Computation of Higher Order Regularisation Parameters

Anna Heffernan
University College Dublin
Collaborators: Adrian Ottewill, University College Dublin
Barry Wardell, AEI, Berlin

embarkinitiative
Investing in People and Ideas

UCD DUBLIN

UCD CASL

Capra 14, Southampton

OUTLINE

- Singular Field Calculation - we expand functions in the coordinates to produce a highly computational efficient method
- Mode Sum Decomposition - we use existing methods
- Results
- Future Work

MOTIVATION

- Make mode sum self force calculations more efficient
- Higher order regularisation parameters lead to faster convergence in l in self force calculations
- As a results, self force calculations don't require as high l's - this is particularly important in Kerr where you do spherical<->spheroidal

Singular Field

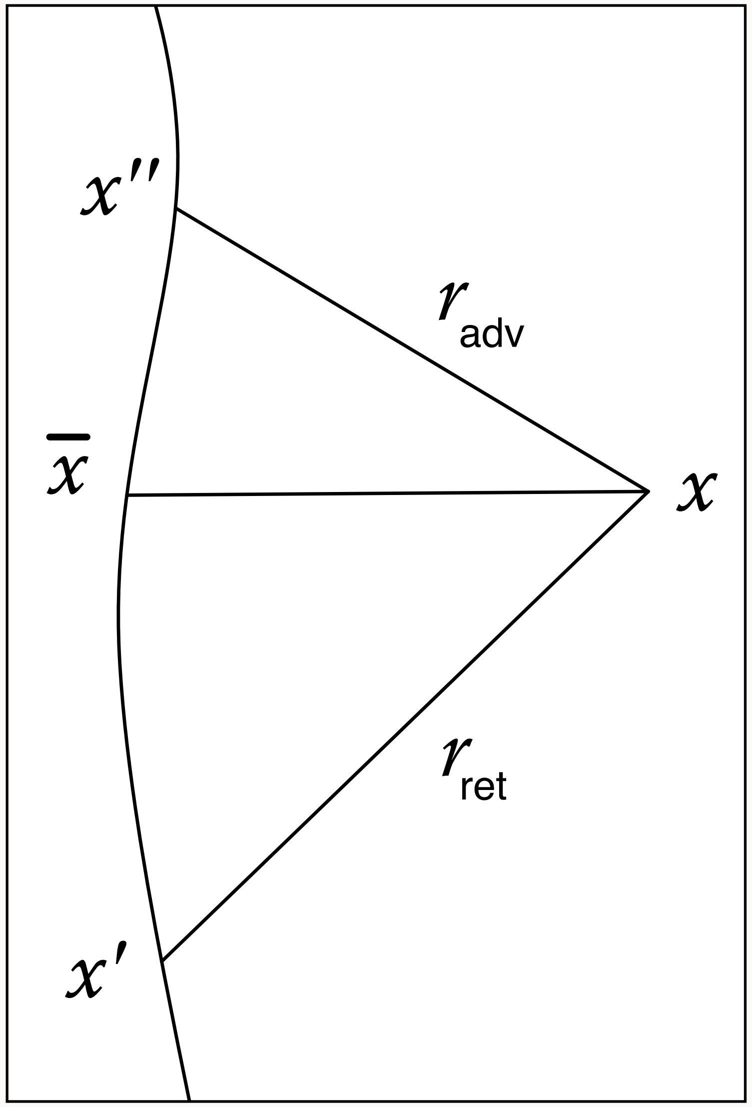
- So

$$\Phi_S(x) = \frac{q}{2}\left[\frac{U(x,x')}{r_{\rm ret}} + \frac{U(x,x'')}{r_{\rm adv}} - \int_{x'}^{x''} V(x,z(\tau'))d\tau'\right]$$

- For x close to the world-line,

$$\begin{aligned} U(x,x') = U(x,x'') &= 1 + \mathcal{O}(\delta x^4) \\ V(x,z(\tau')) &= \mathcal{O}(\delta x^4) \end{aligned}$$

and we expand $r_{\rm ret}$ and $r_{\rm adv}$ in the coordinate distance between x and the world-line (similar to Haas and Poisson)



