Improving Boundary Conditions in Time-Domain Self-Force Calculations

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Motivation

- To develop efficient and accurate Time-Domain Computations of the Self-Force.

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- To develop efficient and accurate Time-Domain Computations of the Self-Force.

- Time Domain Computations are also useful for the computation of Waveforms via Regge-Wheeler and Zerilli-Moncrief master Equations (given a certain spacetime trajectory)

The Particle without Particle scheme

- In order to avoid the presence of singularities in our computational domain and also to avoid introducing an artificial scale in the problem we split our domain:

$$\Omega = [-\infty, \infty]$$

$-\infty \leftarrow r^*$ $\qquad$ $r^* \rightarrow \infty$

The Particle without Particle scheme

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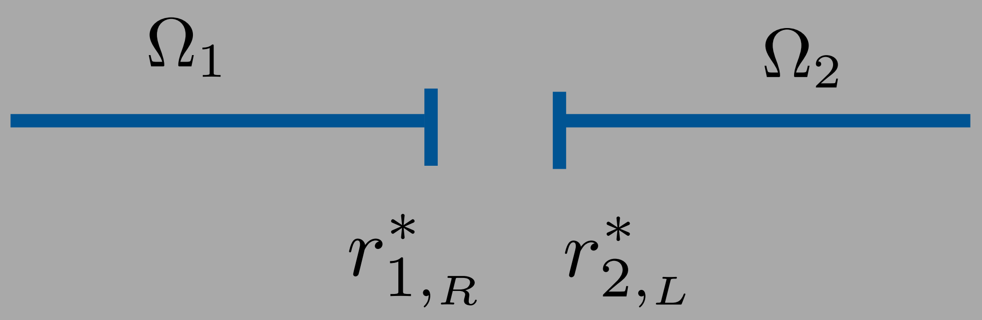
$$\Omega = \Omega_1 \cup \Omega_2$$



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The particle is located at the interface

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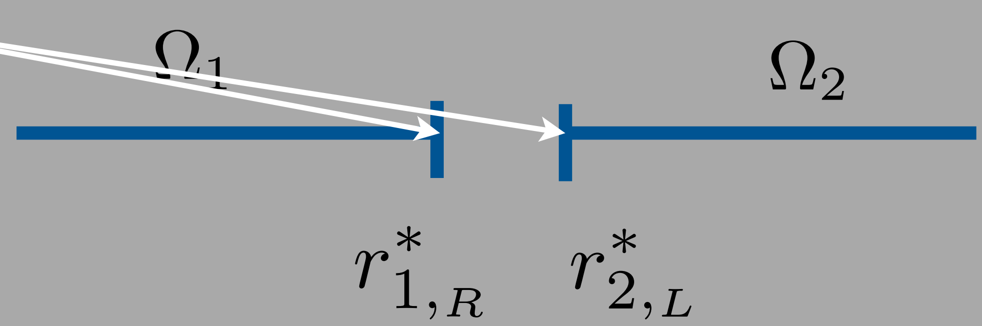

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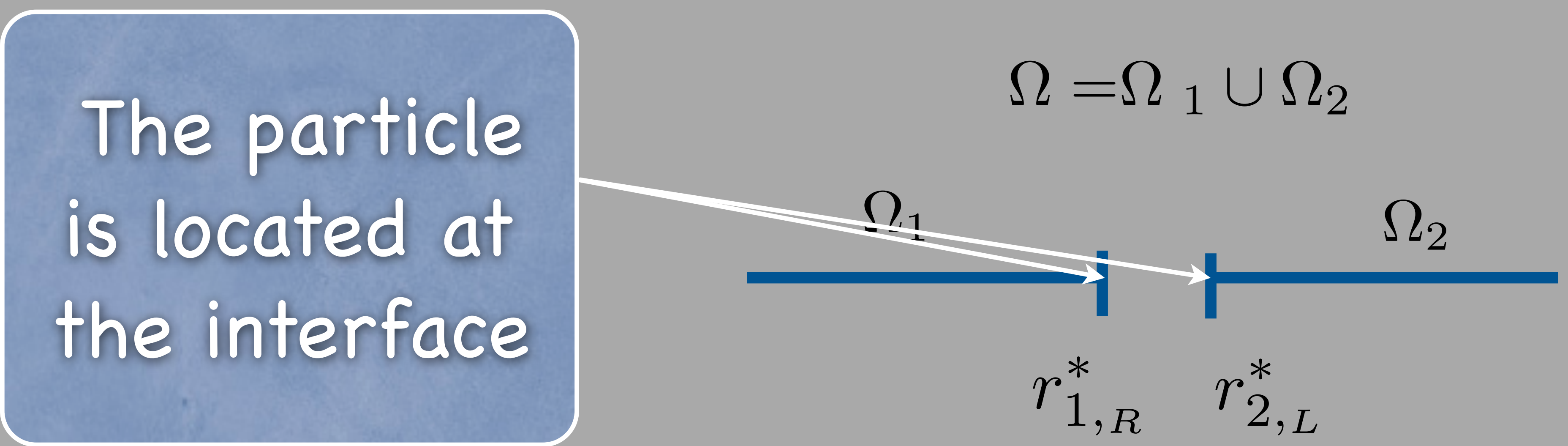

- In this way we are left with homogeneous wave-type equations (i.e. without distributional source terms) at the interiors of the domains. Then, we obtain smooth solutions (in the time domain) in both domains.

