On the presence of Jost solutions in time-domain self-force calculations

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Scheme

1. Problem and Objective
2. Review of the Jost solutions issue
3. Casting *singular sources* as *jump* conditions
   - On the equivalence of the resulting PDE systems
   - Illustration in the charged scalar particle
4. A perspective from late times
5. A by-product: smooth *time switch-on* of the sources
6. Conclusions

Outline

Problem and Objective

General Physical Problem: EMRIs

- Modelling of the dynamics and Gravitational Wave emission of an Extreme Mass Ratio Inspiral.

General Methodology: self-force

- Perturbative approach around a stationary black hole: calculation of the gravitational *self-force*.
- **Time-domain approach** to the resolution of the self-force with **singular sources** (*a posteriori mode-sum* regularization of the solution).
- Use of **trivial Initial Data**.

Specific objective: Jost solutions [JLJ, C. Sopuerta & P. Canizares, PRD 83, 061503(R) (2011)]

Understanding/clarifying the possible emergence of persistent spurious solutions in time-domain schemes, referred to as a *Jost junk solution*, that may contaminate self-force calculations. **Are Time Domain calculations safe?**

Outline

Time domain approach: singular sources and trivial data

Evolution equation

Let us assume a spherically symmetric Black Hole.
Field $\Psi(t,r)$, spherical harmonic component of retarded field (most gauges):

$$\left[-\partial_t^2 + \partial_{r^*}^2 - V(r)\right]\Psi(t,r) = f(r)\left[\,G(t,r)\,\delta(r - r_p(t)) + F(t,r)\,\delta'(r - r_p(t))\,\right]$$

Trivial Initial data

Correct physical initial data not known. Common practice:

$$\Psi|_{t=t_o} = (\partial_t\Psi)|_{t=t_o} = 0$$

Claim: [Field, Hesthaven & Lau, PRD 81, 124030 (2010)]

Jost solutions may appear as a consequence of **inconsistent singular sources** and **trivial initial data**.

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Jost solution: numerical approach I

Impulsive solution: construction

$\Psi_{\text{Impulsive}}$: use sources $F(t,r)$, $G(t,r)$ and trivial data.

Smoothed (*'consistent'*) solution: construction

Define smoothed sources (consistent with trivial data):

$$F^{\mathsf{s}}_{\tau}(t,r) \equiv \alpha(t,\tau)F(t,r) \text{ and } G^{\mathsf{s}}_{\tau}(t,r) \equiv \alpha(t,\tau)G(t,r)$$

with $\alpha(t_o,\tau)=0$ and $\alpha(t,\tau)=1$, for $t \gg \tau$.

Ψ_{Smooth}: use sources $F^{\mathsf{s}}_{\tau}(t,r)$, $G^{\mathsf{s}}_{\tau}(t,r)$ and trivial data.

Numerical Jost solution: construction

$$\boxed{\Psi^{\text{N}}_{\text{Jost}} \equiv \Psi_{\text{Impulsive}} - \Psi_{\text{Smooth}}}$$

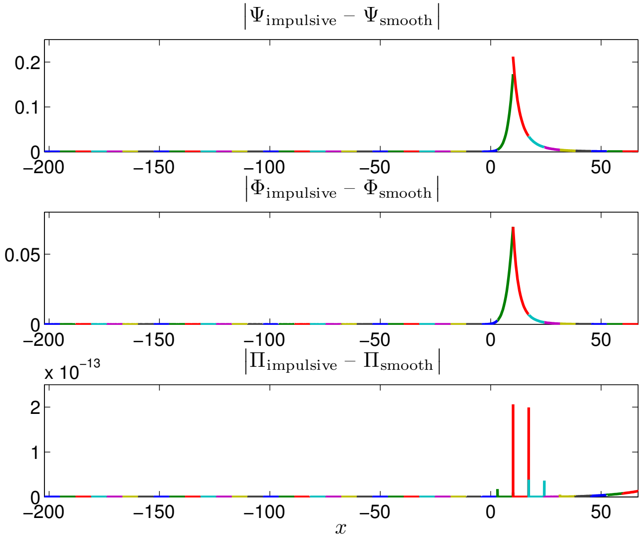
Properties:

(i) It is time independent: $\partial_t \Psi^{\text{N}}_{\text{Jost}} = 0, \forall t$.

