force corrections

Why study self-force?

This subject has been extensively studied in various contexts over the last century. Motivations have changed over time:

- ~~Models for elementary particles~~
- ~~Guidance to fix infinities in QFT~~
- ~~Explain inertia~~
- Obtain a deeper understanding of motion in classical field theories
- Predict behavior of certain astrophysical systems

$\vdots$

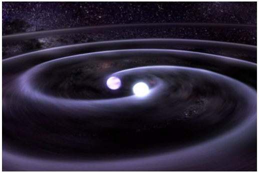
What's so hard?

- Multiple length scales
- Delicate subtractions
- What is a "self"-field anyway?
- What aspects of motion are "interesting?" What is meant by "force?"

Motion in general

The full (classical) description of, say, a binary is provided by coupling together the various PDEs describing the evolution of the fields and the matter.

$$\left.\begin{aligned} G_{ab} = 8\pi T_{ab} \\ u^a \nabla_a \rho + (\rho + p) = 0 \\ (\rho + p) u^b \nabla_b u^a + h^{ab} \nabla_b p = 0 \\ p = p(\rho) \end{aligned}\right\} \Longrightarrow$$



Solving (and providing initial data for) these equations is difficult. Doing so provides a very detailed prediction for the behavior of a very specific system.

What about a coarser description of more generic systems?

Internal and external variables in celestial mechanics

This question is interesting even in Newtonian gravity.

The standard solution is to define "internal and external variables" for (say) an N-body system:

External (or bulk) variables

Center of mass positions
Linear momenta
Angular momenta

Internal variables

Density distributions
Internal velocities
Thermodynamic variables

These two sets couple only very weakly to each other.

The behavior of the external variables is relatively simple and generic. *Focus on these.*

A way forward

One thing to do in relativistic cases is therefore to *define* something like a "center of mass" and figure out how that evolves.

- Center of mass doesn't have an obvious definition in most cases, so one might try looking at point particles (where it is obvious).
- This fails for standard theories (GR, Maxwell EM, ...) unless special rules are invented only for the motion of point particles.
- One might physically argue for such rules – "axiomatizing" the problem – but what do they mean for real (non-point) objects?

A proposal

Derive an effective theory describing some bulk features of realistic objects as quantities evolving on (some notion of) a preferred worldline.

Effective worldlines are complicated

Which "worldline" is interesting?

- Should this even be a collection of points in the physical spacetime? Not if modelling black holes!
- There are multiple approaches in the literature, some of which are quite indirect.
- All definitions are (and must be) nonlocal and coupled to fields with an infinite number of degrees of freedom.
- This means that "effective point particles" constructed from realistic objects can behave in counterintuitive ways if you look too closely... Causality is nontrivial, for example.

Two types of approach to gravitational self-force

Look outside body

- Allows strong internal gravity
- Abstract worldlines (can describe black holes)
- Results only for specific classes of objects
- Requires existence of a buffer region
- No relation between body parameters and internal structure

Look inside body

- Very general objects allowed
- No buffer region needed
- Explicit relations between body parameters and internal structure
- Concrete worldlines (cannot describe black holes)
- Currently no allowance for strong internal gravity

A unifying principle

Regardless of approach, the physical result is:

Equations describing the bulk motion of a **self-interacting** body are those of a **test** body moving in a certain fictitious (usually vacuum) field.

This holds for...

- All objects in Newtonian gravity and electrostatics
- Essentially all objects coupled to linear scalar fields or Maxwell fields in fixed curved spacetimes: AIH [2008, 2009, 2010]
- Well-isolated uncharged masses in GR with slow internal dynamics but possibly strong internal gravity: Gralla & Wald [2008], Pound [2010], Gralla [2011]
- Essentially all uncharged masses in GR whose internal metric can be approximated by a linear perturbation to a vacuum metric: AIH [2011]