Scalar Charged Particle around a MBH

- We consider a simplified model: A charged scalar particle orbiting a non-rotating Black Hole:

$$ds^2 = \left(1 - \frac{2M}{r}\right)\left(-dt^2 + dr^{*2}\right) + r^2 d\Omega^2\,,$$

$$r^* = r + 2M\ln\left(\frac{r}{2M} - 1\right)\,, \qquad d\Omega^2 = d\theta^2 + \sin^2\theta\, d\varphi^2\,.$$

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- The scalar field equation is:

$$\Box\Phi = g^{\mu\nu}\nabla_\mu\nabla_\nu\Phi = -4\pi\rho\,,$$

$$\rho = q\int_\gamma \delta_4[\mathbf{x} - \mathbf{z}(\tau)]\,d\tau\,.$$

$$\gamma = \{x^\mu \;\mid\; x^\mu = z^\mu(\tau)\}$$

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- The equation of motion for the particle is:

$$a^\mu = u^\nu\nabla_\nu u^\mu = \frac{d^2 z^\mu}{d\tau^2} = \frac{q}{m}(g^{\mu\nu} + u^\mu u^\nu)\nabla_\nu\Phi \,.$$

Scalar Charged Particle around a MBH

- Each harmonic mode satisfies a wave-type equation:

$$\left(-\partial_t^2 + \partial_{r^*}^2 - V_\ell\right)\psi^{lm} = S^{\ell m}\,,$$

$$V_\ell(r) = \left(1 - \frac{2M}{r}\right)\left[\frac{2M}{r^3} + \frac{\ell(\ell+1)}{r^2}\right]\,,$$

$$S^{\ell m} = -4\pi q \frac{1 - 2M/r_p}{r_p E_p}\,\bar{Y}^{\ell m}(\frac{\pi}{2},\varphi_p)\,\delta[r^* - r_p^*(t)]\,,$$

$$r_p = r_p(t)\,,\quad \varphi_p = \varphi_p(t)\,.$$

- Each mode is finite at the SCO location (but the total scalar field is divergent). We can then regularize the solution harmonic by harmonic by substracting to the retarded solution an approximation to the singular field valid at the particle location [Mode Sum Scheme]

Scalar Charged Particle around a MBH

- Using the Domain Splitting:

$$r^*_{1,R} = r^*_{2,L} = r^*_p$$

$$\psi(t,r) = \psi_{-}(t,r)\,\Theta\left(r_p(t) - r\right) + \psi_{+}(t,r)\,\Theta\left(r - r_p(t)\right)$$

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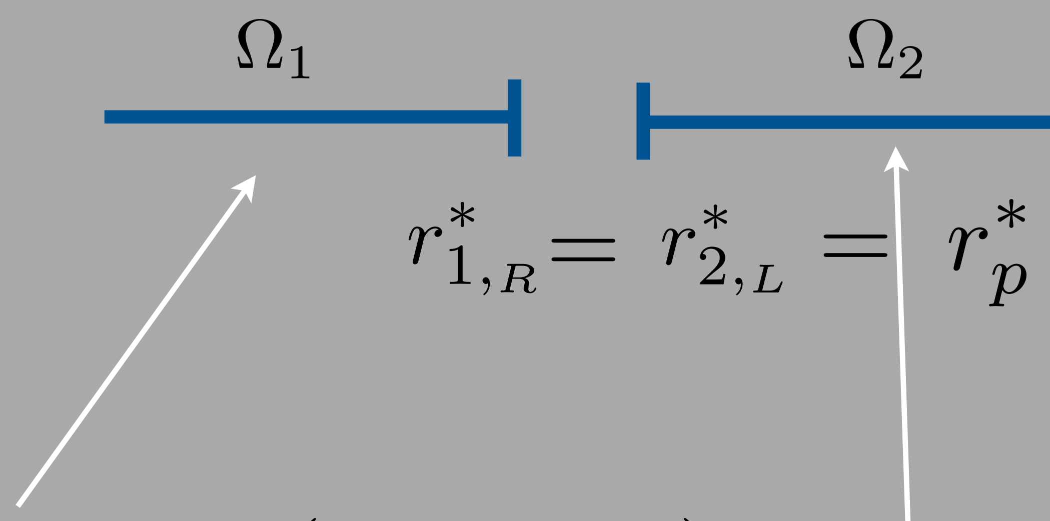


$$\psi(t,r) = \psi_{-}(t,r)\,\Theta\left(r_p(t) - r\right) + \psi_{+}(t,r)\,\Theta\left(r - r_p(t)\right)$$

the problem is reduced to Homogeneous Equations plus Junction/Boundary Conditions:

$$\left(-\partial_t^2 + \partial_{r^*}^2 - V(r)\right)\psi_{\pm} = 0\,,$$

$$[\psi]_p = A\left[r_p(t), \varphi_p(t)\right]\,,$$

$$[\partial_{r^*}\psi]_p = B\left[r_p(t), \varphi_p(t)\right]\,.$$

The Particle without Particle scheme

- We use a first-order reduction of the equations that is symmetric hyperbolic, and hence suitable for imposing the boundary/matching conditions:

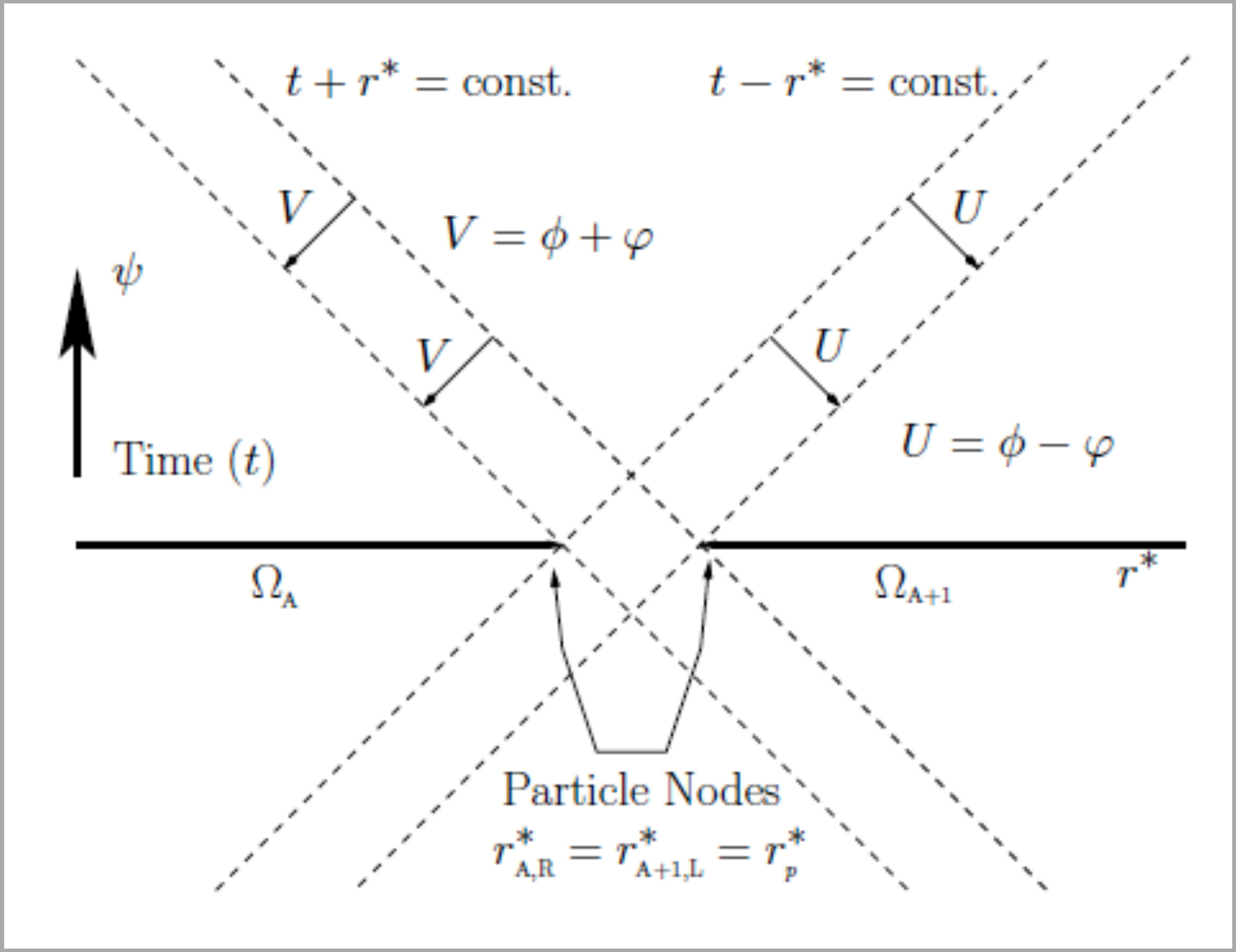
$$\psi^{\ell m}_{\pm} = r\,\Phi^{\ell m}_{\pm}\,,\quad \phi^{\ell m}_{\pm} = \partial_t \psi^{\ell m}_{\pm}\,,\quad \varphi^{\ell m}_{\pm} = \partial_{r^*}\psi^{\ell m}_{\pm}\,,$$

$$\partial_t \mathbf{U}_{\pm} = \mathbb{A}\cdot\partial_{r^*}\mathbf{U}_{\pm} + \mathbb{B}\cdot\mathbf{U}_{\pm}\,,$$

$$\mathbf{U}_{\pm} = (\psi^{\ell m}_{\pm}, \phi^{\ell m}_{\pm}, \varphi^{\ell m}_{\pm})\,,\quad \mathbb{A} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\,,\quad \mathbb{B} = \begin{pmatrix} 0 & 1 & 0 \\ -V_\ell & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\,.$$

The Particle without Particle scheme

- The junction conditions are imposed in two different alternative ways: (i) The penalty method. (ii) The direct communication of the characteristic fields.