CALCULATING SINGULAR FIELD

$$r_{ret} = \sigma_{\alpha'} u^{\alpha'} \qquad r_{adv} = \sigma_{\alpha''} u^{\alpha''}$$

- Expand $\sigma(x, \bar{x})$ (the Synge world-function)
- $$\sigma = \frac{1}{2} g_{\alpha\beta} \Delta x^\alpha \Delta x^\beta + A_{\alpha\beta\gamma} \Delta x^\alpha \Delta x^\beta \Delta x^\gamma + B_{\alpha\beta\gamma\delta} \Delta x^\alpha \Delta x^\beta \Delta x^\gamma \Delta x^\delta + C_{\alpha\beta\gamma\delta\epsilon} \Delta x^\alpha \Delta x^\beta \Delta x^\gamma \Delta x^\delta \Delta x^\epsilon + \dots$$
- Using $2\sigma = \sigma_\alpha \sigma^\alpha$ we solve for coefficients
- Need to calculate intersection points, x′ and x″, of light cone and world line

CALCULATING SINGULAR FIELD

- Expand world line $\gamma_\alpha(\tau') = x'_\alpha$, in terms of an arbitrary parameter around $\bar{x}$

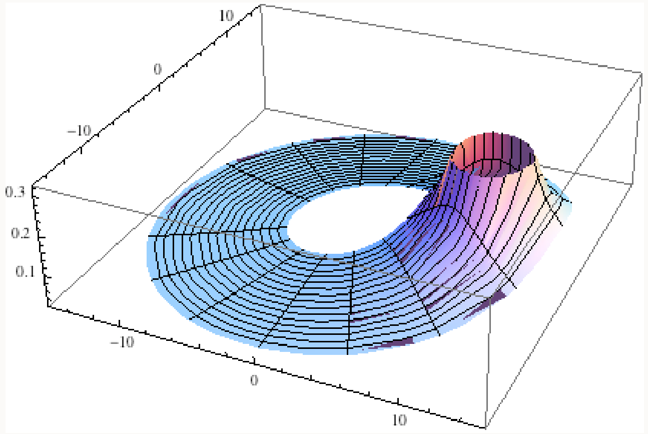
$$\gamma_\alpha(\tau') = \gamma_\alpha(\bar{\tau}) + \gamma'_\alpha(\bar{\tau})(\tau' - \bar{\tau}) + \frac{1}{2}\gamma''_\alpha(\bar{\tau})(\tau' - \bar{\tau})^2 + ...$$

- Expand $(\tau' - \bar{\tau})$, using $\sigma(x, x') = 0$ to calculate coefficients

$$(\tau' - \bar{\tau}) = \tau_1\epsilon + \tau_2\epsilon^2 + \tau_3\epsilon^3 + ...$$

- Can now calculate $\sigma_\alpha(x, x')$ using our formula for

$$\gamma_\alpha(\tau') = x'_\alpha.$$



MODE SUM DECOMPOSITION

- Follow the Barack & Ori method (also done by Detweiler, Messaritaki & Whiting, Haas & Poisson)

- Singular field of a circular orbit in the form

$$\Phi_s = \frac{1}{\rho^2} + O(\epsilon^{-1})$$

$$\rho^2_{Schw} = \frac{r\left(r(r-2m)\left(\frac{\Delta\phi^2(r-2m)}{r-3m} + \Delta\theta^2\right) + \Delta \mathrm{r}^2\right)}{r-2m}$$

$$\begin{aligned}\rho^2_{Kerr} = &-\frac{1}{(-2m\bar{r} + \bar{r}^2 + a^2)\left(am\bar{r}\left(a - 4\sqrt{m\bar{r}}\right) + 3m\bar{r}^3 - \bar{r}^4 + 2a^2m^2\right)} \\ &\times\bar{r}^2(\bar{r}(a^4\Delta\phi^2\bar{r} + \bar{r}^3\left(\Delta\theta^2\left(a^2+6m^2\right) + 2\Delta\phi^2\left(a^2+2m^2\right) + \Delta\mathrm{r}^2\right) + 4am\sqrt{m\bar{r}}\left(a^2\Delta\theta^2 + \Delta\mathrm{r}^2\right) \\ &-8am^{5/2}\Delta\theta^2\bar{r}^{3/2} - m\bar{r}^2\left(4a\left(a\left(\Delta\theta^2+\Delta\phi^2\right) - \Delta\theta^2\sqrt{m\bar{r}}\right) + 3\Delta\mathrm{r}^2\right) - m\bar{r}^4\left(5\Delta\theta^2 + 4\Delta\phi^2\right) + \bar{r}^5\left(\Delta\theta^2+\Delta\phi^2\right) \\ &-a^2m\left(\Delta\theta^2\left(a^2-4m^2\right) + \Delta\mathrm{r}^2\right)) - 2a^2m^2\left(a^2\Delta\theta^2 + \Delta\mathrm{r}^2\right))\end{aligned}$$