Effective test body motion in Newtonian gravity

The total force and torque acting on a Newtonian mass is

$$\frac{\mathrm{d}p_i}{\mathrm{d}t} = -\int \rho \nabla_i \phi \mathrm{d}^3 x =: F_i[\phi]$$

$$\frac{\mathrm{d}S_i}{\mathrm{d}t} = -\int \rho[(x-z) \times \nabla \phi]_i \mathrm{d}^3 x =: N_i[\phi]$$

Define an effective field $\hat{\phi}$ satisfying $\nabla^2 \hat{\phi} = 0$:

$$\hat{\phi} := \phi - \left(-\int \frac{\rho(x')}{|x-x'|} \mathrm{d}^3 x' \right)$$

There's no self-force or self-torque:

$$F_i[\phi] = F_i[\hat{\phi}] \qquad N_i[\phi] = N_i[\hat{\phi}]$$

Newtonian multipole expansion

$\hat{\phi}$ is a vacuum field and usually varies slowly: $\hat{\phi}(x) = \langle \hat{\phi} \rangle_{S^2(r,x)}$.

It can therefore be expanded about the center of mass $z(t)$:

$$\frac{\mathrm{d}p_i}{\mathrm{d}t} = -\int \rho \nabla_i \hat{\phi} \mathrm{d}^3 x = -m \nabla_i \hat{\phi}(z) - \frac{1}{2} Q^{jk}(t) \nabla_{ijk} \hat{\phi}(z) + \ldots$$

$$\frac{\mathrm{d}S_i}{\mathrm{d}t} = -\int \rho [(x-z) \times \nabla \hat{\phi}]_i \mathrm{d}^3 x = \epsilon_{ijk} Q^{kl}(t) \nabla^j{}_l \hat{\phi}(z) + \ldots$$

These are the same as the equations satisfied by an extended test mass moving in the fictitious vacuum potential $\hat{\phi}$ (the "external field").

Interpretations of the effective field

There are various ways to compute $\hat{\phi}$...

- Subtract self-field from full field inside body:

$$\hat{\phi} := \phi - \left(- \int \frac{\rho(x')}{|x - x'|} \mathrm{d}^3 x' \right).$$

- "Average" the physical field over a surface:

$$\begin{aligned} \hat{\phi}(x) &= \frac{1}{4\pi} \oint \left[\frac{\nabla' \phi(x')}{|x - x'|} - \phi(x') \nabla' \left(\frac{1}{|x - x'|} \right) \right] \cdot \mathrm{d}S' \\ &= \langle \phi \rangle_{S^2(r,x)} - \left(- \frac{m}{r} \right). \end{aligned}$$

Coupling relativistic fields

Similar ideas hold for non-Newtonian systems. Generically, self-forces no longer vanish, but are still "simple."

They do two things to the test body laws of motion :

1. Relations between the multipole moments and the matter distribution change ("stress-energy of the self-field").
2. The field in which the object appears to move is no longer purely "external" (except under very special boundary conditions).

Non-Newtonian effective fields

Physically-motived axioms for point particles coupled to relativistic fields proposed by Dirac [1938], Quinn & Wald [1997], Quinn [2000], Detweiler and Whiting [2003], Poisson [2004/2011]. In one form,

"Detweiler-Whiting axiom"

Choose a certain Green function G_{S} satisfying $D[G_{\mathrm{S}}] = \delta$ and define $\hat{\phi} := \phi - \int \rho' G_{\mathrm{S}} \mathrm{d}V'$. Then $D[\hat{\phi}] = 0$ and $m\ddot{z}^a = f^a_{\mathrm{test,mon}}[\hat{\phi}]$.

The remaining self-field is interpreted only as renormalizing m.

This is now a **derived** result for very general **non-point** objects.

Effective test bodies in General relativity

Given a compact uncharged matter distribution in GR, non-perturbative definitions of linear and angular momentum (p_a, S^{ab}) have been provided in AIH [2011]. A non-perturbative "S-field" h^{S}_{ab} has also been defined.