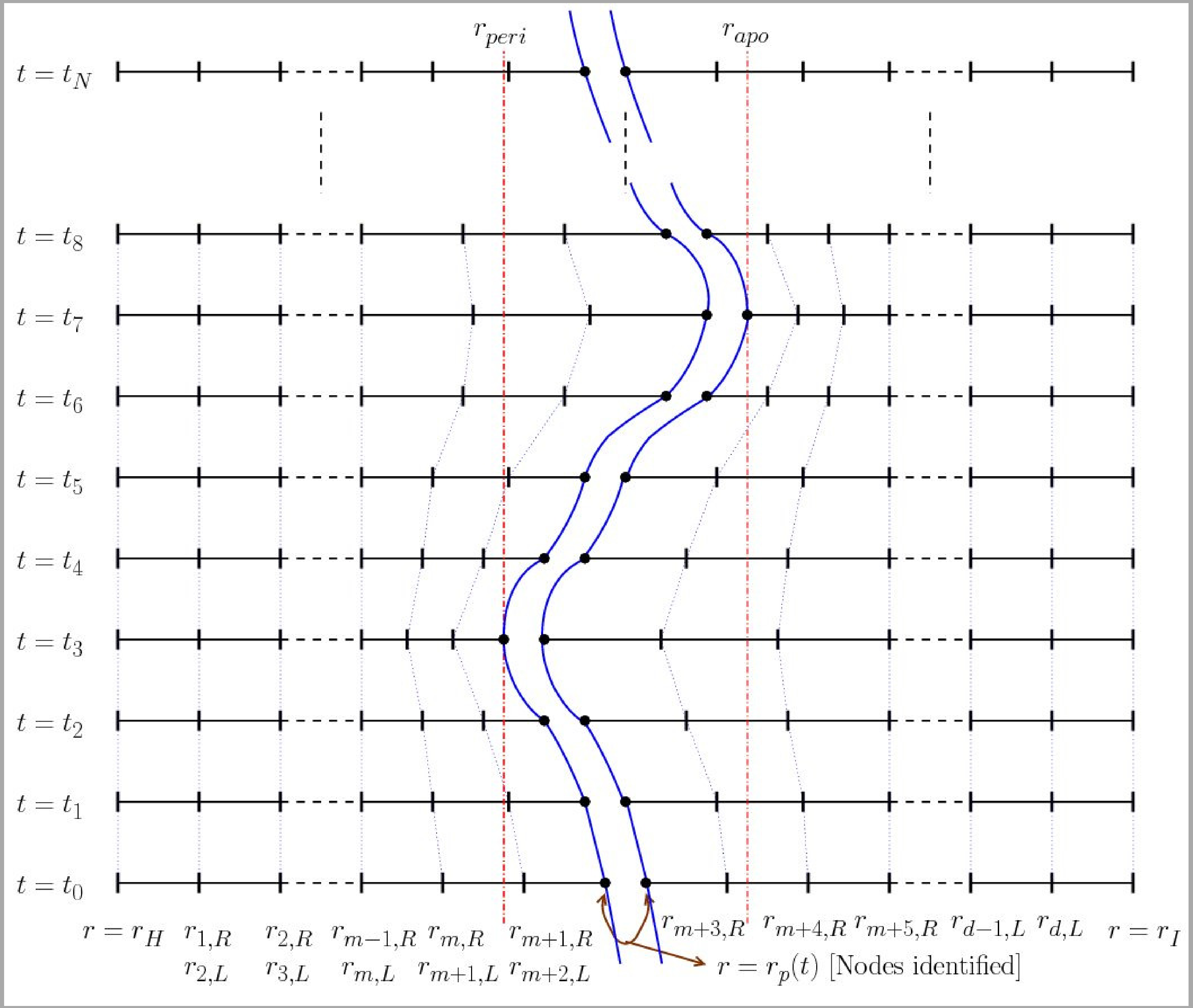
$$\psi^{\ell m} = r\,\Phi^{\ell m}$$

$$U^{\ell m} = \phi^{\ell m} - \varphi^{\ell m}$$

$$V^{\ell m} = \phi^{\ell m} - \varphi^{\ell m}$$

The Particle without Particle scheme

- In practice we use multiple domains (some of them with dynamical boundaries):



Improving the PwP scheme

- Two ways of improving the method are:

- Improve the “global” boundary conditions: At present we are using standard outgoing wave conditions at the boundaries of the “truncated” physical domain.

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- Improve the “global” boundary conditions: At present we are using standard outgoing wave conditions at the boundaries of the “truncated” physical domain.

- To reduce the number of domains without reducing the spatial resolution: The aim is to have faster computations of the self-force but maintaining the same accuracy.