MODE SUM DECOMPOSITION

- Rotate coordinates

$$\sin\theta\cos\phi - \Omega t = \cos\Theta$$
$$\sin\theta\sin\phi - \Omega t = \sin\Theta\cos\Phi$$
$$\cos\theta = \sin\Theta\sin\Phi$$

- Use coordinate change to get ρ^2 in the form

$$\rho^2 = \xi^2(\delta^2 + 1 - \cos\Theta)$$

- where ξ^2 is defined by

$$\xi^2_{Schw} = \frac{2\chi\bar{r}^2\left(\bar{r} - 2m\right)}{\bar{r} - 3m}$$

- $$\xi^2_{Kerr} = \frac{2\chi\left(2a\bar{r}\sqrt{M\bar{r}} + \bar{r}^3 + a^2M\right)\left(\bar{r}\left(\bar{r} - 2M\right) + a^2\right)}{\bar{r}\left(2a\sqrt{M\bar{r}} - 3M\bar{r} + \bar{r}^2\right)}$$

MODE SUM DECOMPOSITION

- δ^2 is defined by

$$\delta^2_{Schw} = \frac{\Delta \mathrm{r}^2 \left(\bar{r} - 3m\right)}{2\chi \bar{r} \left(\bar{r} - 2m\right)^2} \qquad \delta^2_{Kerr} = \frac{\Delta \mathrm{r}^2 \bar{r}^3 \left(2a\sqrt{M\bar{r}} - 3M\bar{r} + \bar{r}^2\right)}{2\chi \left(2a\bar{r}\sqrt{M\bar{r}} + \bar{r}^3 + a^2 M\right) \left(\bar{r}\left(\bar{r} - 2M\right) + a^2\right)^2}$$

- χ is defined by

$$\chi = 1 - \alpha \sin \Phi^2$$

- α is defined by

$$\alpha_{Schw} = \frac{m}{\bar{r} - 2m} \qquad \alpha_{Kerr} = \frac{2a^3\bar{r}\sqrt{m\bar{r}} - 2a^2m^2\bar{r} + a^2m\bar{r}^2 + a^2\bar{r}^3 - 4am\bar{r}^2\sqrt{m\bar{r}} + m\bar{r}^4 + a^4m}{\left(2a\bar{r}\sqrt{m\bar{r}} + \bar{r}^3 + a^2m\right)\left(\bar{r}\left(\bar{r} - 2m\right) + a^2\right)}$$

- Rewrite higher order $(\Phi_s)_r$ in terms of ρ by subbing out $\cos \Theta$

MODE SUM DECOMPOSITION

- New form makes integration straight forward using

$$\rho^{-n-\frac{1}{2}} = \xi^{-n-\frac{1}{2}} \left(\delta^2 + 1 - \cos\Theta\right)^{-n-\frac{1}{2}}$$
$$= \xi^{-n-\frac{1}{2}} \sum_l A_l^n(\delta)\, P_l(\cos\Theta)$$

- with

$$A_l^{n+1} = -\frac{1}{(2n+1)\delta}\frac{d}{d\delta}A_l^n$$

- Also

$$\langle \chi^{-p} \rangle = {}_2F_1\left(p, \frac{1}{2}; 1; \alpha\right) = F_p$$

