Two-timescale methods and transient resonances

in the inspirals of point particles into massive black holes

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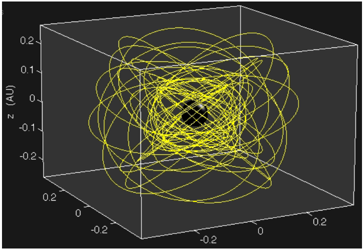
Motivation

- ▷ Inspirals of compact objects $\mu \sim 10 M_\odot$ into massive black holes $M \sim 10^3 - 10^6 M_\odot$ are a promising source of gravitational waves.
- ▷ Last $\sim$ year of inspiral contains $\sim M/\mu \sim 10^2 - 10^5$ cycles of waveform in the relativistic regime.

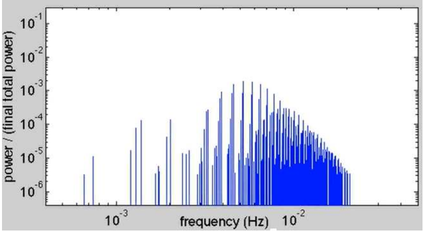
Many science payoffs: map spacetime, learn about black hole growth history and galaxy cores, cosmology

Required: theoretical waveforms with phase accuracy $\sim 10^{-2} - 10^{-5}$.

Geodesics in Kerr



Emitted GW spectrum



(from S. Drasco's black hole movies)

Timescales in the problem

$\epsilon = \mu/M \ll 1$

- On short timescales:

$$\tau_{\rm orb} \sim M \qquad \sim 50 \text{ s} \left(\frac{M}{10^7 M_\odot}\right)$$

μ moves on a geodesic of M's background spacetime.

- On longer timescales

$$\tau_{\rm rr} \sim M/\epsilon \qquad \sim 1.6 \text{ yrs} \left(\frac{M}{10^7 M_\odot}\right)\left(\frac{10^{-6}}{\epsilon}\right)$$

gravitational radiation reaction causes the orbit to gradually evolve.

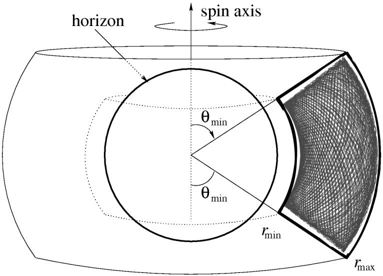
- Geodesic orbits and true orbits dephase by ~ 1 cycle after

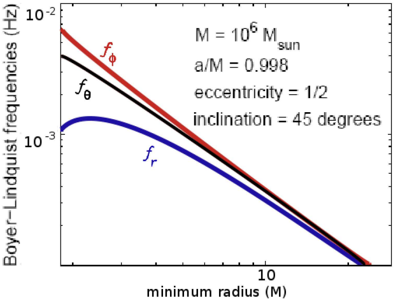
$$\tau_{\rm deph} \sim M/\sqrt{\epsilon} \qquad \sim 13 \text{ hrs} \left(\frac{M}{10^7 M_\odot}\right)\left(\frac{10^{-6}}{\epsilon}\right)^{1/2}$$

Bound geodesic orbital dynamics in Kerr

Conserved quantities:

- Energy $E = -\xi^a u_a$, azimuthal angular momentum $L_z = \varphi^a u_a$, (special cases of conserved currents $T_{ab}\xi^b$, $T_{ab}\varphi^b$)
- Carter constant $Q = K_{ab}u^a u^b$, $\nabla_{(a}K_{bc)} = 0$
- Reparameterization: $(E, L_z, Q) \leftrightarrow (p, e, \iota)$





(Steve Drasco 2005)

Overview of the talk

- Motivation for a self-consistent two-timescale treatment of inspirals in Kerr
- Stage I: Two-timescale orbital motion
- Qualitatively new feature for generic orbits: transient resonances
- Properties, effects of the resonances
- Implications for gravitational wave science
- Stage II: Sketch of two-timescale Einstein Equations