Generalized gravitational Detweiler-Whiting axiom (AIH [2011])

If $g_{ab}|_{\mathrm{supp}\ T_{ab}}$ is sufficiently near a "background metric" $\bar{g}_{ab}$ satisfying $\bar{G}_{ab} + \Lambda \bar{g}_{ab} = 0$ and if $\hat{g}_{ab} := g_{ab} - h^{\mathrm{S}}_{ab}$ varies sufficiently slowly inside the body, (p_a, S^{ab}) evolve via Dixon's [1974] **test body** multipole expansions in the **fictitious metric** $\hat{g}_{ab}$ (to all multipole orders).

All multipole moments (not only mass!) are shifted by h^{S}_{ab}:

$$p_a = p_a^{\mathrm{bare}} + \delta p_a, \qquad S_{ab} = S_{ab}^{\mathrm{bare}} + \delta S_{ab}, \qquad J_{abcd} = J_{abcd}^{\mathrm{bare}} + \delta J_{abcd}$$

The simplest example

Apply the center of mass condition $p_a S^{ab} = 0$ to pick out a unique worldline $z(s)$ about which to evaluate the momenta.

The simplest test bodies move on geodesics wrt the background metric. Appropriate self-gravitating bodies do the same wrt $\hat{g}_{ab}$:

$$\frac{\hat{\mathrm{D}}^2 z^a}{\mathrm{d}s^2} = 0, \qquad m = \mathrm{const.}$$

"Self-force" at this level is geodesic motion in a certain vacuum metric...

Some approximations

If. . .

- g_{ab} can be approximated by a retarded Lorenz-gauge metric perturbation to a vacuum background (then $\hat{g}_{ab} \to g^{\mathrm{R}}_{ab}$),
- All timescales are long compared to object's diameter:

$$\|\ddot{z}\| d \ll 1, \qquad \omega_{\mathrm{int}} d \ll 1,$$

- External length scales are large compared to the diameter:

$$d \|\partial^{n+1} \hat{g}\| \ll \|\partial^n \hat{g}\|,$$

then

$$\frac{\hat{\mathrm{D}}^2 z^a}{\mathrm{d}s^2} \to \frac{\mathrm{D}^2_{\mathrm{R}} z^a}{\mathrm{d}s^2} \to 0 \Longleftrightarrow \text{MiSaTaQuWa equation}$$

MiSaTaQuWa equation

An equation (due to Mino, Sasaki, & Tanaka [1997] and Quinn & Wald [1997]) that allows self-interaction of a real object to be computed as though it were a point particle experiencing a force wrt a background metric $\bar{g}_{ab}$ (related to g_{ab} via Lorenz gauge)

$$\frac{\bar{\mathrm{D}}\dot{z}^a}{\mathrm{d}s} = (\bar{g}^{ad} + \dot{z}^a \dot{z}^d)\dot{z}^b \dot{z}^c (H_{dbc} - 2H_{bcd}) + \dots$$

$$H_c{}^{ab} := 4m \lim_{\epsilon \to 0} \int_{-\infty}^{s-\epsilon} \bar{\nabla}_c \bar{G}_{\mathrm{ret}}^{aba'b'} \dot{z}_{a'} \dot{z}_{b'} \mathrm{d}s'$$

This is the simplest *self-force* effect. Test body corrections to geodesic motion (like the Papapetrou force) can be comparable or larger!

Self-interaction and spin

Spinning bodies (with negligible higher moments) satisfy the Papapetrou equations:

$$\frac{\hat{\mathrm{D}} p^a}{\mathrm{d}s} = \frac{1}{2} \hat{R}_{bcd}{}^a S^{bc} \dot{z}^d$$

$$\frac{\hat{\mathrm{D}} S^{ab}}{\mathrm{d}s} = 2 p^{[a} \dot{z}^{b]}$$

If the spin is small, it is parallel-transported wrt $\hat{g}_{ab}$. Self-torque equivalent of MiSaTaQuWa (AIH [2011]):

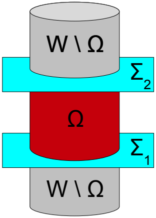
$$0 = \frac{\hat{\mathrm{D}} S_a}{\mathrm{d}s} = \frac{\bar{\mathrm{D}} S_a}{\mathrm{d}s} + 2m \dot{z}^b \dot{z}^c \bar{R}_{abc}{}^d S_d + \frac{1}{2} \dot{z}^b S^c (H_{cab} - 2H_{(ab)c})$$

Why doesn't h^S_{ab} matter?

