Gauge-invariant quantity in a radiation gauge for a particle in circular, equatorial orbit around a Kerr black hole

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PLAN

- Wald's generalization of CCK formalism
- Summarize the procedure in a radiation gauge
- Spheroidal-spherical problem
- Results

Wald's generalization of CCK formalism

Suppose one wants to solve the linearized vacuum Einstein equation:

$[\mathcal{E}(h)]_{\mu\nu} = 0$ that is

$$-\nabla_\mu \nabla_\nu h - \Box h_{\mu\nu} + \nabla^\alpha \nabla_\nu h_{\alpha\mu} + \nabla^\alpha \nabla_\mu h_{\alpha\nu} + g_{\mu\nu}(\Box h - \nabla^\alpha \nabla^\beta h_{\alpha\beta}) = 0$$

Suppose a decoupled equation is derived for a new variable which is a function of the metric perturbation, h - Teukolsky equation.

Teukolsky equation

Newman-Penrose equations (Bianchi identities):
Derivative operators acting on the Weyl scalars
= Derivative operators acting on the Ricci tensor

Their combination then gives us:
Derivative operator acting on ψ_0, i.e., $\mathcal{O}\psi_0$
=Derivative operator acting on $R_{\mu\nu}$, i.e., $S^{\mu\nu}R^{(1)}_{\mu\nu}$

If one writes, $\psi_0 = \mathcal{T}(h)$, and $R^{(1)}_{\mu\nu} \sim \mathcal{E}(h)$
we get $\mathcal{O}\mathcal{T}(h) = \mathcal{S}\mathcal{E}(h)$

$\mathcal{O}_s \mathcal{T}_s(h) = \mathcal{S}_s \mathcal{E}_s(h)$

Example: $s = 2$ case

$$\mathcal{T}_2(h) = -l^\alpha m^\beta l^\gamma m^\delta C_{\alpha\beta\gamma\delta} = \boxed{\psi_0}$$

$$= \frac{-1}{2} l^\alpha m^\beta l^\gamma m^\delta \Big[h_{\alpha\gamma;\beta\delta} + h_{\beta\delta;\alpha\gamma} - h_{\beta\gamma;\alpha\delta} - h_{\alpha\delta;\beta\gamma}$$

$$+ R^{(0)}_{\alpha\epsilon\gamma\delta} h^\epsilon_\beta - R^{(0)}_{\beta\epsilon\gamma\delta} h^\epsilon_\alpha \Big]$$

$$\mathcal{O}_2 = (\boldsymbol{D} - 3\epsilon + \bar{\epsilon} - 4\varrho - \bar{\varrho})(\boldsymbol{\Delta} - 4\gamma + \mu)$$

$$- (\boldsymbol{\delta} + \bar{\pi} - \bar{\alpha} - 3\beta - 4\tau)(\bar{\boldsymbol{\delta}} + \pi - 4\alpha) - 3\psi_2$$

$\mathcal{O}_s\mathcal{T}_s(h) = \mathcal{S}_s\mathcal{E}_s(h)$

$$[\mathcal{E}_2(h)]_{\mu\nu} = G_{\mu\nu} = 8\pi T_{\mu\nu}$$

$$\begin{aligned}[\mathcal{S}_2]^{\mu\nu} = \frac{1}{2}(\boldsymbol{\delta} + \bar{\pi} - \bar{\alpha} - 3\beta - 4\tau)[(\boldsymbol{D} - 2\epsilon - 2\bar{\varrho})l^\mu m^\nu \\ - (\boldsymbol{\delta} + \bar{\pi} - 2\bar{\alpha} - 2\beta)l^\mu l^\nu] \\ + \frac{1}{2}(\boldsymbol{D} - 3\epsilon + \bar{\epsilon} - 4\varrho - \bar{\varrho})[(\boldsymbol{\delta} + 2\bar{\pi} - 2\beta)l^\mu m^\nu \\ - (\boldsymbol{D} - 2\epsilon + 2\bar{\epsilon} - \bar{\varrho})m^\mu m^\nu]\end{aligned}$$

$$\boldsymbol{D} = l^\mu \nabla_\mu$$
$$\boldsymbol{\Delta} = n^\mu \nabla_\mu$$
$$\boldsymbol{\delta} = m^\mu \nabla_\mu$$