Improving the PwP scheme

- One way of achieving this is by using a compactification scheme:

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- The easiest thing is to compactify spatial infinity. Let us consider the example of the advection equation:

$$(\partial_t + \partial_x)\, u(t,x) = 0\,, \qquad x \in [0, +\infty)\,,$$

$$u(0,x) = u_o(x)\,, \qquad u(t,0) = b(t)\,.$$

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- We can compactify the spatial domain using the mapping:

$$x \longmapsto \rho = \frac{x}{1+x}\,, \qquad \rho \in [0,1)$$

$$\partial_t u + (1-\rho)^2 \partial_\rho u = 0\,.$$

Improving the PwP scheme

- Spatial Compactification of the advection equation:

$$(\partial_t + \partial_x)\, u(t,x) = 0\,, \qquad x \in [0, +\infty)\,,$$

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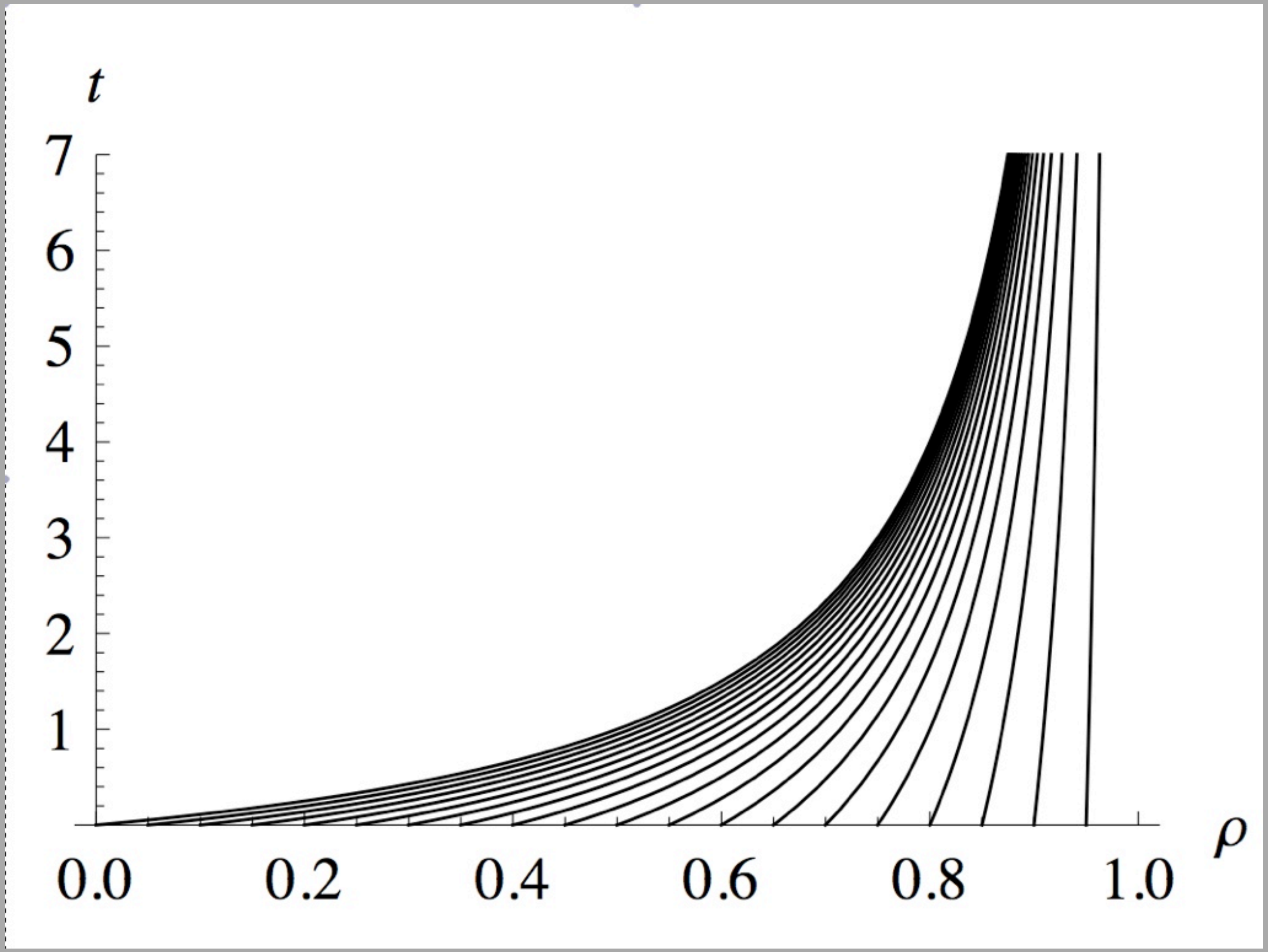

Characteristics of the transformed advection equation.

The coordinate speed approaches 0 near spatial infinity.