(ii) It has a jump at the particle: $\left[\, \Psi^{\text{N}}_{\text{Jost}} \,\right]_p = -f_p^{-1} F(t_o, r_p)$, where $f_p \equiv f(r_p)$.

(iii) The spatial derivative, $\partial_{r^*}\Psi^{\text{N}}_{\text{Jost}}$, is continuous at $r = r_p$.

Jost solution: numerical approach II



[Field, Hesthaven & Lau, PRD 81, 124030 (2010)]

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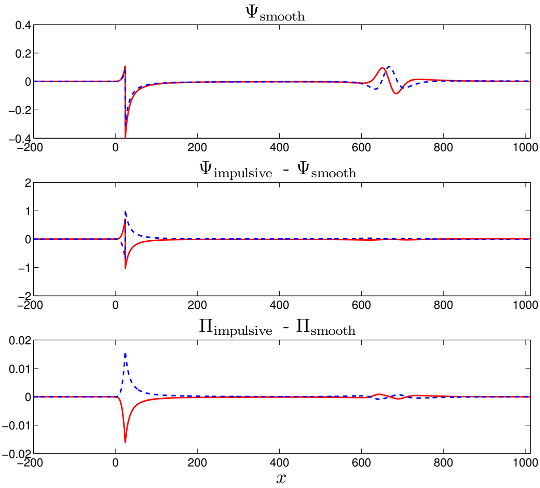

[Field, Hesthaven & Lau, PRD 81, 124030 (2010)]

Jost solution: analytical approach

Analytical Jost solution

Motivated by the numerical solution, construct: $\boxed{\Psi^{\rm A}_{\rm Jost} \equiv \Psi^{\rm A,-}_{\rm Jost}\,\Theta_- + \Psi^{\rm A,+}_{\rm Jost}\,\Theta_+}$
with: $\Theta_+ \equiv \Theta(r^* - r^*_p)$, $\Theta_- \equiv \Theta(r^*_p - r^*)$.
(Θ the Heaviside step function)

Stationary multi-domain equation

$\Psi^{\rm A,-}_{\rm Jost}$ and $\Psi^{\rm A,+}_{\rm Jost}$ satisfy

$$\left[\partial^2_{r^*} - V(r)\right]\Psi^{\rm A,\pm}_{\rm Jost} = 0$$

with $\left[\,\Psi^{\rm A}_{\rm Jost}\,\right]_p = -f_p^{-1}F(t_o, r_p)$. This means $\Psi^{\rm A}_{\rm Jost}$ satisfy:

$$\left[\partial^2_{r^*} - V(r)\right]\Psi^{\rm A}_{\rm Jost} = -f_p F(t_o, r_p)\delta'(r - r_p)\,,$$

Equivalence Numerical and Analytical approach

Differences between $\Psi^{\rm A}_{\rm Jost}$ and $\Psi^{\rm N}_{\rm Jost}$ vanish to numerical precision.

Posed questions

The Jost solution Problem:

There exists a Difference between the smoothed (consistent) and the impulsive (inconsistent) solutions:

(numerical) *Artifact* or *Real*?

An early *warning*...

Sources are identical at late times $\Longrightarrow$ Difference of solutions should be a solution of the homogeneous..

But the Jost solution has a delta source...

Are we actually solving *different* problems?

Outline

From delta source to jumps I: idea

Particle-Without-Particle approach [Canizares & Sopuerta, PRD79, 084020 (2009)]

Basic idea to deal with the *two-scales problem* in EMRIs:

Transform:

- The problem with a **singular (delta) sources**.

Into:

- **Jump conditions** between two homogeneous wave-equations in neighbouring domains, by placing the particle at the boundary between two domains.

From delta source to jumps II: first-order and multidomain system

Evolution system: Second-order formulation

$$\left[-\partial_t^2 + \partial_{r^*}^2 - V(r)\right]\Psi(t,r) = \underbrace{f(r)\left[G(t,r)\,\delta(r-r_p(t)) + F(t,r)\,\delta'(r-r_p(t))\right]}_{S(t,r)}$$

From delta source to jumps II: first-order and multidomain system

Evolution system: First-order formulation

Introduce fields: $\phi \equiv \partial_t \Psi \, , \; \varphi \equiv \partial_{r^*} \Psi$

Evolution equations:

$$
\begin{aligned}
\partial_t \Psi &= \phi \, , \\
\partial_t \phi &= \partial_{r^*} \varphi - V(r)\Psi - S(t,r) \\
\partial_t \varphi &= \partial_{r^*} \phi
\end{aligned}
$$