Metric Perturbation

Suppose $\mathcal{S}\mathcal{E} = \mathcal{O}\mathcal{T}$ holds where $\mathcal{S}$, $\mathcal{O}$, $\mathcal{E}$ and $\mathcal{T}$ are linear partial differential operators and suppose Ψ satisfies $\mathcal{O}^\dagger \Psi = 0$. If $\mathcal{E}$ is self-adjoint, then $\mathcal{S}^\dagger \Psi$ satisfies $\mathcal{E}(h) = 0$. Taking the adjoint of $\mathcal{S}\mathcal{E} = \mathcal{O}\mathcal{T}$, we have

$$\mathcal{E}^\dagger \mathcal{S}^\dagger = \mathcal{T}^\dagger \mathcal{O}^\dagger$$
$$\mathcal{E} \mathcal{S}^\dagger = \mathcal{T}^\dagger \mathcal{O}^\dagger$$

If $\mathcal{O}^\dagger \Psi = 0$, then $\mathcal{E}(\mathcal{S}^\dagger \Psi) = 0$, i.e., h $= \mathcal{S}^\dagger \Psi$
Writing the above equations with the appropriate spin-weights, we have

$$h_{\mathrm{ORG}} = \mathcal{S}^\dagger_{+2} \Psi_{\mathrm{ORG}}$$
$$h_{\mathrm{IRG}} = \mathcal{S}^\dagger_{-2} \Psi_{\mathrm{IRG}}$$

Weyl scalar

$$\begin{aligned} \mathcal{S}\mathcal{E}(\mathcal{S}^\dagger\Psi) &= \mathcal{O}\mathcal{T}(\mathcal{S}^\dagger\Psi) \\ 0 &= \mathcal{O}[\mathcal{T}\mathcal{S}^\dagger\Psi] \end{aligned}$$

$\mathcal{T}\mathcal{S}^\dagger$ maps solutions of $\mathcal{O}^\dagger\Psi = 0$ to $\mathcal{O}\psi = 0$.

This gives us the appropriate Weyl scalar in terms of the Hertz potential ($\psi = \mathcal{T}\mathcal{S}^\dagger\Psi$) as follows

$$\psi_0 = \mathcal{T}_2\mathcal{S}^\dagger_{+2}\Psi_{\rm ORG}$$

$$\psi_0 = \mathcal{T}_2\mathcal{S}^\dagger_{-2}\Psi_{\rm IRG} \qquad h_{\rm ORG} = \mathcal{S}^\dagger_{+2}\Psi_{\rm ORG}$$

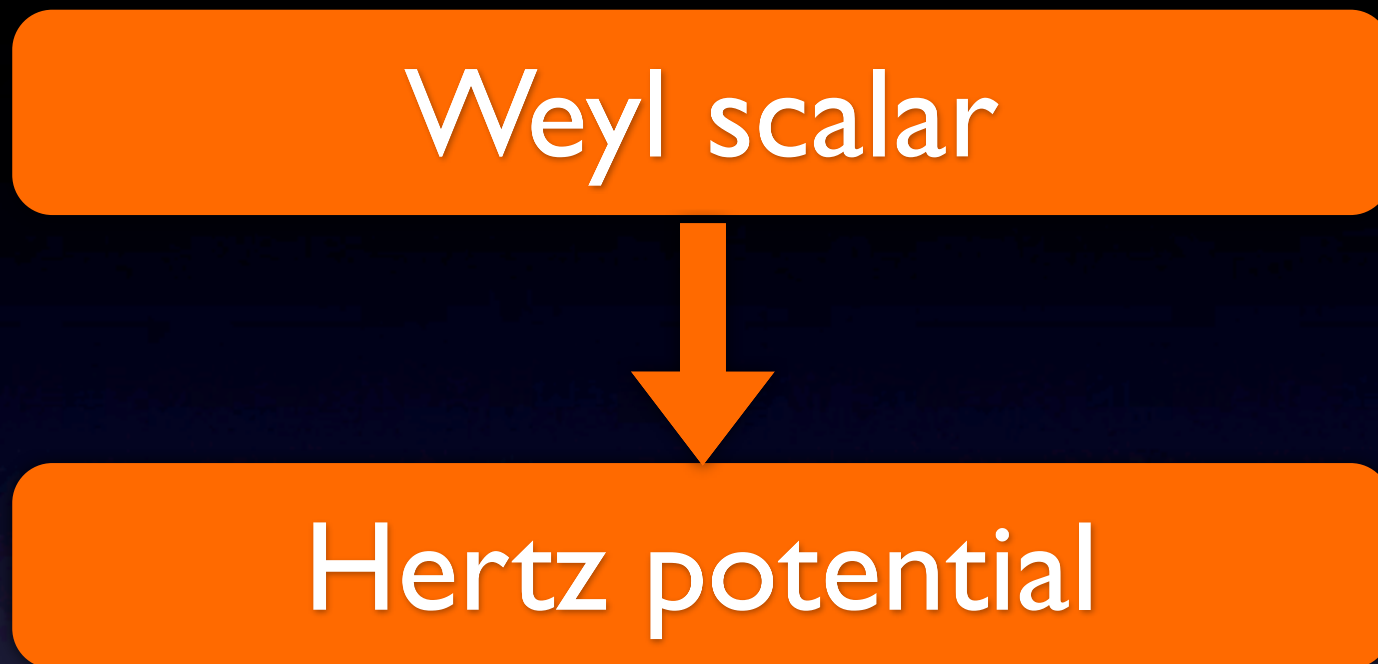
$$\psi_4 = \mathcal{T}_{-2}\mathcal{S}^\dagger_{+2}\Psi_{\rm ORG} \qquad h_{\rm IRG} = \mathcal{S}^\dagger_{-2}\Psi_{\rm IRG}$$