RESULTS

- Aterm

$$A_{Schw} = -\frac{(2l+1)\mathrm{sgn}(\Delta \mathrm{r})\sqrt{\bar{r}\left(\bar{r}-3m\right)}}{2\bar{r}^2\left(2m-\bar{r}\right)}$$

$$A_{Kerr} = -\frac{(2l+1)\mathrm{sgn}(\Delta \mathrm{r})\sqrt{\frac{\bar{r}\left(2a\sqrt{m\bar{r}}-3m\bar{r}+\bar{r}^2\right)}{\left(\bar{r}^{3/2}+a\sqrt{m}\right)^2}}}{2\left(\bar{r}\left(\bar{r}-2m\right)+a^2\right)}$$

- Bterm

$$B = b_{\frac{3}{2}}F_{\frac{3}{2}} + b_{\frac{1}{2}}F_{\frac{1}{2}}$$

$$b_{Schw\frac{1}{2}} = -\frac{\sqrt{\frac{3m-\bar{r}}{2m-\bar{r}}}}{\bar{r}^2}$$

$$b_{Kerr\frac{1}{2}} = -\frac{m\sqrt{\bar{r}}\sqrt{\frac{1}{\left(\bar{r}^{3/2}+a\sqrt{m}\right)^2\left(\bar{r}(\bar{r}-2m)+a^2\right)}}\left(a^2\left(\bar{r}-m\right)-2a\sqrt{m}\bar{r}^{3/2}+2\bar{r}^3\right)}{2\sqrt{\frac{1}{2a\sqrt{m\bar{r}}-3m\bar{r}+\bar{r}^2}}\left(2a^3\sqrt{m}\bar{r}^{3/2}+a^2\bar{r}\left(m\bar{r}+\bar{r}^2-2m^2\right)-4am^{3/2}\bar{r}^{5/2}+m\bar{r}^4+a^4m\right)}$$

$$b_{Schw\frac{3}{2}} = \frac{1}{2\bar{r}^2\left(\frac{2m-\bar{r}}{3m-\bar{r}}\right)^{3/2}}$$

$$b_{Kerr\frac{3}{2}} = \frac{\bar{r}^3\left(-a^2\sqrt{\bar{r}}\left(\bar{r}-3m\right)+2am^{3/2}\bar{r}+m\bar{r}^{3/2}\left(\bar{r}-3m\right)-2a^3\sqrt{m}\right)\sqrt{\frac{1}{\left(\bar{r}^{3/2}+a\sqrt{m}\right)^2\left(\bar{r}(\bar{r}-2m)+a^2\right)}}}{2\sqrt{\frac{1}{2a\sqrt{m\bar{r}}-3m\bar{r}+\bar{r}^2}}\left(\bar{r}\left(\bar{r}-2m\right)+a^2\right)\left(2a^3\bar{r}\sqrt{m\bar{r}}+a^2\bar{r}\left(m\bar{r}+\bar{r}^2-2m^2\right)-4a\bar{r}\left(m\bar{r}\right)^{3/2}+m\bar{r}^4+a^4m\right)}$$

RESULTS

- Cterm $$C = 0$$
- Dterm $$D = \frac{2}{(2l-1)(2l+3)}(d_{-\frac{1}{2}}F_{-\frac{1}{2}} + d_{\frac{1}{2}}F_{\frac{1}{2}} + d_{\frac{3}{2}}F_{\frac{3}{2}} + d_{\frac{5}{2}}F_{\frac{5}{2}})$$

$$d_{Schw-\frac{1}{2}} = \frac{m\left(\bar{r}-2m\right)^{3/2}}{\bar{r}^3\left(\bar{r}-3m\right)^{3/2}}$$

$$d_{Schw\frac{1}{2}} = \frac{\left(\bar{r}-4m\right)\left(\bar{r}-m\right)}{4\bar{r}^3\sqrt{-5m\bar{r}+\bar{r}^2+6m^2}}$$

$$d_{Schw\frac{3}{2}} = -\frac{\sqrt{\bar{r}-3m}\left(-7m\bar{r}+5\bar{r}^2-14m^2\right)}{8\bar{r}^3\left(\bar{r}-2m\right)^{3/2}}$$

$$d_{Schw\frac{5}{2}} = \frac{3\left(\bar{r}-3m\right)^{3/2}\left(\bar{r}+m\right)}{8\bar{r}^3\left(\bar{r}-2m\right)^{3/2}}$$

