Periastron Advance in Black Hole Binaries

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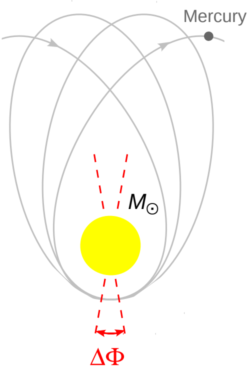
Relativistic perihelion advance of Mercury

- Observed anomalous precession of Mercury's perihelion of $\sim$ 43" /century
- Accounted for by the leading order relativistic angular advance per orbit

$$\Delta\Phi_{\mathrm{GR}} = \frac{6\pi G M_{\odot}}{c^2 a\,(1-e^2)}$$

- One of the first successes of Einstein's theory of general relativity
- Relativisic periastron advance of $\sim$ $^{\circ}$/year now measured in binary pulsars



Periastron advance in black hole binaries

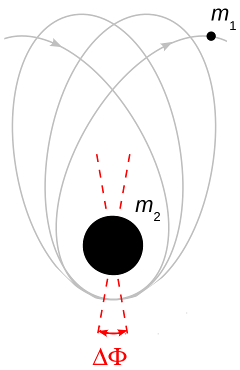
- Conservative part of the dynamics only
- Generic non-circular orbit parametrized by the two invariant frequencies

$$\Omega_r = \frac{2\pi}{P}\,, \quad \Omega_\varphi = \frac{1}{P}\int_0^P \dot{\varphi}(t)\,\mathrm{d}t$$

- Periastron advance per radial period

$$K \equiv \frac{\Omega_\varphi}{\Omega_r} = 1 + \frac{\Delta\Phi}{2\pi}$$

- In the circular orbit limit $e \to 0$, the relation $K(\Omega_\varphi)$ is coordinate invariant



Post-Newtonian and EOB results

- Third post-Newtonian result [Damour, Jaranowski & Schäfer 2000]

$$K_{3\mathrm{PN}} = 1 + 3x + \left(\frac{27}{2} - 7\nu\right) x^2 + \left(\frac{135}{2} - \left[\frac{649}{4} - \frac{123}{32}\pi^2\right]\nu + 7\nu^2\right) x^3 + \mathcal{O}(x^4)\,,$$

where $\nu \equiv m_1 m_2/m^2$ and $x \equiv (m\Omega_\varphi)^{2/3} \sim v^2$

- Effective-one-body result [Buonanno & Damour 1999; Damour 2010]

$$K_{\mathrm{EOB}} = \sqrt{\frac{AA'B}{AA' + 2u(A')^2 - uAA''}}\,,$$

where A and B are the EOB potentials

Gravitational self-force result(s)

- The conservative SF correction ρ to the Schwarzschild result has recently been computed numerically [Barack & Sago 2010]

$$W \equiv \frac{1}{K^2} = \underbrace{1-6x}_{\text{Schw.}} + \underbrace{q\,\rho(x)}_{\text{SF effect}} + \mathcal{O}(q^2)$$

- Usual SF result expressed in terms of mass ratio $q \equiv m_1/m_2$:

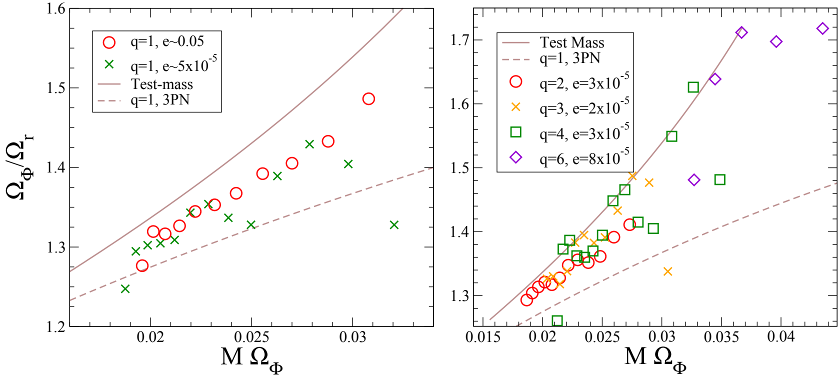
$$K^q_{\text{GSF}} = \frac{1}{\sqrt{1-6x}}\left[1 - \frac{q}{2}\frac{\rho(x)}{1-6x} + \mathcal{O}(q^2)\right]$$

- Since at first order $q = \nu + \mathcal{O}(\nu^2)$, we also have

$$K^\nu_{\text{GSF}} = \frac{1}{\sqrt{1-6x}}\left[1 - \frac{\nu}{2}\frac{\rho(x)}{1-6x} + \mathcal{O}(\nu^2)\right]$$

“Early” numerical relativity results for $K(\Omega_\varphi)$

[Mroué, Pfeiffer, Kidder & Teukolsky 2010]



Improved measurement of $K(\Omega_\varphi)$ in NR (1/2)

- Extract the orbital frequency $\Omega(t)$ from the coordinate separation $\mathbf{r}(t)$ *via*

$$\Omega \equiv \frac{|\mathbf{r} \times \dot{\mathbf{r}}|}{r^2}$$

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- Fit the NR data for $\Omega(t)$ to a model of the form

$$\begin{aligned} \Omega_\varphi(t) &= p_0 \, (p_1 - t)^{p_2} \\ \delta\Omega(t) &= p_3 \cos\left[p_4 + p_5(t - T) + p_6(t - T)^2\right] \end{aligned}$$

for a time interval of width $W \times 2\pi/\Omega(T)$, centered on $t = T$

Improved measurement of $K(\Omega_\varphi)$ in NR (2/2)