$$\psi_4 = \mathcal{T}_{-2}\mathcal{S}^\dagger_{-2}\Psi_{\rm IRG}$$

H in a radiation gauge

Weyl scalar

H in a radiation gauge



H in a radiation gauge

Weyl scalar

↓

Hertz potential

↓

Metric perturbation

H in a radiation gauge

Weyl scalar

Numerically solve the separable Teukolsky equation

$$\begin{aligned}\mathcal{T}_s\psi_s \;:=\; & \left\{\left[\frac{(r^2+a^2)^2}{\Delta} - a^2\sin^2\theta\right]\frac{\partial^2}{\partial t^2} - 2s\left[\frac{M(r^2-a^2)}{\Delta} - r - ia\cos\theta\right]\frac{\partial}{\partial t} + \frac{4Mar}{\Delta}\frac{\partial^2}{\partial t\partial\phi} - \Delta^{-s}\frac{\partial}{\partial r}\left(\Delta^{s+1}\frac{\partial}{\partial r}\right)\right. \\ & \left. - \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) - 2s\left[\frac{a(r-M)}{\Delta} + \frac{i\cos\theta}{\sin^2\theta}\right]\frac{\partial}{\partial\phi} + \left[\frac{a^2}{\Delta} - \frac{1}{\sin^2\theta}\right]\frac{\partial^2}{\partial\phi^2} + (s^2\cot^2\theta - s)\right\}\psi_s \\ = \; & 4\pi(r^2 + a^2\cos^2\theta)T_s,\end{aligned}$$

H in a radiation gauge

Weyl scalar ✔

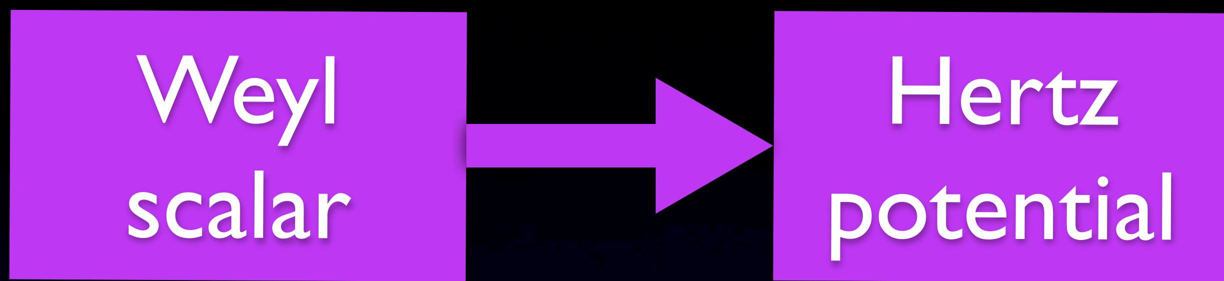
↓

Hertz potential

↓

Metric perturbation

H in a radiation gauge



$$\psi_0 = \mathcal{T}_2 \mathcal{S}^\dagger_{+2} \Psi_{\rm ORG}$$

$$\psi_0 = \frac{1}{8}\left(\mathcal{L}^4 \bar{\Psi} + 12 M \partial_t \Psi\right)$$