Methods of computing waveforms

- Numerical relativity:
  - ▷ impractical, despite much recent progress (H. Pfeiffer, C. Lousto)
- Post-Newtonian methods:
  - ▷ invalid in the relativistic regime, useful for scoping out capabilities (Gair et al 2004, Brown et al 2006, Barack & Cutler 2004)
- Effective One-body methods
  - ▷ require calibration, useful for detection templates (Yunes et al, T. Damour)
- Black hole perturbation theory, first order

$$h_+ - ih_\times = \sum_{lmkn} Z_{lmkn} h_{lmkn}$$

  - ▷ Self-consistency requires a geodesic worldline source
  - ▷ produces "snapshots" valid for $\tau_{\mathrm{deph}} \ll \tau_{\mathrm{inspiral}}$ (Drasco et al 2005)

Methods of computing waveforms cont.

Use of conservation laws:

- ▷ compute fluxes $\langle \dot{E} \rangle$ and $\langle \dot{L}_z \rangle$ to infinity and down the horizon, infer evolution of orbital E, L_z
- ▷ works only for circular or equatorial orbits, leading order
- ▷ can track an entire inspiral either in the frequency or time domain

(Shibata 1994, Glampedakis et al 2002, Hughes 2000, Krivan et al 1997, Burko et al 2003, Martel 2003, Souperta et al 2005, Sundararajan et al 2007)

Use averaged self-force: Adiabatic waveforms

- ▷ requires only $\tilde{a}^{\mathrm{SF}} = P \cdot \nabla(h^{\mathrm{ret}} - h^{\mathrm{adv}})/2$ (Mino 2003)

$$\langle \dot{Q} \rangle = \sum_{lmkn} c^{\mathrm{EH}}_{lmkn} |Z^{\mathrm{EH}}_{lmkn}|^2 + c^{\infty}_{lmkn} |Z^{\infty}_{lmkn}|^2, \qquad \langle \dot{E} \rangle = \sum_{lmkn} |Z^{\mathrm{EH}}_{lmkn}|^2 + |Z^{\infty}_{lmkn}|^2$$

- ▷ correct to leading order, waveforms in hand (ignoring inconsistencies)

Computing waveforms cont.

Black Hole Perturbation Theory, Second Order:

▷ validity still limited to $\tau_{\rm deph} \ll \tau_{\rm inspiral}$

sourced by a geodesic of $g^{(0)}$

$$g_{\alpha\beta} = g^{(0)}_{\alpha\beta} + \epsilon h^{(1)}_{\alpha\beta} + \epsilon^2 h^{(2)}_{\alpha\beta} + O(\epsilon^3)$$

After $\tau_{\rm deph}$: $\epsilon h^{(1)}_{\alpha\beta} \sim \epsilon^2 h^{(2)}_{\alpha\beta}$

Related: use of the gravitational self force:

▷ Given inspiralling orbit, how to compute waveforms?

linearized theory + non-geodesic source for waveforms is inconsistent, results expected to be gauge dependent

Possible fixes:

▷ geodesic deviation Gralla, Wald '10

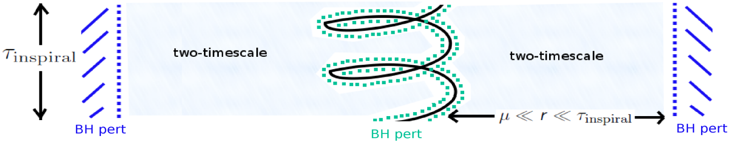
▷ relax the gauge condition Quinn, Wald '97, Mino '03, Pound '10

▷ stitch together many small evolution steps

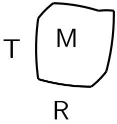
▷ this talk: two-timescale expansions

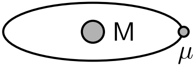
Two-timescale expansion

- Systematic framework, gives rigorous derivation of adiabatic prescription at leading order.
- Gives method for computing post-adiabatic corrections.
- Two stages: (i) Orbital motion, (ii) Einstein eqns.
- Basis of method:
  Posit a one-parameter family of worldlines $z^\alpha(\tau, \epsilon)$ and metrics $g_{\alpha\beta}(x, \epsilon)$ of a specific form in a specific region of spacetime
- locally in time: ansatz is compatible with geodesic motion and first order perturbation theory



Context: Approximation Schemes





Minkowski → Linearized Pert. → 2nd Order Pert.