Multidomain approach

Ansatz for field splitting (only two domains, circular orbit, **just for simplicity!**):

$$
\begin{aligned}
\Psi &= \Psi^- \, \Theta_- + \Psi^+ \, \Theta_+ \\
\phi &= \phi^- \, \Theta_- + \phi^+ \, \Theta_+ \\
\varphi &= \varphi^- \, \Theta_- + \varphi^+ \, \Theta_+ + [\, \Psi \,]_p \, \delta(r^* - r^*_p)
\end{aligned}
$$

with a jump for a field χ: $[\, \chi \,]_p \equiv \lim_{\epsilon \to 0} (\chi(r_p + \epsilon) - \chi(r_p - \epsilon))$

From delta source to jumps III: the system with jumps

Evolution system

Homogeneous evolution equations AND **jump conditions**:

$$\begin{aligned} \partial_t \Psi^\pm &= \phi^\pm \\ \partial_t \phi^\pm &= \partial_{r^*} \varphi^\pm - V(r)\Psi^\pm \\ \partial_t \varphi^\pm &= \partial_{r^*} \phi^\pm \end{aligned} \qquad \begin{aligned} [\,\Psi\,]_p\,(t) &= f_p^{-1} F(t, r_p) \\ [\,\phi\,]_p\,(t) &= f_p^{-1} (\partial_t F)(t, r_p) \\ [\,\varphi\,]_p\,(t) &= G(t, r_p) - f_p^{-1} (\partial_{r^*} F)(t, r_p) \end{aligned}$$

Remarks:

- Not all initial conditions are consistent with singular sources.
- Jumps $[\,\Psi\,]_p$ and $[\,\phi\,]_p$ contain redundant information: one is the derivative of the other.
  BUT if $[\,\Psi\,]_p\,(t_o) = f_p^{-1} F(t_o, r_p)$ is not enforced, equivalence to original system is lost.

Outline

Evolution system with jumps integrated: first version

Reinserting the sources in the system [Field, Hesthaven & Lau, PRD 81, 124030 (2010)]

$$
\begin{aligned}
\partial_t \Psi &= \varphi \\
\partial_t \phi &= \partial_{r^*}\varphi - V(r)\Psi - J_\varphi\,\delta(r^* - r^*_p)\,, \\
\partial_t \varphi &= \partial_{r^*}\phi + J_\phi\,\delta(r^* - r^*_p)
\end{aligned}
$$

with $J_\varphi = [\,\varphi\,]_p\,, J_\phi = [\,\phi\,]_p$

Remarks:

- $[\,\Psi\,]_p\,(t)$ is not explicitly enforced.
- It is only enforced through $[\,\varphi\,]_p$: i.e. only $\partial_t\,[\,\Psi\,]_p\,(t)$ is enforced.
- $[\,\Psi\,]_p\,(t_o) = f_p^{-1}F(t_o, r_p)$ is never enforced.
- The resulting system is inequivalent to the original one.
- *Jost solutions* appear in this system.
- But note that $[\,\Psi\,]_p\,(t_o) = 0$ is consistent with the trivial initial data. A **choice** is necessary: either Initial Data or the Sources...

Evolution system with jumps integrated: second version

Remarks:

Redefine: $\tilde{\varphi} = \varphi + [\,\Psi\,]_p\, \delta(r^* - r^*_p)$ The system is then:

$$
\begin{aligned}
\partial_t \phi &= \partial_{r^*} \tilde{\varphi} - V(r)\Psi - J_\varphi \delta(r^* - r^*_p) - J_\Psi \delta'(r^* - r^*_p) \\
\partial_t \tilde{\varphi} &= \partial_{r^*} \phi \,,
\end{aligned}
$$

with $J_\varphi = [\,\varphi\,]_p$, $J_\Psi = [\,\Psi\,]_p$.

Remarks:

- $[\,\Psi\,]_p\,(t)$ **IS** explicitly enforced.
- The system **IS** equivalent to the original one.
- No Jost solutions appear in this system.

Main observation

Jost junk solutions appear when implementing a *finite* jump condition, $[\,\Psi\,]_p$, through an infinitesimal equation $(\partial_t\,[\,\Psi\,]_p = [\,\phi\,]_p)$ *without* simultaneously imposing the initial value $[\,\Psi\,]_p\,(t_o)$ consistent with the singular source.