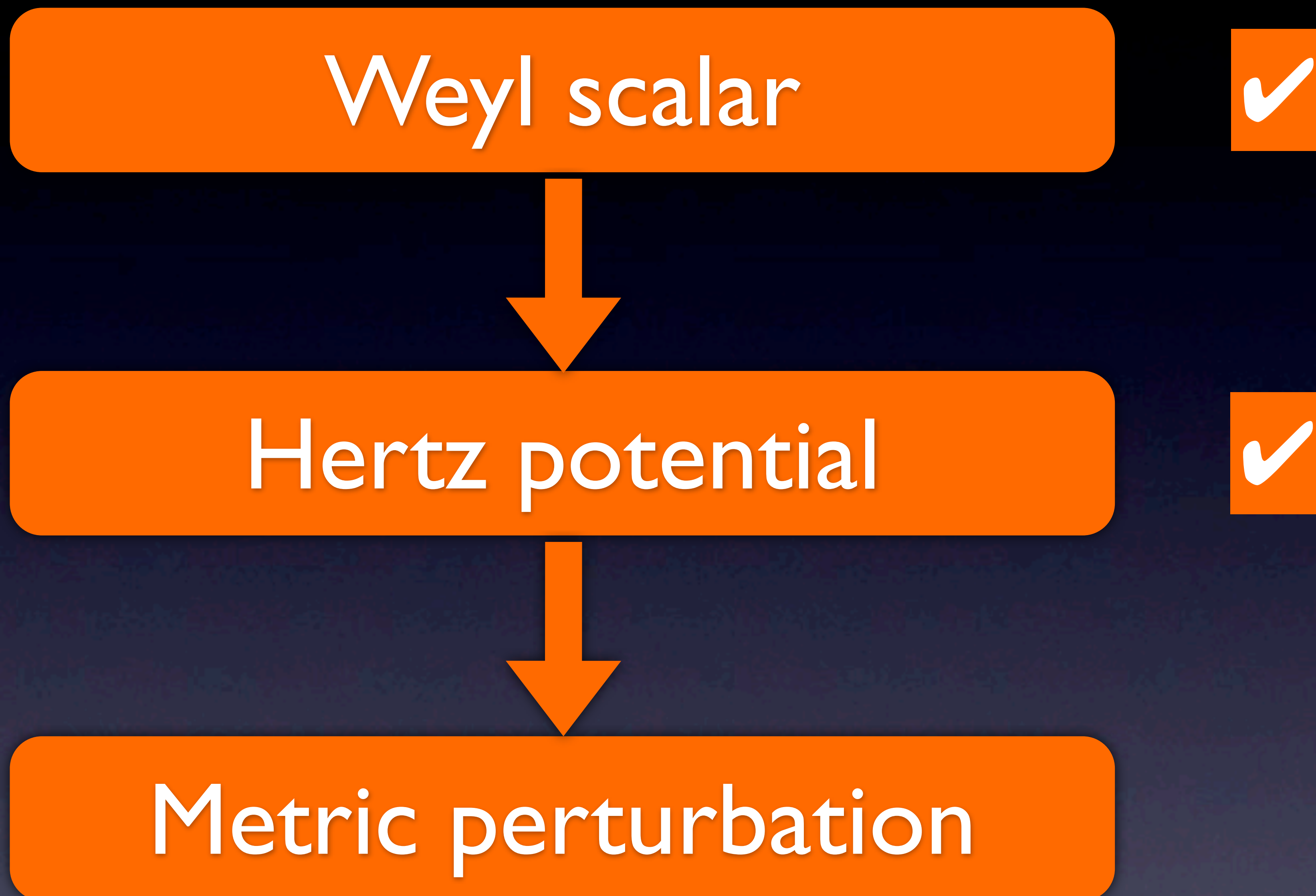
$$\Psi = \sum_{\ell,m,\omega} \Psi_{\ell,m,\omega}(r)\, {}_2S^\omega_{\ell,m}(\theta,\phi) e^{-i\omega t} \qquad \psi_0 = \sum_{\ell,m,\omega} \psi_{0\,\ell,m,\omega}(r)\, {}_2S^\omega_{\ell,m}(\theta,\phi) e^{-i\omega t}$$