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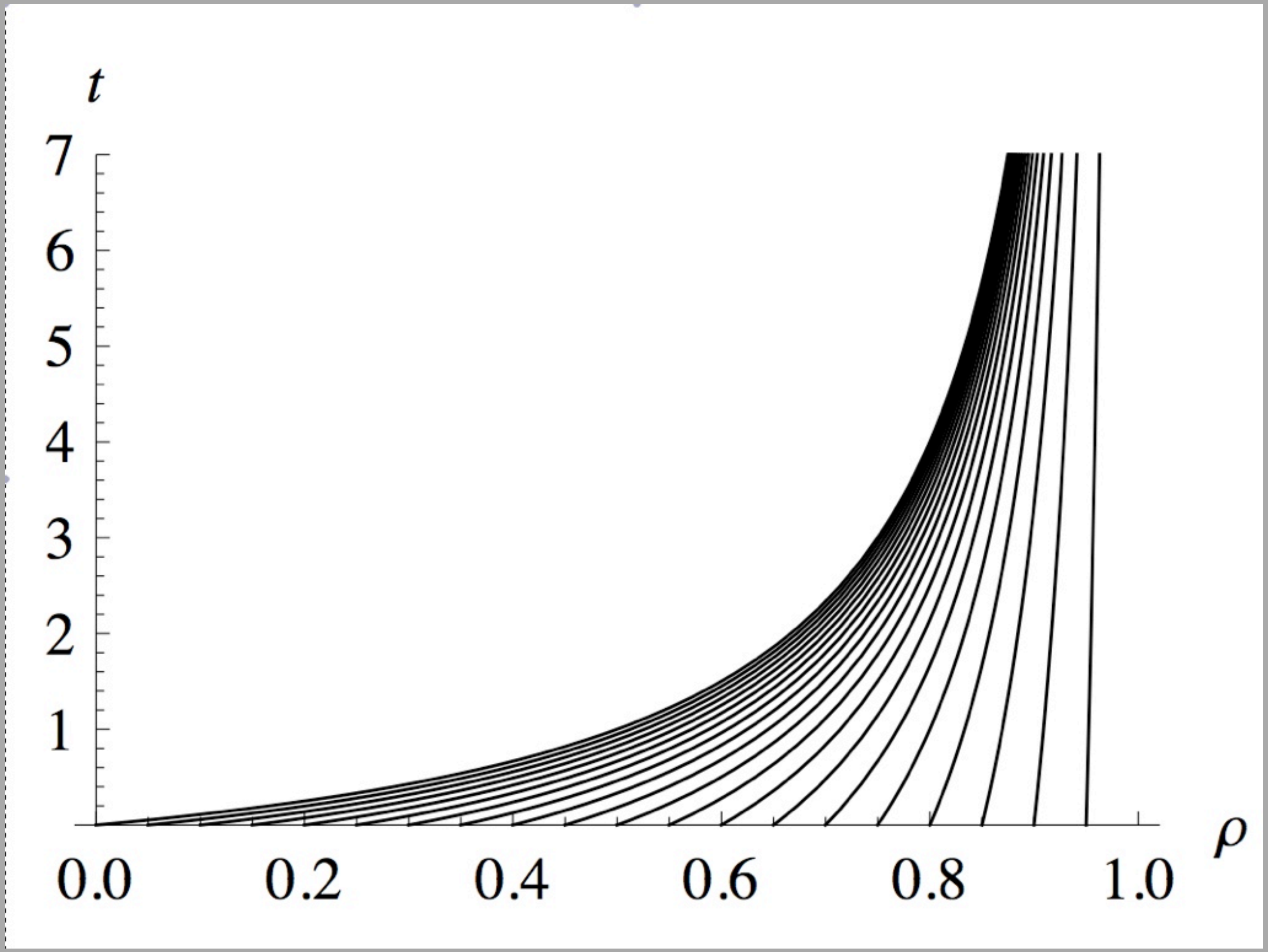


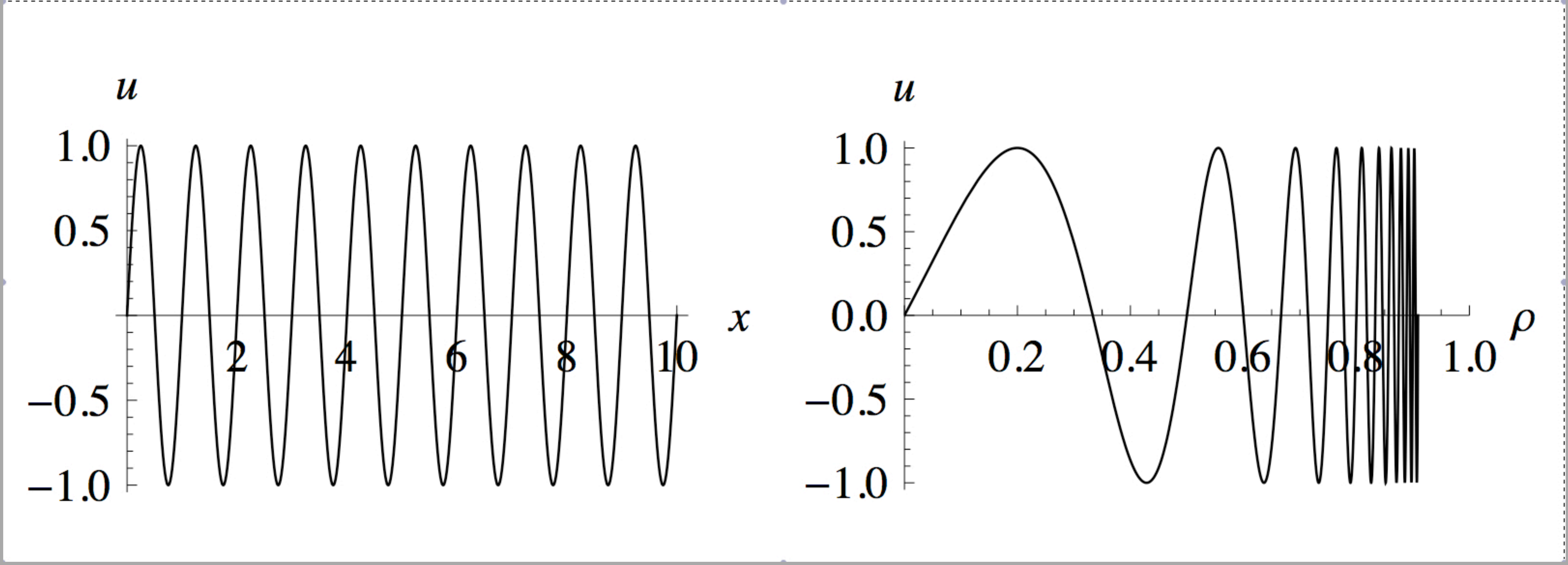
This means that the field cannot reach infinity in a finite time.