Minkowski → Newtonian → Post-1 Newt.

Kerr → Linearized Pert. → 2nd Order Pert.

Kerr → Adiabatic → Post-1 Adiabatic

$M/R \ll 1$	$M/R \ll 1$
$M/R \ll R^2/T^2$	$R/T \ll 1$

Orbital motion part of the two-timescale formalism

Action angle variables $q_\alpha = (q_t, q_r, q_\theta, q_\phi)$, $J_\lambda = (E/\mu, L_z/\mu, Q/\mu^2)$

Geodesic equation in Kerr with self-force:

$$\frac{dq_\alpha}{d\tau} = \boldsymbol{\omega}_\alpha(J_\lambda) \; + \epsilon\, g_\alpha^{(1)}(q_r, q_\theta, J_\lambda) + O(\epsilon^2),$$

$$\frac{dJ_\lambda}{d\tau} = 0 \; + \epsilon\, G_\lambda^{(1)}(q_r, q_\theta, J_\nu) + \epsilon^2\, G_\lambda^{(2)}(q_r, q_\theta, J_\nu) + O(\epsilon^3)$$

▷ tori in phase space

▷ fundamental frequencies $\boldsymbol{\omega}_r$, $\boldsymbol{\omega}_\theta$ (t and φ symmetries)

Two-timescale expansions for the solutions:

- ansatz for the dependence on ϵ
- initially based on adiabaticity ($\tau_{\rm orb} \ll \tau_{\rm rr}$),
  later augmented by treatments of resonances, separatrices

Two-timescale expansion

Introduce:
- a "slow" variable $\tilde{\boldsymbol{\tau}} = \boldsymbol{\epsilon}\,\boldsymbol{\tau}$
- auxiliary phase variables $\boldsymbol{\psi}_\alpha$

Ansatz: asymptotic expansion of the solutions at fixed $\tilde{\tau}$:

$$\psi_\alpha(\tilde{\tau}, \epsilon) = \frac{1}{\epsilon}\,\psi_\alpha^{(0)}(\tilde{\tau}) + \frac{1}{\sqrt{\epsilon}}\,\psi_\alpha^{(1/2)}(\tilde{\tau}) + \ln\epsilon\,\psi_\alpha^{(\ln)} + \psi_\alpha^{(1)}(\tilde{\tau}) + O(\epsilon^{1/2})$$

$$J_\lambda(\tau, \epsilon) = \mathcal{J}_\lambda^{(0)}(\tilde{\tau}) + \sqrt{\epsilon}\,\mathcal{J}_\lambda^{(1/2)}(\tilde{\tau}) + \epsilon\ln\epsilon\, J^{(\ln)} + \epsilon\, J_\lambda^{(1)}(\psi_\alpha, \tilde{\tau}) + O(\epsilon^{3/2})$$

$$q_\alpha(\tau, \epsilon) = \psi_\alpha(\tilde{\tau}) + O(\epsilon)$$

Adiabatic approximation

$$J_\lambda(\tau,\epsilon) = \mathcal{J}_\lambda^{(0)}(\tilde\tau) + \sqrt{\epsilon}\mathcal{J}_\lambda^{(1/2)}(\tilde\tau) + \epsilon J_\lambda^{(1)}(\psi_\alpha,\tilde\tau) + O(\epsilon^{3/2})$$

$$\psi_\alpha(\tilde\tau,\epsilon) = \frac{1}{\epsilon}\psi_\alpha^{(0)}(\tilde\tau) + \frac{1}{\sqrt{\epsilon}}\psi_\alpha^{(1/2)}(\tilde\tau) + \psi_\alpha^{(1)}(\tilde\tau) + \ldots$$