$$\Psi_{\ell,m,\omega} = 8\frac{(-1)^m D \bar{\psi}_{0\ell,-m,-\omega} + 12 i M \omega \psi_{0\ell,m,\omega}}{D^2 + 144 M^2 \omega^2}$$

$$\mathcal{L}^4 S_{-2} = D S_{+2} \qquad D = \sqrt{(\ell+2)(\ell+1)\ell(\ell-1)} \quad (a=0)$$

Thursday, July 7, 2011

H in a radiation gauge



H in a radiation gauge

$$h_{\mathrm{ORG}} = \mathcal{S}^{\dagger}_{+2}\Psi_{\mathrm{ORG}}$$

$$n \cdot l = 1$$

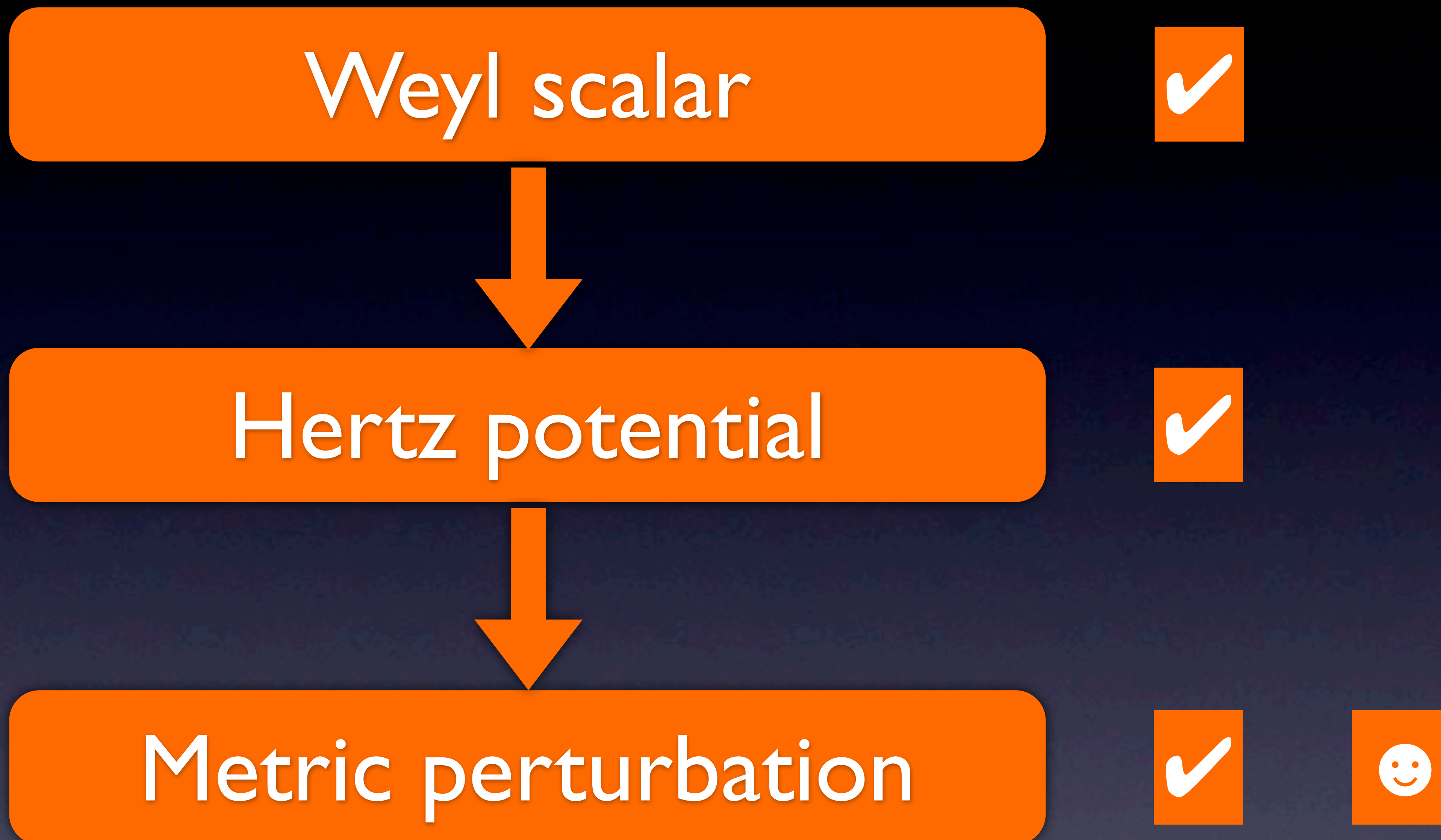