Improving the PwP scheme

- Spatial Compactification of the advection equation:

$$u_o(x) = \sin(2\pi\, x)\,, \qquad b(t) = -\sin(2\pi\, t)$$

$$u(t,x) = -\sin\left[2\pi(t-x)\right]\,, \qquad u(t,\rho) = -\sin\left[2\pi\left(t - \frac{\rho}{1-\rho}\right)\right]$$



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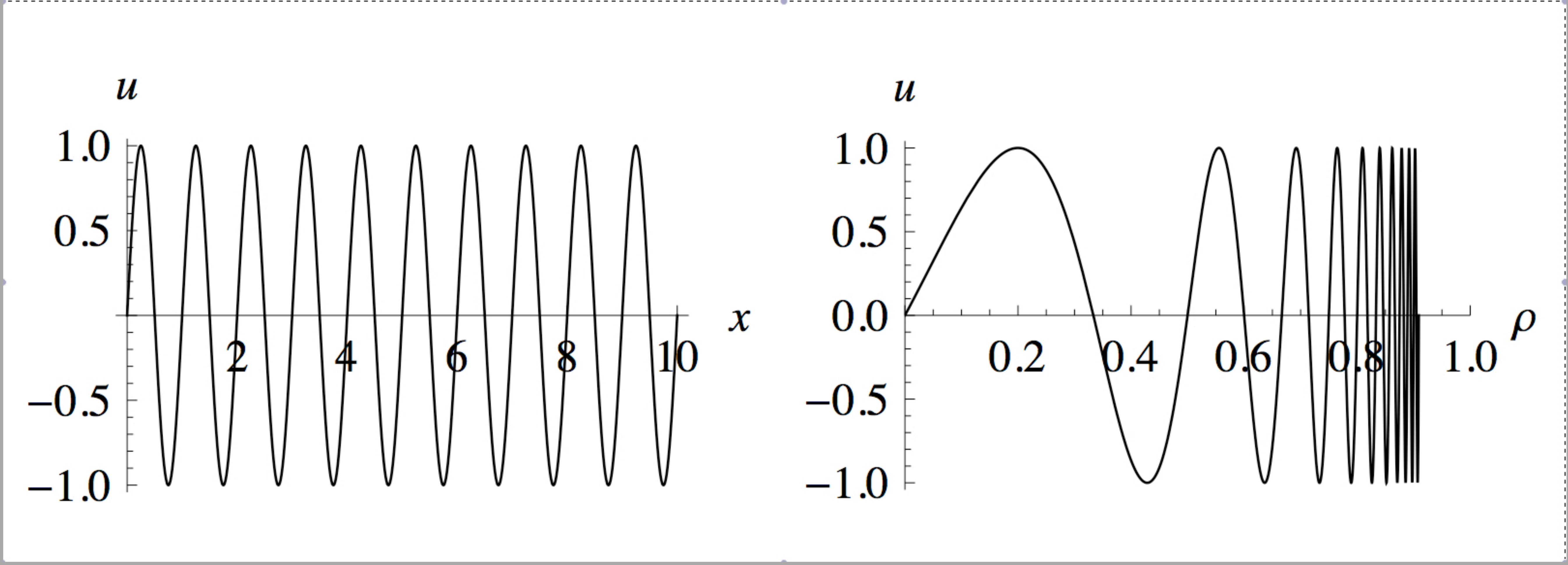
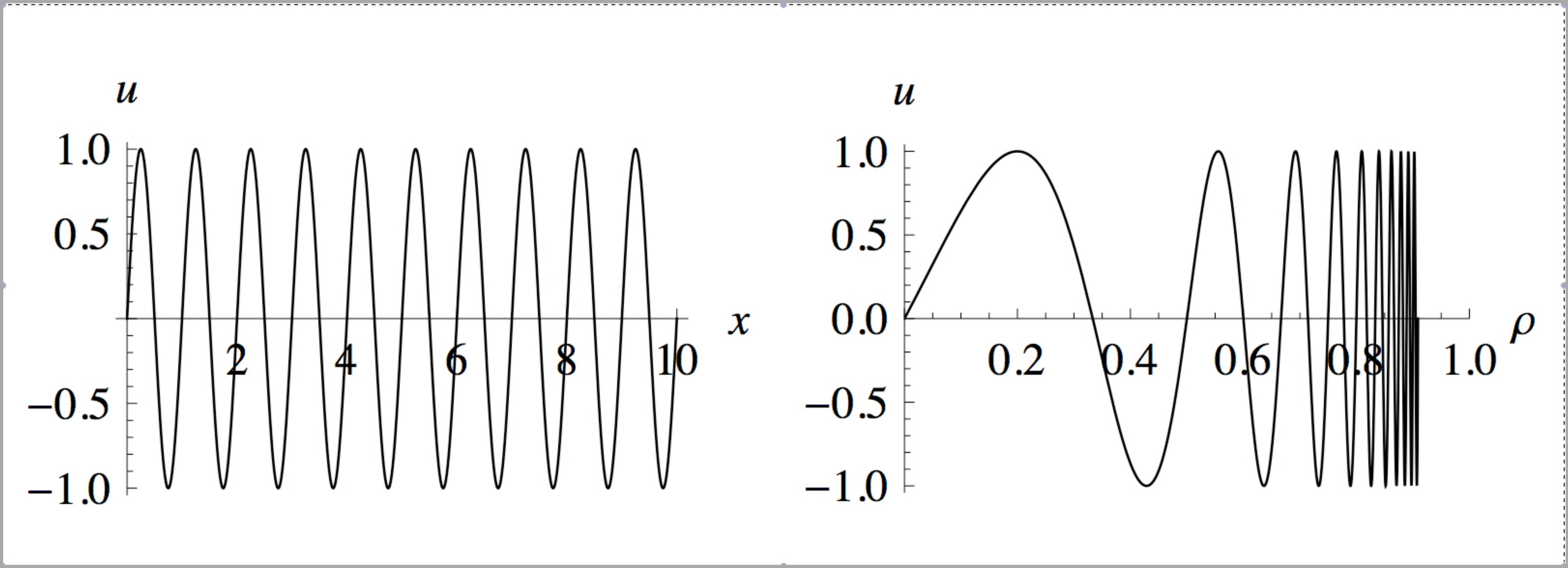