RESULTS

$$d_{Kerr-\frac{1}{2}} = \frac{-\left(\bar{r}^{3/2}+a\sqrt{m}\right)^4\sqrt{-2m\bar{r}+\bar{r}^2+a^2}}{24\bar{r}^4\left(2a^3\bar{r}\sqrt{m\bar{r}}-2a^2m^2\bar{r}+a^2m\bar{r}^2+a^2\bar{r}^3-4a\bar{r}\left(m\bar{r}\right)^{3/2}+m\bar{r}^4+a^4m\right)^7}$$

$$\times\left(2a^3\sqrt{m}\bar{r}^{3/2}-2a^2m^2\bar{r}+a^2m\bar{r}^2+a^2\bar{r}^3-4am^{3/2}\bar{r}^{5/2}+m\bar{r}^4+a^4m\right)^4\left(\frac{m}{\left(\bar{r}^{3/2}+a\sqrt{m}\right)^2\left(2a\sqrt{m\bar{r}}-3m\bar{r}+\bar{r}^2\right)}\right)^{3/2}$$

$$\times(-24m^{5/2}\bar{r}^{27/2}+48m^{7/2}\bar{r}^{25/2}+216am^3\bar{r}^{12}+56a^3m\bar{r}^{12}-468a^2m^{5/2}\bar{r}^{23/2}+8a^5\bar{r}^{11}-432am^4\bar{r}^{11}$$

$$+28a^3m^2\bar{r}^{11}+1164a^2m^{7/2}\bar{r}^{21/2}+292a^4m^{3/2}\bar{r}^{21/2}-372a^3m^3\bar{r}^{10}-132a^5m\bar{r}^{10}-576a^2m^{9/2}\bar{r}^{19/2}$$

$$-712a^4m^{5/2}\bar{r}^{19/2}+24a^7\bar{r}^9+216a^3m^4\bar{r}^9+494a^5m^2\bar{r}^9+594a^4m^{7/2}\bar{r}^{17/2}-265a^6m^{3/2}\bar{r}^{17/2}+1032a^3m^5\bar{r}^8+410a^5m^3\bar{r}^8-20a^7m\bar{r}^8$$

$$-2484a^4m^{9/2}\bar{r}^{15/2}+46a^6m^{5/2}\bar{r}^{15/2}-690a^5m^4\bar{r}^7-1052a^7m^2\bar{r}^7+1998a^4m^{11/2}\bar{r}^{13/2}+3716a^6m^{7/2}\bar{r}^{13/2}-23a^8m^{3/2}\bar{r}^{13/2}-3174a^5m^5\bar{r}^6$$

$$+272a^7m^3\bar{r}^6+360a^9m\bar{r}^6-2010a^6m^{9/2}\bar{r}^{11/2}-2302a^8m^{5/2}\bar{r}^{11/2}+1044a^5m^6\bar{r}^5+4352a^7m^4\bar{r}^5-14a^9m^2\bar{r}^5-1335a^6m^{11/2}\bar{r}^{9/2}+670a^8m^{7/2}\bar{r}^{9/2}$$