Outline

Charged particle in a circular orbit around Schwarzschild

Evolution system

Spherical harmonic modes: $\Phi_{\ell m}$, satisfy our model equation with: $\Psi = r\Phi_{\ell m}$, $F = 0$, and $G = f^{-1}S_{\ell m}$.
Characteristic variables (Ψ, u, v): $(u,v) = (\phi - \varphi, \phi + \varphi)$.

$$
\begin{aligned}
\partial_t \Psi^\pm &= (u^\pm + v^\pm)/2\,, & [\,\Psi\,]_p\,(t) &= 0 \\
\partial_t u^\pm &= -\partial_{r^*} u^\pm - V(r)\Psi^\pm\,, & [\,u\,]_p\,(t) &= -G(t, r_p) \\
\partial_t v^\pm &= \partial_{r^*} v^\pm - V(r)\Psi^\pm\,, & [\,v\,]_p\,(t) &= G(t, r_p)
\end{aligned}
$$

Enforcing jump conditions [C. Sopuerta, P. Canizares & JLJ, PRD 82, 044023 (2010)]

Two different strategies:

i) **Approach I**: Finite jumps $[\,u\,]_p\,(t)$ and $[\,v\,]_p\,(t)$ directly enforced with triv. ID.

ii) **Approach II**: Jump time derivatives, $d\,[\,u\,]_p\,/dt$ and $d\,[\,v\,]_p\,/dt$, enforced as extra evolution equations with appropriately modified (non-trivial) ID *guaranteeing correct initial values of the jumps*: $[\,u\,]_p\,(t_o)$ and $[\,v\,]_p\,(t_o)$.

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Characteristic variables (Ψ, u, v): $(u, v) = (\phi - \varphi, \phi + \varphi)$.

$$\begin{aligned} \partial_t \Psi^\pm &= \phi^\pm\,, & \left[\,\Psi\,\right]_p(t) &= 0 \\ \partial_t \phi^\pm &= \partial_{r^*}\varphi^\pm - V(r)\Psi^\pm\,, & \left[\,\phi\,\right]_p(t) &= 0 \\ \partial_t \varphi^\pm &= \partial_{r^*}\phi^\pm\,, & \left[\,\varphi\,\right]_p(t) &= G(t, r_p) \end{aligned}$$

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Generating a *Jost* solution

Testing our understanding of the Jost problem

If our understanding of the problem is correct, a *Jost-like* solution should emerge when calculating the **difference** between the solutions in **Approches I and II**, but using trivial ID in **both** cases.

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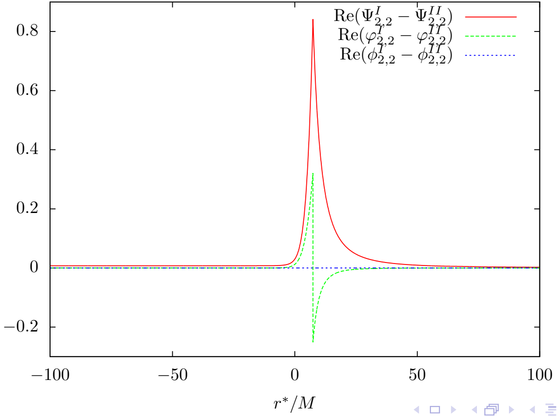


Outline

Perspective from late times I

Hitherto, focus on *initial data*. But...

Crucial point: sources faithfully implemented at late times

- Late times: in the studied problem, the only relevant element in late times analysis is the correct implementation of the sources.
- Choice of initial data should not play any role: any intermediate-time data can be taken as valid ID.

But then, why do the *smooth time switch-on* solutions work...?

The role of the smooth time switch-on $\alpha(t, \tau)$

It **guarantees the correct implementation of the late time sources** *when the source is implemented through an evolution equation for the jumps, with zero initial values.*

Perspective from late times II

Sources implemented through *smoothed jumps* with trivial ID

- Consider a field χ with jump condition and smoothed jump condition
$$[\chi]_p = J_\chi(t), \qquad [\tilde{\chi}]_p = \alpha(t,\tau) J_\chi(t)$$
- Implement the jumps using evolution equations:
$$d[\chi]_p/dt = J'_\chi, \qquad d[\tilde{\chi}]_p/dt = (\alpha J_\chi)'$$
with zero initial values: $[\chi]_p(t_o) = [\tilde{\chi}]_p(t_o) = 0$,
- Then

$$\begin{aligned} [\chi]_p &= J_\chi(t) - J_\chi(t_o), \\ [\tilde{\chi}]_p &= \alpha(t,\tau) J_\chi(t) - \alpha(t_o,\tau) J_\chi(t_o) \\ &= \alpha(t,\tau) J_\chi(t) \simeq J_\chi(t) \text{ (for } t \gg \tau) \end{aligned}$$

Conclusion

Smooth solution Ψ_{Smooth} solves the correct dynamical equation, whereas the Impulsive solution $\Psi_{\mathrm{Impulsive}}$ does not (it is *contaminated by a Jost solution*).