$$m \cdot \bar{m} = -1$$

$$\mathrm{h}_{\alpha\beta}\,\mathrm{l}^{\alpha}\,\mathrm{l}^{\beta} = \rho^{-4}\,(\bar{\delta} - 3\,\alpha - \bar{\beta} + 5\,\pi)\,(\bar{\delta} - 4\,\alpha + \pi)\,\Psi + \mathrm{c.c.}$$

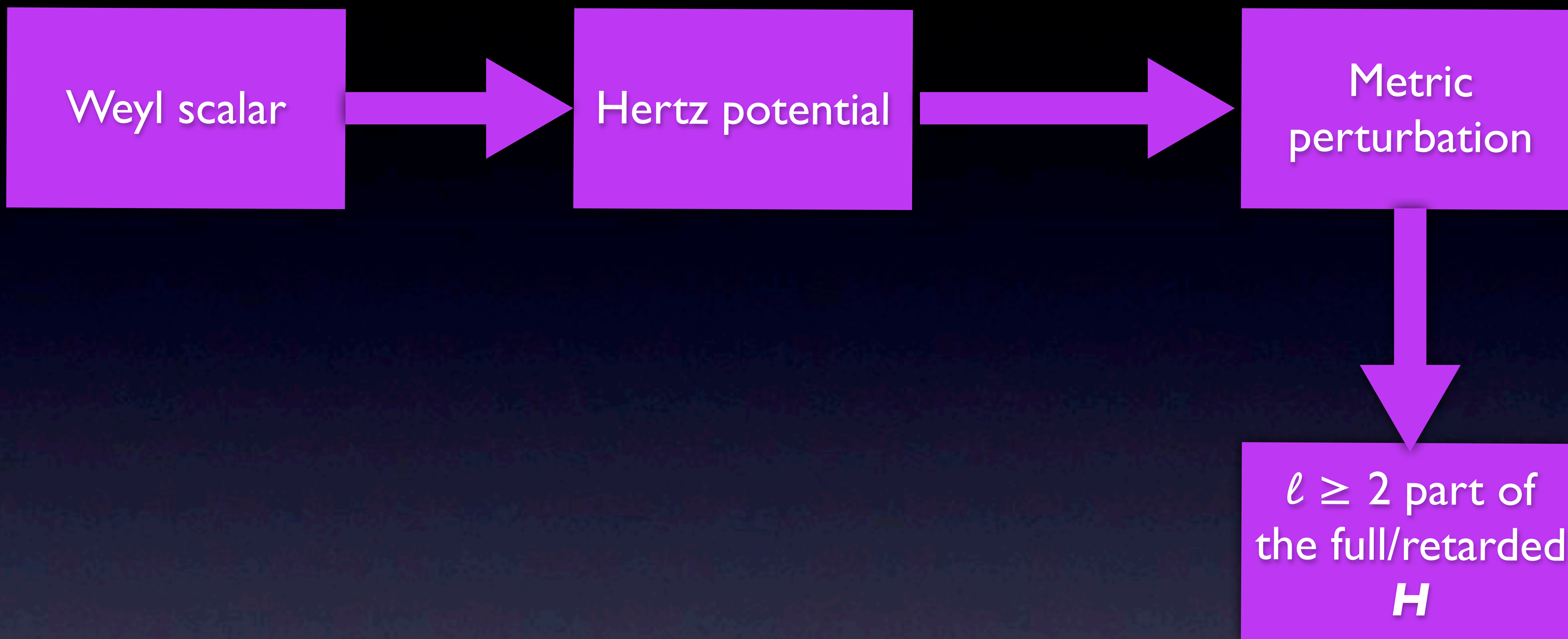
$$\mathrm{h}_{\alpha\beta}\,\mathrm{m}^{\alpha}\,\mathrm{m}^{\beta} = \rho^{-4}\,(\Delta + 5\,\mu - 3\,\gamma + \bar{\gamma})\,(\Delta + \mu - 4\,\gamma)\,\Psi$$