$$\frac{d\mathcal{J}_\lambda^{(0)}}{d\tilde\tau} = \langle G_\lambda^{(1)}\rangle[\mathcal{J}_\nu^{(0)}(\tilde\tau)], \qquad \psi_\alpha^{(0)}(\tilde\tau) = \int^{\tilde\tau}\omega_\alpha[\mathcal{J}_\lambda^{(0)}(\tilde\tau')]d\tilde\tau'.$$

nonresonant tori are ergodic:

$$\langle G_\lambda^{(1)}\rangle \equiv \frac{1}{(2\pi)^2}\int_0^{2\pi} dq_r \int_0^{2\pi} dq_\theta \, G_\lambda^{(1)}(q_r, q_\theta, J_\lambda)$$

Adiabatic prescription:

- Replace $G_\lambda^{(1)}$ by $\langle G_\lambda^{(1)}\rangle$, drop all other forcing terms.
- Closed system, simpler: no need to specify ϵ to solve the ODEs

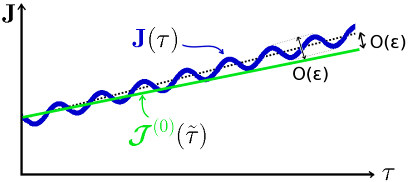
Properties of the adiabatic approximation

$$\frac{d\mathcal{J}_\lambda^{(0)}}{d\tilde{\tau}} = \langle G_\lambda^{(1)} \rangle [\mathcal{J}_\nu^{(0)}(\tilde{\tau})], \qquad \psi_\alpha^{(0)}(\tilde{\tau}) = \int^{\tilde{\tau}} \omega_\alpha [\mathcal{J}_\lambda^{(0)}(\tilde{\tau}')] d\tilde{\tau}'$$

Fourier expansion of the forcing functions:

$$G^{(1)}(q_r, q_\theta, J_\lambda) = G_{00}^{(1)} + \sum_{k_r \neq 0} \sum_{k_\theta \neq 0} G_{k_r k_\theta}^{(1)}(J_\lambda) e^{i(k_r q_r + k_\theta q_\theta)}$$

Full solution & adiabatic approx.



Post-1-adiabatic corrections

$$J_\lambda(\tau,\epsilon) = \mathcal{J}_\lambda^{(0)}(\tilde\tau) + \sqrt{\epsilon}\mathcal{J}_\lambda^{(1/2)}(\tilde\tau) + \epsilon\, J_\lambda^{(1)}(\psi_\alpha,\tilde\tau) + O(\epsilon^{3/2})$$

$$\psi_\alpha(\tilde\tau,\epsilon) = \frac{1}{\epsilon}\psi_\alpha^{(0)}(\tilde\tau) + \frac{1}{\sqrt{\epsilon}}\psi_\alpha^{(1/2)}(\tilde\tau) + \psi_\alpha^{(1)}(\tilde\tau) + O(\epsilon^{1/2})$$

- require $g^{(1)}$, $G^{(1)}$ and $\langle G^{(2)}\rangle$ (currently unknown)
- give rise to phase errors of $O(1)$ over an inspiral

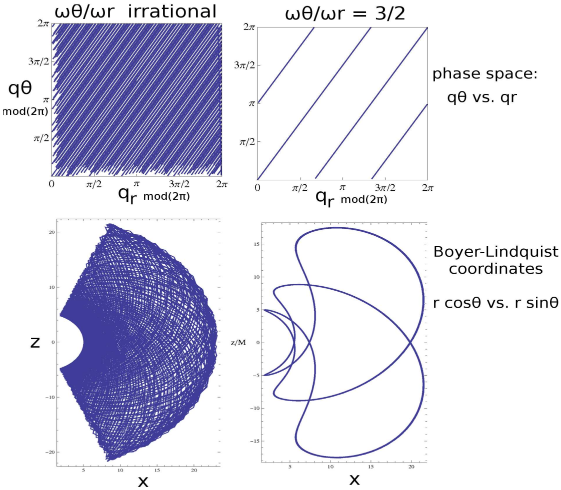
Decompose the forcing terms according to parity properties under $q_r \to 2\pi - q_r, \quad q_\theta \to 2\pi - q_\theta$