Outline

The smooth switch-on as a high-frequency noise filter

We have learned:

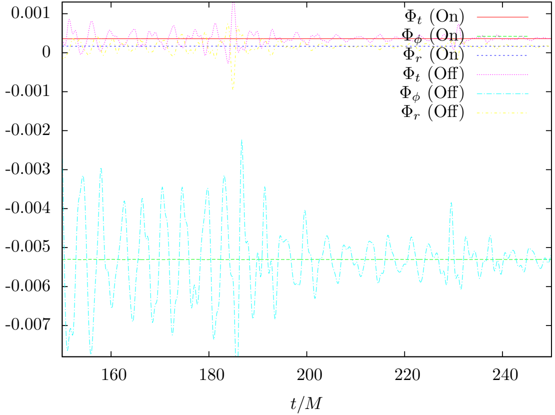
The *smooth switch-on* is not the critical element in the discussion of the Jost solution: it is a *sufficient* prescription guaranteeing consistency in a particular approach.

But it certainly improves the consistency between the source and the ID...

By-products of using $\alpha(t, \tau)$ [Field, Hesthaven & Lau, Class. Quantum Grav. 26, 165010 (2009)]

- Reduces the burst of junk radiation [Field, Hesthaven & Lau, CQG 26, 165010 (2009)].
- More importantly: it can be critical in the calculation of the self-force in certain approaches. In the **Approach I** [C. Sopuerta, P. Canizares & JLJ, PRD 82, 044023 (2010), JLJ, C. Sopuerta & P. Canizares, PRD 83, 061503(R) (2011)]:
  - High-frequency numerical noise due to instantaneous switch-on of the source. Very slow time decay (much larger than the orbital time scale).
  - Noise can be eliminated by: a) increasing the resolution, or b) using a numerical filter (**remark**: this provides an explicit example of correct solution with both singular sources AND trivial data).
  - The use of the *smooth switch-on* of the source provides a natural filter.

Caluclation of self-force in **Approach I** to the jumps



Components of the self-force, $(\Phi_t, \Phi_r, \Phi_\phi)$, with ("On") and without ("Off") time smooth switch-on: No recovery of the second significant digit in $\Phi_r = 1.677 \cdot 10^{-4}$, $\Phi_t = 3.609 \cdot 10^{-4}$ in the "Off" case.

$$\left(r_p = 6M, t_{\mathrm{fin}} = 250M > 2 \cdot T_{\mathrm{orb}} = 2 \cdot 2\pi (r_p^3/M)^{\frac{1}{2}} \approx 184.69M\right)$$

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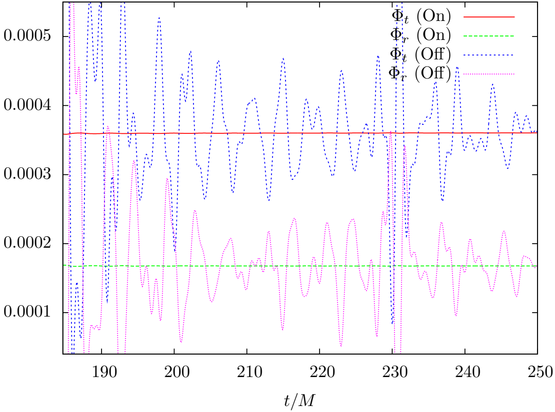

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Conclusions

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- **No contamination of the retarded solution by Jost junk solutions, as long as the sources are faithfully implemented at late times**.
- It is NOT a numerical issue.
- The conclusion above is conceptually independent of the Initial Data and/or the use of a *smooth time switch-on*.
- The use of a *smooth time switch-on* of the sources prevents the high-frequency noise that spoils self-force calculations in some numerical schemes.
- Possible extension of this last point to other scenarios (e.g. junk radiation in numerical relativiy initial data...)?