- Measure the average angular frequency Ω_φ and the radial frequency Ω_r using

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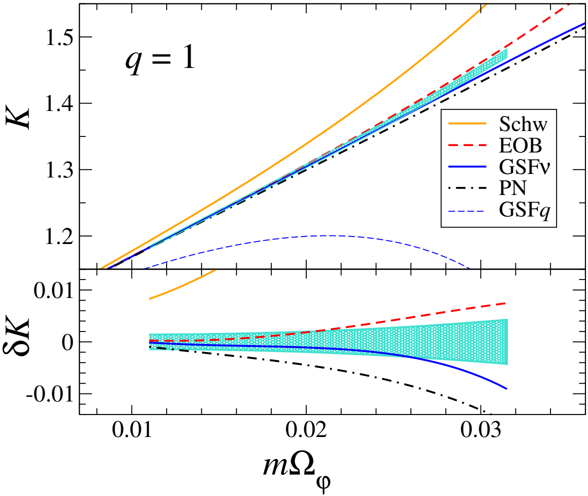
- An error estimate is provided by changing the window size W
- Error introduced by non-zero eccentricity is subdominant
- Motion well approximated by a sequence of quasi-circular orbits because $0.3\% \lesssim \dot{\Omega}_\varphi/\Omega_\varphi^2 \lesssim 1.7\%$

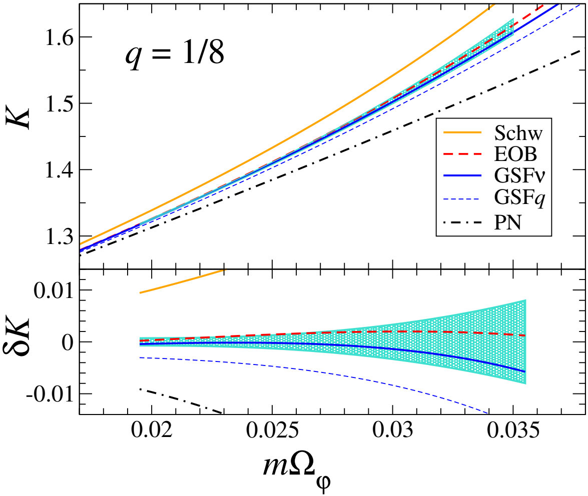
Extensive comparison for a mass ratio 1:1

[Le Tiec, Mroué, Barack, Buonanno, Pfeiffer, Sago & Taracchini 2011]



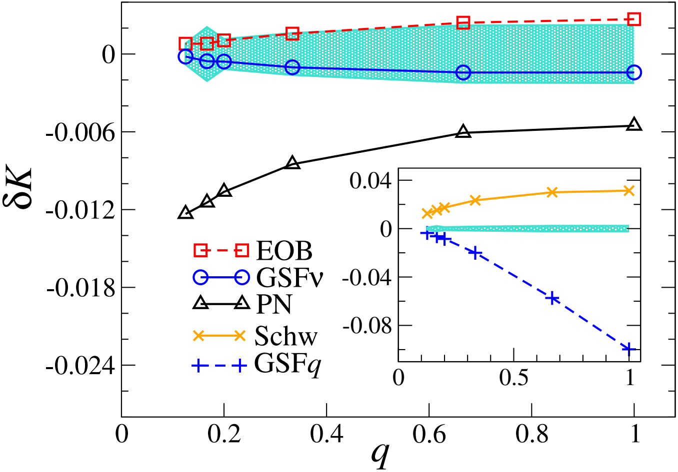
Extensive comparison for a mass ratio 1:8

[Le Tiec, Mroué, Barack, Buonanno, Pfeiffer, Sago & Taracchini 2011]



Variation with respect to the mass ratio

[Le Tiec, Mroué, Barack, Buonanno, Pfeiffer, Sago & Taracchini 2011]



Why does the GSFν result perform so well?

A few *a posteriori* heuristic explanations:

- K is linear in ν up to 3PN order
- The 3PN term $\propto \nu^2$ contributes less than 1% to K_{3PN}
- The exact $K(\Omega_\varphi; m_1, m_2)$ must be symmetric by exchange $1 \leftrightarrow 2$ of the black holes
- Assuming the coefficients a_n in the series $K = \sum_n a_n(x)\, \nu^n$ do not increase quickly with n, this expansion will exhibit a fast convergence since $0 < \nu \leqslant 1/4$

Summary and outlook

- Good convergence of the PN series in the comparable mass regime $q \lesssim 1$, unlike in the extreme mass ratio limit $q \ll 1$
- The EOB (3PN) prediction is in very good agreement with the NR results for all mass ratios and frequencies considered
- The standard GSF result in terms of the mass ratio q agrees with the NR data up to a difference $\mathcal{O}(q^2)$
- The GSF result with $q \to \nu$ compares remarkably well to the NR results, even for binaries with $m_1 \simeq m_2$
- First order BH perturbation theory may be good enough to model the GW emission from IMRIs with $q \sim 10^{-4} - 10^{-2}$