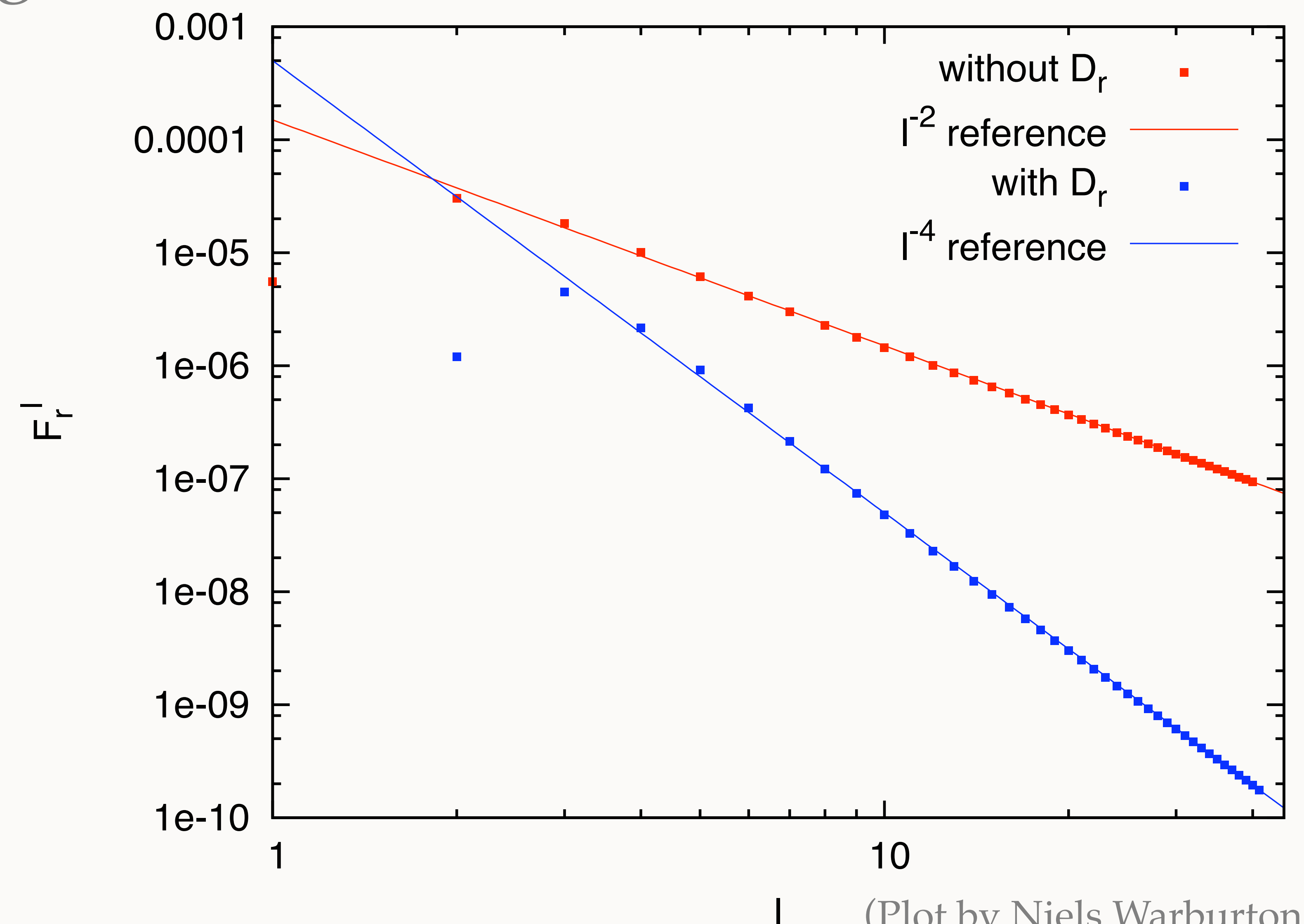
$$-1332a^7m^5\bar{r}^4-2114a^9m^3\bar{r}^4+174a^6m^{13/2}\bar{r}^{7/2}+1898a^8m^{9/2}\bar{r}^{7/2}-180a^7m^6\bar{r}^3+558a^9m^4\bar{r}^3+360a^{11}m^2\bar{r}^3+480a^{10}\left(m\bar{r}\right)^{3/2}\bar{r}^3-267a^8m^{11/2}\bar{r}^{5/2}$$

$$+274a^9m^5\bar{r}^2-76a^{11}m^3\bar{r}^2-74a^{10}m\left(m\bar{r}\right)^{3/2}\bar{r}^2+144a^{12}m^{5/2}\bar{r}^{3/2}-140a^{11}m^4\bar{r}-904a^{10}m^2\left(m\bar{r}\right)^{3/2}\bar{r}$$

$$+24a^{13}m^3+138a^{10}m^3\left(m\bar{r}\right)^{3/2}-24a^{12}\sqrt{m^7\bar{r}}+144a^8\sqrt{m\bar{r}^{15}}+10a^6\sqrt{m\bar{r}^{19}}-48a^4\sqrt{m\bar{r}^{23}})$$

RESULTS

- Regularised l-modes of radial self force



(Plot by Niels Warburton)

FUTURE WORK

- Work out E and F in Schwarzschild and Kerr
- Look at eccentric or general geodesics
- Look at spheroidal harmonic decomposition in Kerr
- Calculate higher order regularisation parameters in (Lorenz gauge) gravity