Letting $\mathcal{F}_\xi(x,x') =$ ("force density at x due to matter at x'"), the interesting part of $P_\xi(\Sigma_2) - P_\xi(\Sigma_1)$ has the form:

$$\int_\Omega \mathrm{d}V \int_W \mathrm{d}V' \mathcal{F}_\xi(x,x') = \frac{1}{2}\int_\Omega \mathrm{d}V \Big(\int_W \mathrm{d}V' [\mathcal{F}_\xi(x,x') + \mathcal{F}_\xi(x',x)] + \int_{W\backslash\Omega} \mathrm{d}V' [\mathcal{F}_\xi(x,x') - \mathcal{F}_\xi(x',x)] \Big).$$

- The first line looks like Newton's 3rd law. It gives a force involving $\mathcal{L}_\xi \hat{G}_S^{aba'b'}$, which is a (nonlocal) linear functional of $\mathcal{L}_\xi \hat{g}_{ab}$. It renormalizes the quadrupole and higher moments.
- The second line has a contribution localized to small finite regions around Σ_1 and Σ_2. It renormalizes the momenta.



Comments and cautions

- MiSaTaQuWa (and more) has now been derived from first principles in various ways.
- "Worldlines" that this equation produces do not have a trivial interpretation.
- Many approximations are required for MiSaTaQuWa. They may or may not apply to your problem of interest.
- Other effects (like spin) can easily be as important as gravitational SF.
- MiSaTaQuWa is a general-purpose equation. Better starting points may exist for special problems (symmetric backgrounds, quasicircular orbits in specific spacetimes, etc.)

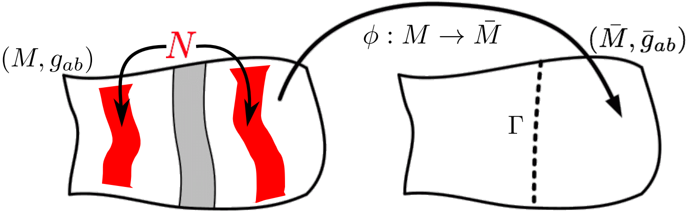
Open problems

- What about gravitational SF in the presence of non-gravitational forces (like EM)?
- Nonlinear effects?
    - Notion of "self-field" much less clear (and may not be useful)
    - Is there still a sense in which things "move on an effective vacuum metric?"
    - Or does this require assumptions on sphericity or internal equilibrium?
    - Is there a useful notion of "motion?"
- Sharp long-term error estimates
- Computational methods

Perturbative worldlines

The most popular choices for worldlines are intrinsically perturbative (Gralla & Wald [2008], Pound [2010], etc.).

Use variants of matched asymptotic expansions to ask where the far-field metric perturbations come from if they were produced by a point source coupled to the linearized (Lorenz-reduced) Einstein equation.



Choose Γ such that $g_{ab} \approx \phi^*(\bar{g}_{ab} + h^{\mathrm{lin}}_{ab}[\Gamma])$ in some buffer region $N \subset M$.

Non-perturbative worldlines

One can also non-perturbatively choose a physical worldline inside a matter distribution (e.g., Ehlers & Rudolph [1977], AIH [2011]).

- Look for something with nice properties and argue that it's representative.
- Usually consider points z satisfying $p_a(z,\Sigma)S^{ab}(z,\Sigma) = 0$.
- Problem is pushed into defining momenta p_a and S^{ab}.