$$G_\lambda = \epsilon \underbrace{\left[\langle G^{(1)}_{\lambda,\ \mathrm{dissipative}}\rangle\right.}_{\text{known}} + \underbrace{\delta G^{(1)}_{\lambda,\ \mathrm{dissipative}}}_{\text{“known”}} + \underbrace{\left. G^{(1)}_{\lambda,\ \mathrm{conservative}}\right]}_{\text{“known”}} + \underbrace{O(\epsilon^2)}_{\text{unknown}}$$

Transient Resonances

Occur when $\omega_\theta / \omega_r = \mathbf{k}/\mathbf{n}$ with **k**, **n** small integers.

Geometric picture:
trajectories fill up lower-dimensional tori



Resonance Timescale

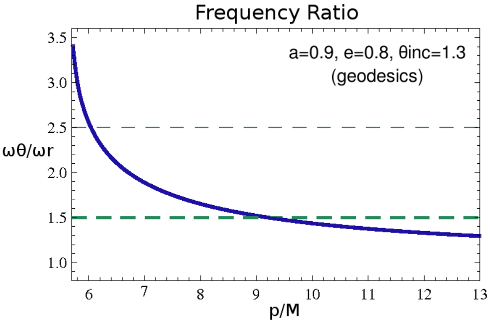
Timescale: intermediate between $\tau_{\rm rr}$ and $\tau_{\rm orb}$:

$$\tau_{\rm res} \sim 1/\sqrt{\epsilon\, \mathbf{s}}, \qquad \mathbf{s} = \mathbf{k} + \mathbf{n}$$

near a resonance at $\tau = 0$:

$$G^{(1)} \sim G^{(1)}_{00} + \sum_{v} G^{(1)}_{v(\mathbf{kn})} e^{iv\left[\cancel{(\mathbf{n}\omega_r - \mathbf{k}\omega_\theta)\tau} + (\mathbf{n}\dot{\omega}_r - \mathbf{k}\dot{\omega}_\theta)\tau^2/2 + \dots\right]} + \text{oscill.}$$

$$\dot{\omega} \sim \epsilon$$



strong resonances:

$\mathbf{s} \lesssim O(|\ln(\epsilon)|)$

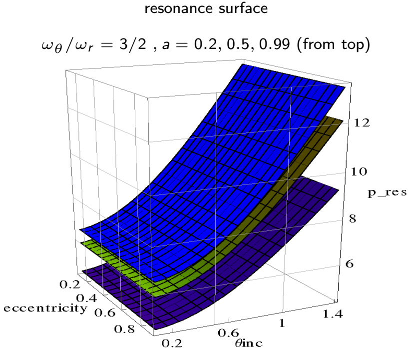
▷ few and isolated

Estimates for relevant resonances and locations

- Resonant frequency combination:

$$\sigma = n\omega_r - k\omega_\theta = 0$$

for Kerr inspirals: $n > k$

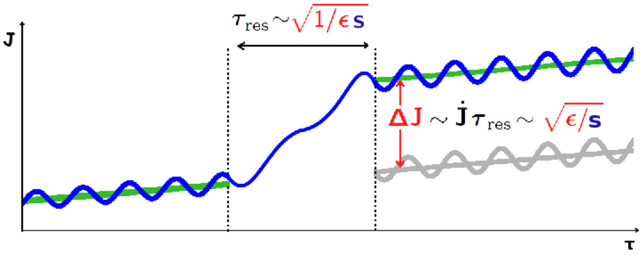


- Scaling estimates of $G^{(1)}_{kn}$ based on post-Newtonian orbits:

$$\frac{\dot{E}^{\rm PN}_{kn}}{E^{\rm PN}} \sim -\frac{e^n\ a^2\ \sin^2\iota}{(1-e^2)p^6}\ \delta_{k,2}\ f_E(e), \qquad \frac{\dot{Q}^{\rm PN}_{kn}}{Q^{\rm PN}} \sim \frac{e^n\ a^2\ \sin^2\iota}{p^6}\ \delta_{k,2}\ f_Q(e),$$

$$\frac{\dot{L}^{\rm PN}_{z\ kn}}{L^{\rm PN}_z} \sim \frac{e^n \sin^2\iota}{p^{11/2}}\ \delta_{k,2}\left[\frac{a}{\cos\iota}\ f_1(e) + \frac{a^2}{\sqrt{p}}\ f_2(e)\right]$$

Effect of a transient resonance



- $O(1)$ corrections to the adiabatic approximation during $\tau_{\rm res}$
- net jumps $\Delta J = O(\sqrt{\epsilon})$ across a resonance, affect the frequencies: $\Delta\omega \sim \omega_{,J}\,\Delta J$
- leads to phase errors $\sim 1/\sqrt{\epsilon}$ after further inspiral

Analytic Treatment

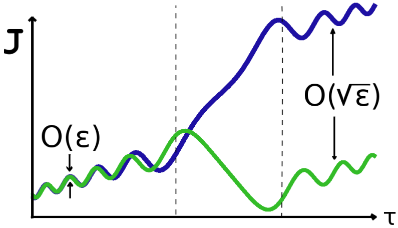
- Relevant phase in the resonance zone: $Q = \mathbf{k}\ q_r - \mathbf{n}\ q_\theta$
- Use matched asymptotic expansions to compute

$$\boldsymbol{\Delta J}_\lambda \sim \sqrt{\epsilon}\ N(Q_0,\ G^{(1)}_{kn},\ G^{(1)}_{00},\ \partial\dot{Q}/\partial J_\lambda),$$

phase entering the resonance to $O(1)$

- For the post-resonance phase to $O(1)$, also need $\Delta J^{(1)}_\lambda$ and ΔQ (✓)

Effect of differences in initial conditions



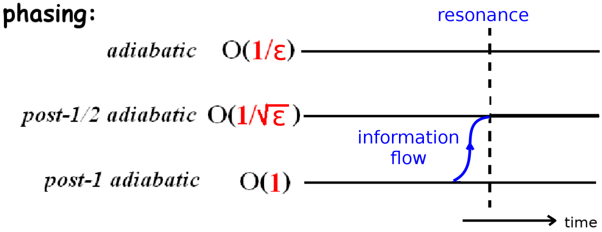
linearized approximation :

$$\boldsymbol{\Delta J}_\lambda \sim \sqrt{\epsilon}\ 2\sqrt{\pi} \sum_{v \neq 0} \frac{G^{(1)}_{vk,vn}}{\sqrt{v|\sigma_{,\tilde{\tau}}|}}$$

$$\times \left[\cos(v\,Q_0) - \mathrm{sgn}(\sigma_{,\tilde{\tau}}) \sin(v\,Q_0)\right]$$

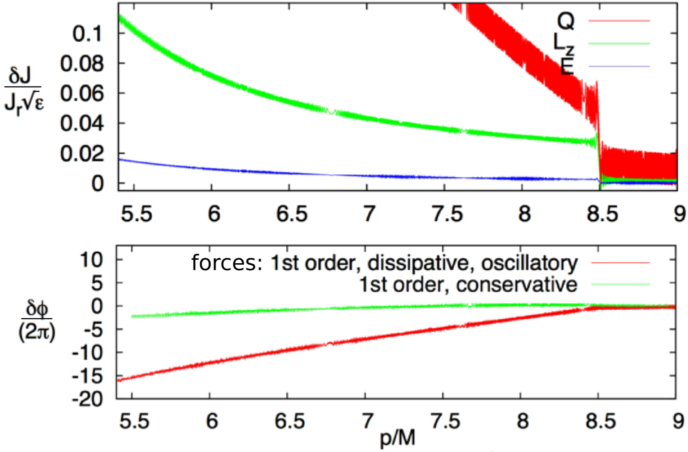
Properties of transient resonances

- Non-perturbative effects in v/c.
- Occur for eccentric, inclined orbits around spinning black holes.
- Driven by spin-dependent, instantaneous, dissipative pieces of the self-forces.
- Cause increased sensitivity to initial conditions:



Numerical estimates

Integrate Kerr geodesics + approximate post-Newtonian forcing terms



a=0.95, ε=3x10 -6

ωθ/ωr=3/2

Matching the phase evolution

Can idealize the phase evolution as:

$$\psi_\alpha(\tau;\epsilon) = \begin{cases} \epsilon^{-1}\psi_\alpha^{(0)}(\tilde{\tau}) + O(1), & \tau - \tau_0 \ll \tau_{\rm res} \\ \epsilon^{-1}\psi_\alpha^{(0)}(\tilde{\tau} + \sqrt{\epsilon}\Delta\tilde{\tau}) - \omega_\alpha(\tilde{\tau}_0)\Delta\tilde{\tau}/\sqrt{\epsilon} + O(1), & \tau - \tau_0 \gg \tau_{\rm res} \end{cases}$$

- effective $O(\sqrt{\epsilon})$ time shift: $\Delta\psi_\alpha \sim \omega_\alpha(\tilde{\tau}_0)\Delta\tilde{\tau}/\sqrt{\epsilon}$,
- equivalently: match to post-resonance phasing, phase error is accumulated before the resonance:

define $\tilde{\psi}_\alpha(\tau) = \psi_\alpha(\tau + \sqrt{\epsilon}\Delta\tau) - \Delta\psi_\alpha$, then:

$$\psi_\alpha(\tau;\epsilon) = \begin{cases} \epsilon^{-1}\tilde{\psi}_\alpha^{(0)}(\tilde{\tau}) - \Delta\tilde{\tau}/\sqrt{\epsilon}\,[\omega_\alpha(\tilde{\tau}) - \omega_\alpha(\tilde{\tau}_0)] + O(1), & \tau - \tau_0 \ll \tau_{\rm res} \\ \epsilon^{-1}\tilde{\psi}_\alpha^{(0)}(\tilde{\tau}) + O(1), & \tau - \tau_0 \gg \tau_{\rm res} \end{cases}$$

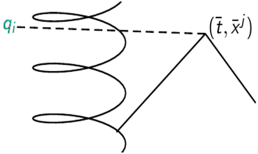
Summary: Transient resonances

- Qualitatively new feature of two-body problem in the relativistic, small mass ratio regime.
- Occur for generic orbits.
- Make the orbit more sensitive to changes in the initial data.
- Give rise to phase corrections that scale as $\sqrt{M/\mu}$, sudden jumps in the time derivative of the phases.
- Require currently unknown pieces of the gravitational self force.
- Numerical estimates: $\delta\phi \sim 10$ cycles for mass ratios $\sim 10^{-6}$.
- Further investigations in progress.

Two-timescale metric (non-resonant case)

Ansatz for the metric in the region $\mu \ll r \ll M/\epsilon$

$$\begin{aligned} g_{\alpha\beta}(\bar{t}, \bar{x}^j, \epsilon) &= g^{(0)}_{\alpha\beta}(\bar{x}^i) + \epsilon h^{(1)}_{\alpha\beta}\left(q_r, q_\theta, q_\phi, \tilde{t}, \bar{x}^i\right) \\ &+ \epsilon^2 h^{(2)}_{\alpha\beta}\left(q_r, q_\theta, q_\phi, \tilde{t}, \bar{x}^i\right) + O(\epsilon^3) \end{aligned}$$



assumes periodicity in $q_i = (q_r, q_\theta, q_\phi)$

$$\tilde{t} \equiv \epsilon \bar{t}, \quad q_i(\bar{t}, \epsilon) = \frac{1}{\epsilon} f_i^{(0)}(\epsilon \bar{t}) + f_i^{(1)}(\epsilon \bar{t}) + \ldots$$

$\partial / \partial \bar{t}$ = Killing vector

Self-consistency verified by substitution into Einstein's equations.