$$\mathrm{h}_{\alpha\beta}\,\mathrm{l}^{\alpha}\,\mathrm{m}^{\beta} = -\frac{1}{2\,\rho^{4}}[(\bar{\delta} - 3\,\alpha - \bar{\beta} + 5\,\pi)\,(\Delta + \mu - 4\,\gamma) + (\Delta + 5\,\mu - 3\,\gamma + \bar{\gamma})\,(\bar{\delta} - 4\,\alpha + \pi)]\,\Psi$$

H in a radiation gauge



H in a radiation gauge



$$H_\ell^{\rm ret} = B + \frac{C}{(\ell + \frac{1}{2})} + \frac{D}{P_2(\ell)} + \cdots + H_\ell^{\rm R}$$

$$H^{\text{ret}} = \sum_{\ell=0}^{\infty} H_\ell'^{\text{ret}} = \sum_{\ell=0}^{\infty} H_\ell^{\text{ret}}$$

$$H_\ell'^{\text{ret}} = \sum_{m=-\ell}^{\ell} \tilde{R}_{\ell,m} S_{\ell,m}$$

$$H_\ell^{\text{ret}} = \sum_{m=-\ell}^{\ell} R_{\ell,m} Y_{\ell,m}$$

$$\text{where } R_{\ell,m} = \sum_{\ell'} b^m_{\ell',\ell} \tilde{R}_{\ell',m}$$

As one goes to higher ℓs, one sees that the difference between H_ℓ' and H_ℓ becomes smaller and smaller, i.e., the difference converges to zero.

$$H_\ell^{\text{ret}} = B + \frac{C}{\left(\ell + \frac{1}{2}\right)} + \frac{D}{P_2(\ell)} + \cdots + H_\ell^{\text{R}}$$

