A Kirchhoff integral approach to the calculation of Green's functions beyond the normal neighbourhood.

Brien Nolan
Dublin City University

Joint work with Marc Casals, Perimeter Institute/UCD

Outline

- Self-force via MiSaTaQuWa.
- The retarded Green's function and the Kirchhoff integral.
- Plebański-Hacyan spacetime and its world function.
- Implementation of the Kirchhoff formula.

Self-force

- Various binary systems are of interest to gravitational wave astronomers.
- Need for accurate results describing the motion of a small black hole (m) in the field of a large black hole (M), taking into account the self-gravity of m.
- Equation of motion of a scalar particle in a curved spacetime (scalar version of MiSaTaQuWa):

$$ma^\alpha = q(g^{\alpha\beta} + u^\alpha u^\beta)\nabla_\beta \Phi_{rad}.$$

- We consider a massless scalar field, non-minimally coupled, sourced by a scalar charge $q(\tau)$ that is confined to a time-like world-line $\gamma : x = z(\tau)$:

$$(\Box - \xi R)\Phi = -4\pi \int q(\tau)\delta_4(x, z(\tau))d\tau.$$

- Then the term required in MiSaTaQuWa is

$$\nabla_\alpha \Phi_{rad} = \text{ some local terms } + q \lim_{\epsilon \to 0^+} \int_{-\infty}^{\tau - \epsilon} \nabla_\alpha G_R(z(\tau), z(\tau'))d\tau',$$

  where $G_R(x, x')$ is the retarded Green's function.
- NB G_R is required *globally*.

Retarded Green's function

- This satisfies

$$(\Box - \xi R) G_R(x, x') = -4\pi\delta_4(x, x')$$

  and the boundary condition

$$G_R(x, x') = 0 \text{ if } x \notin J^+(x').$$

- "The field due to a unit mass concentrated at x'".
- $\Phi(x) = q \int_\gamma G_R(x, z(\tau)) d\tau$.

Kirchhoff formula

- Consider the homogeneous equation $(\Box - \xi R)\Phi = 0$.
- Let x lie to the future of an initial data hypersurface Σ'. Then

$$\Phi(x) = -\frac{1}{4\pi}\int_{\Sigma'}(G_R(x,x')\nabla^{\alpha'}\Phi(x') - \Phi(x')\nabla^{\alpha'}G_R(x,x'))d\Sigma'_{\alpha'}.$$

  - $d\Sigma'_{\alpha'} = -n_{\alpha'}dV'$ is the surface element on Σ';
  - $n_{\alpha'}$ the future directed normal one-form;
  - dV' is the invariant volume element on Σ'.
- This is Kirchhoff's formula for the solution of the Cauchy initial value problem.
- G_R as the propagator for the data.

Hadamard form of G_R

- Let $\mathcal{N}$ be a *convex normal neighbourhood* of x: every pair of points in $\mathcal{N}$ is connected by a unique geodesic that stays in $\mathcal{N}$. Then within $\mathcal{N}$,

  $$G_R(x,x') = [U(x,x')\delta(\sigma(x,x')) + V(x,x')\theta(-\sigma(x,x'))]\theta_+(x,x'),$$

  where $\sigma(x,x')$ is the world function (well-defined when x, x' are connected by a unique geodesic), δ, θ are the usual distributions and

  $$\theta_+(x,x') = \begin{cases} 1 & \text{if } x \text{ lies to the future of } x'; \\ 0 & \text{otherwise,} \end{cases}$$

- Given σ, there is an algorithm for generating U, V (involves solving transport equations).

So...

- Start with your favourite black hole spacetime.
- Calculate σ.
- Apply Hadamard's algorithm to calculate U, V and so generate G_R.
- Write down the RHS of MiSaTaQuWa and solve.
- Generate wave forms, call up your observer friends and tell them what to look for.

Caustics

- In black hole spacetimes, the past light cone of a particle intersects multiple times with the past world line of the particle.
- The 'normal neighbourhood condition' doesn't last very long.
- The Hadamard form is useful only for calculating the 'quasi-local' part of the retarded Green's function (*cf.* Ottewill and Wardell); other methods (e.g. quasi-normal modes) can be used to calculate the 'distant past' contribution. Then apply matched asymptotic expansions to generate the global form (*cf.* Casals, Dolan, Ottewill and Wardell).
- An alternative: apply the Kirchhoff formula.

Kirchhoff for Green

- An observation: if x'' lies to the future of x, then $G_R(x'', x)$ satisfies the *homogeneous* wave equation.
- Apply Kirchhoff: Σ' a time slice lying to the future of Σ and to the past of Σ''.

$$G_R(x'', x) = -\frac{1}{4\pi} \int_{\Sigma_{t'}} \left(G_R(x'', x')\partial_{t'} G_R(x', x) - \partial_{t'} G_R(x'', x') G_R(x', x) \right) dV'.$$

- Convolution (rather than product) of distributions; valid as the distributions have compact support (cf. Friedlander).
- Think of $G_R(x', x)$ (and its time derivative) as data, $G_R(x'', x')$ as the propagator.
- Choose t, t', t'' so that Hadamard forms may be used throughout, but x'' lies outside the maximal convex normal neighbourhood of x.