Double expansion of the Einstein Equation

- Metric:

$$g_{\alpha\beta}(\bar{t}, \bar{x}^j; \epsilon) = g^{(0)}_{\alpha\beta}(\bar{x}^j) + \epsilon h^{(1)}_{\alpha\beta}(q_r, q_\theta, q_\phi, \tilde{t}, \bar{x}^j) + O(\epsilon^2)$$

- Connection:

$$\nabla = \nabla^{(0,0)} + \epsilon \left[\nabla^{(0,1)} + \nabla^{(1,0)}_{[h^{(1)}]} \right] + O(\epsilon^2)$$

factors of h derivatives involving $\tilde{t}$ and $\Omega^{(s)}$

- Einstein tensor:

$$G_{\alpha\beta}[g] = G_{\alpha\beta}[g^{(0)}] + \epsilon G^{(1,0)}_{\alpha\beta}[h^{(1)}] + O(\epsilon^2)$$

- $O(\epsilon)$ stress-energy conservation:

$$\nabla^{(0,0)} G^{(1,0)}[h^{(1)}] = 0$$

Solving the Einstein Eqs. (I)

$$G^{(1,0)}_{\alpha\beta}[h^{(1)}] = 0 \qquad \rightarrow \qquad \mathcal{D} h^{(1)}_{\alpha\beta} = 0,$$

linear differential operator on the 6-dim manifold $(q_r, q_\theta, q_\phi, \bar{x}^j)$

solution that matches to $T^{(1)}_{\alpha\beta}(q_i, \bar{x}^j, \tilde{t})$ is:

$$\begin{aligned} h^{(1)}_{\alpha\beta} &= \frac{\partial g^{(0)}_{\alpha\beta}}{\partial M} \delta M(\tilde{t}) + \frac{\partial g^{(0)}_{\alpha\beta}}{\partial a} \delta a(\tilde{t}) + \ldots \\ &+ \mathcal{F}_{\alpha\beta}[q_r, q_\theta, q_\phi, \bar{x}^j, E(\tilde{t}), L_z(\tilde{t}), Q(\tilde{t})], \end{aligned}$$

same as in standard linear pert. theory with a geodesic source

gauge freedom:

(i) $x^\alpha \rightarrow x^\alpha + \epsilon \xi^\alpha(q_r, q_\theta, q_\phi, \tilde{t}, x^j) + O(\epsilon^2)$

(ii) $\epsilon-$independent transformations that preserve $\partial/\partial\bar{t}$

Required solutions to get q_i to $O(1)$:

$$G^{(1,1)}[h^{(1)}] + G^{(2,0)}[h^{(1)}, h^{(1)}] + G^{(1,0)}[h^{(2)}] = T^{(2)}$$

the 00 Fourier component determines the $\tilde{t}$-dependence of the adiabatic solution (similar to conservation law approach),
oscillatory pieces give schematically

$$h^{(2)} = \mathcal{G}^{(2)}(\tilde{t}, \bar{x}^j) + H_{\alpha\beta}(q_i, \tilde{t}, \bar{x}^j).$$

Last missing piece: the 00 Fourier component of:

$$G^{(0,1)}[h^{(2)}] + \langle G^{(3,0)}[h^{(1)}, h^{(1)}, h^{(1)}] + 2G^{(2,1)}[h^{(1)}, h^{(2)}] + G^{(1,2)}[h^{(1)}] \rangle = 0$$

Summary

- Systematic 2-timescale approximation scheme, resolves the difficulties with the standard perturbation theory.
- Framework for computing higher order corrections to the adiabatic evolution.
- Identification of which pieces of the forcing functions are required to compute the motion at each order
- Treatment of resonances.