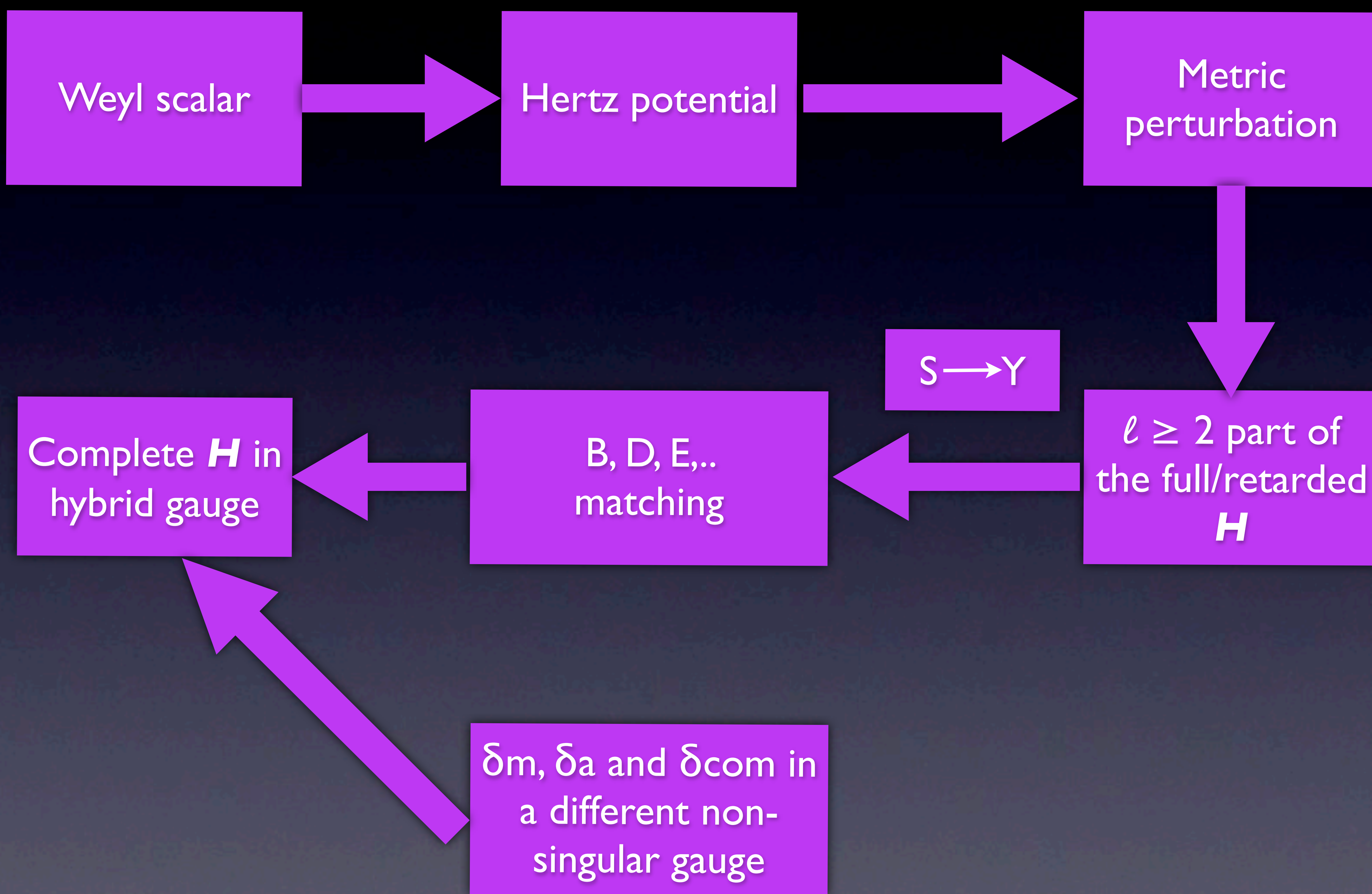
$$H^{\mathrm{R}} = \lim_{\ell_{\max}\to\infty} \sum_{\ell=0}^{\ell_{\max}} (H'_\ell - B - C/L)$$

$$H^{\mathrm{R}} = \lim_{\ell_{\max}\to\infty} \sum_{\ell=0}^{\ell_{\max}} (H_\ell - B - C/L)$$

The above sums, whether one uses H_ℓ or H'_ℓ gives us the same renormalized field but instead of infinity if one is restricted to a certain $\ell_{\max}$ which is 75 in our case, the first sum does not converge.

hybrid

H in a radiation gauge



Gauge-invariant results for a particle in circular, equatorial orbit around a Kerr BH

$$u^\alpha u^\beta (g_{\alpha\beta} + h_{\alpha\beta}) = 1$$

$$u^\alpha = [u_0^t + u_1^t + O(\mu^2)]k^\alpha$$

$$\Delta U = u_1^t = u_0^t\, H$$

r0/M	a = 0.7 M	a = 0.9 M
4	-1.4748811719	-1.4633559752
6	-1.01878981134	-1.0078165302
7	-0.8760106461	-0.8679363173
8	-0.7672776106	-0.7612477750
10	-0.6136003896	-0.6100017577
15	-0.4076336292	-0.4062824668
20	-0.3047875811	-0.3041226445
30	-0.2023749186	-0.2021320671
50	-0.1209396770	-0.1208717776
70	-0.0862158376	-0.0861865457

Gauge-invariant results

	r = 15 M a = 0.5 M (Kerr)	r = 15 M a = 0 M (Schwarzschild)
Singular term (B) in H	0.130773	0.130679
Renormalized H	-0.295911	-0.300533

$$\Omega = \Omega_0 \left(1 - \frac{r^2 \left(r^{3/2} - 3Mr^{1/2} + 2aM^{1/2} \right)}{2M\mu \left(r^{3/2} + aM^{1/2} \right)} F_r \right)$$

$$U = U_0 \left(1 - \frac{r^{1/2} \left(r^2 - 2aM^{1/2} r^{1/2} + a^2 \right)}{2\mu \left(r^{3/2} + aM^{1/2} \right)} F_r \right)$$