Test case - Plebański-Hacyan spacetime

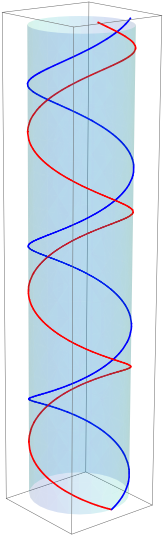
- Want to apply the formula in a spacetime where σ can be calculated, but where caustics form.
- Spherical symmetry is desirable, to allow us to study Ori's four-fold singularity structure in G_R, seen semi-analytically in Nariai and Schwarzschild.
- Plebański-Hacyan spacetime is just $\mathbb{M}_2 \times \mathbb{S}^2$, with the direct sum of the standard metrics on each:

$$ds^2 = -dT^2 + dY^2 + d\Theta^2 + \sin^2\Theta d\Phi^2.$$

- $\sigma(x', x) = -\frac{1}{2}(t' - t)^2 + \frac{1}{2}(y' - y)^2 + \frac{1}{2}\gamma^2$, where γ is the geodesic distance on the 2-sphere:

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi').$$

An $\mathbb{S}^1$−worth of future-pointing null geodesics emerging from $\vec{x}$ converge at the antipodal point at times $t' - t = (2n+1)\pi$, and at $\vec{x}$ at times $2n\pi$.



Hadamard form for Plebański-Hacyan spacetime

- $U = \left(\dfrac{\gamma}{\sin\gamma}\right)^{1/2}$
- $V = \displaystyle\sum_{n=0}^{\infty} \nu_n(\gamma)\sigma^n$ (analytic)
- $\nu_0(\gamma) = V|_{\sigma=0} = \dfrac{1}{8}U(\gamma)(1 - 4\xi R + \dfrac{1}{\gamma^2} - \dfrac{\cot\gamma}{\gamma})$
- $\nu_n(\gamma), n \geq 1$ are determined recursively by solving a sequence of first order linear ODEs.
- $G_R(x', x) = (U_1\delta(\sigma_1) + V_1\theta(-\sigma_1))\,\theta(t' - t)$.
- $G_R(x'', x') = (U_2\delta(\sigma_2) + V_2\theta(-\sigma_2))\,\theta(t'' - t')$.

Some simplifications

- Lorentz transformation on $\mathbb{M}_2$ to get $y'' = y$; translation to get $y'' = y = 0$.
- Rotation to get $\theta = 0$, so that $\gamma_1 = \gamma(x', x) = \theta'$.
- Kirchhoff formula is independent of the value of $t' \in (t, t'')$. Choose midpoint; $2(t' - t) = 2(t'' - t') = t'' - t = \tau$.
- Note that normal neighbourhood restrictions yield $0 < \tau < 2\pi$. For $\tau > \pi$, x'' is outside the maximal normal neighbourhood of x.
- $\theta(t'' - t') \equiv \theta(t' - t) \equiv 1$.
- Take $\alpha := \gamma(x', x)$ and $\beta := \gamma(x'', x')$ to be the coordinates on $\mathbb{S}^{2\prime}$.

Then

$$G_R(x'', x) = \frac{1}{4\pi} \sum_{k=1}^{6} G_k,$$

$$G_k = \int_{\mathbb{R}} dy' \int_{\mathbb{S}^2} \omega(\alpha, \beta) d\alpha d\beta \, \mathcal{G}_k, \quad 1 \leq k \leq 6,$$

with

$$\begin{aligned}
\mathcal{G}_1 &= \frac{\tau}{2} U_1 U_2 (\delta(\sigma_1)\delta'(\sigma_2) + \delta'(\sigma_1)\delta(\sigma_2)), \\
\mathcal{G}_2 &= -\frac{\tau}{2} (U_1 V_2 + U_2 V_1) \delta(\sigma_1)\delta(\sigma_2), \\
\mathcal{G}_3 &= \frac{\tau}{2} (U_1 V_2 \delta'(\sigma_1)\theta(-\sigma_2) + U_2 V_1 \delta'(\sigma_2)\theta(-\sigma_1)), \\
\mathcal{G}_4 &= U_1 \delta_{t'} V_2 \delta(\sigma_1)\theta(-\sigma_2) - U_2 \partial_{t'} V_1 \delta(\sigma_2)\theta(-\sigma_1), \\
\mathcal{G}_5 &= -\frac{\tau}{2} V_1 V_2 (\delta(\sigma_1)\theta(-\sigma_2) + \delta(\sigma_2)\theta(-\sigma_1)), \\
\mathcal{G}_6 &= (V_1 \partial_{t'} V_2 - V_2 \partial_{t'} V_1)\theta(-\sigma_1)\theta(-\sigma_2).
\end{aligned}$$

Sample calculation - G_1

- Integration by parts;
- use one of the δ's to do the y' integration, another to integrate over one of the angles;
- a step function arises from the condition that the integral contains the support of the other delta;
- the remaining integral needs to be regularized $G_1 \to G_1^{(\epsilon)}$.

$$
\begin{aligned}
\frac{1}{4\tau} G_1^{(\epsilon)} \quad = \quad & \frac{\theta(\tau - 2\pi + \theta'' + 2\epsilon)}{\sqrt{\tau^2 - (2\pi - \theta'' - 2\epsilon)^2}} \frac{1}{(2\pi - \theta'')\sqrt{\sin\theta''}} \left[\frac{1}{\sqrt{\epsilon}} + O(\epsilon^{1/2}) \right] \\
- \quad & \frac{\theta(\tau - \theta'' - 2\epsilon)}{\sqrt{\tau^2 - (\theta'' + 2\epsilon)^2}} \frac{1}{\theta''\sqrt{\sin\theta''}} \left[\frac{1}{\sqrt{\epsilon}} + O(\epsilon^{1/2}) \right] \\
- \quad & \int_{\theta''/2+\epsilon}^{\pi-\theta''/2-\epsilon} \frac{\partial_\alpha H(\alpha,\alpha)}{\sqrt{\tau^2 - 4\alpha^2}} \theta(\tau - 2\alpha) d\alpha
\end{aligned}
$$

Sample result - G_1

- For $\tau < \pi$ (normal neighbourhood),

$$G_1 = \lim_{\epsilon \to 0} G_1^{(\epsilon)} = -4 \left(\frac{\theta''}{\sin \theta''} \right)^{1/2} \frac{\delta(\tau - \theta'')}{\theta''} + \mathcal{O}(1).$$

- For $\pi < \tau < 2\pi$ (beyond normal neighbourhood),

$$\begin{aligned} G_1 \quad = \quad & \frac{4\sqrt{2}\tau}{(2\pi - \theta'')\sqrt{\tau + 2\pi - \theta''}\sqrt{\sin \theta''}} \times \\ & \left\{ PV \left(\frac{1}{\tau - 2\pi + \theta''} \right) - \frac{B(\theta'')}{2\sqrt{2}} \ln |\tau - 2\pi + \theta''| \right\} + \mathcal{O}(1), \end{aligned}$$

where $PV\left(\frac{1}{x}\right)$ (the principal value distribution) is defined by

$$< PV \left(\frac{1}{x} \right), \phi >= \lim_{\epsilon \to 0} \int_{\mathbb{R} \setminus [-\epsilon, \epsilon]} \frac{\phi(x)}{x} dx,$$

for $\phi \in C_0^\infty(\mathbb{R})$.

Overall result

- Normal neighbourhood:

$$G_R = U(\gamma)\delta(\sigma) + V(\gamma,\sigma)\theta(-\sigma)$$

- Past first caustic, before second caustic:

$$G_R = \left[U(2\pi-\gamma)PV\left(\frac{1}{\hat{\sigma}}\right) + \nu_0(2\pi-\gamma)\ln|\hat{\sigma}| + \mathcal{O}(1)\right]\theta(\tau-\pi),$$

  where $\hat{\sigma}$ is σ with $\gamma \to 2\pi-\gamma$.

- $\hat{\sigma}=0$ along the null geodesics returning from the antipodal point.
- $\theta(\tau-\pi)$ is a causality preserving term; τ is the geodesic distance on $\mathbb{M}_2$.

Comments

- Ori's observation: spherical symmetry induces a 4-fold recursion in the singularity structure of the retarded Green's function as successive caustics are met:

$$\delta(\sigma) \to PV\left(\frac{1}{\sigma}\right) \to -\delta(\sigma) \to -PV\left(\frac{1}{\sigma}\right) \to \delta(\sigma) \to \cdots$$

- This has been observed in Naria spacetime (Casals et al) and in Schwarzschild (Dolan and Ottewill) using large$-\ell$ asymptotics and QNM's.
- First step proven here for Plebański-Hacyan spacetime; likewise for the first step of a corresponding sequence

$$\theta(-\sigma) \to \ln|\sigma| \to -\theta(-\sigma) \to -\ln|\sigma| \to \theta(-\sigma) \to \cdots$$

- Conjecture: U picks up a factor of i at every caustic (Landau contour argument - Casals); V would then necessarily do the same.
- The transitions induced in the Feynman Green's function would then give exactly the sequences above:

$$\begin{aligned} G_F &= \frac{U}{\sigma + i0} + V \ln(\sigma + i0) + W \\ &= U \left[PV \left(\frac{1}{\sigma} \right) - i\pi\delta(\sigma) \right] + V \left[\ln|\sigma| + i\pi\theta(-\sigma) \right] + W. \end{aligned}$$

- NB both the general structure and the *details of the coefficients* found above are consistent with this hypothesis.

Conclusions and future work.

- Kirchhoff formula can be used to determine the singularity structure in G_R beyond the first caustic. Work is continuing to extend the result beyond the second caustic.
- Various remainders - $\mathcal{O}(1)$ terms - were neglected here in order to determine the singularity structure. It should be possible to determine these.
- Is singularity structure for the first inter-caustic zone enough to rebuild an amended Hadamard form for G_R? (Answer is yes for the pre-caustic zone.)
- The global retarded Green's function as a sum over caustics of Hadamard forms.
- Schwarzschild?