Improving the PwP scheme

- Spatial Compactification of the advection equation:

  The oscillations can not be resolved in the compactifying coordinate near infinity due to infinite blueshift in spatial frequency.



Hyperboloidal Compactification

- An Alternative: Hyperboloidal Compactification

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- The idea is also to transform time, that is, to change the time slicing:

$$t \longmapsto \tau = t - \left(x + \frac{C}{1+x}\right)\,,$$

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- We obtain the following transformed advection equation:

$$\partial_\tau u + \frac{1}{C}\partial_\rho u = 0$$

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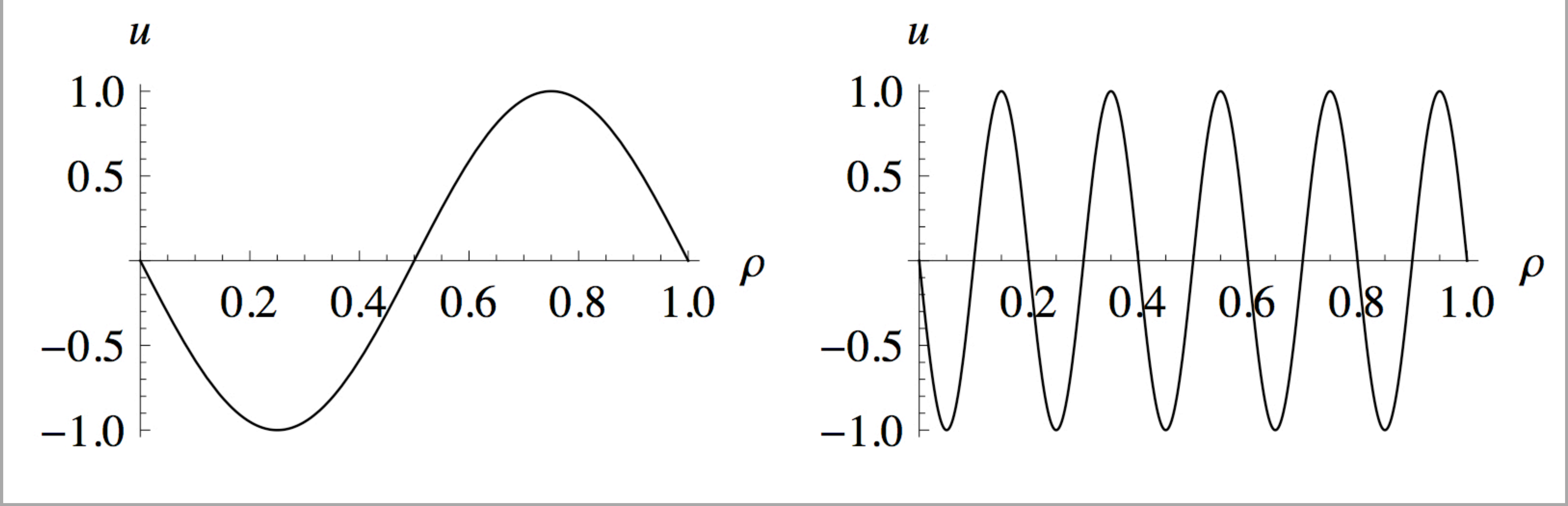

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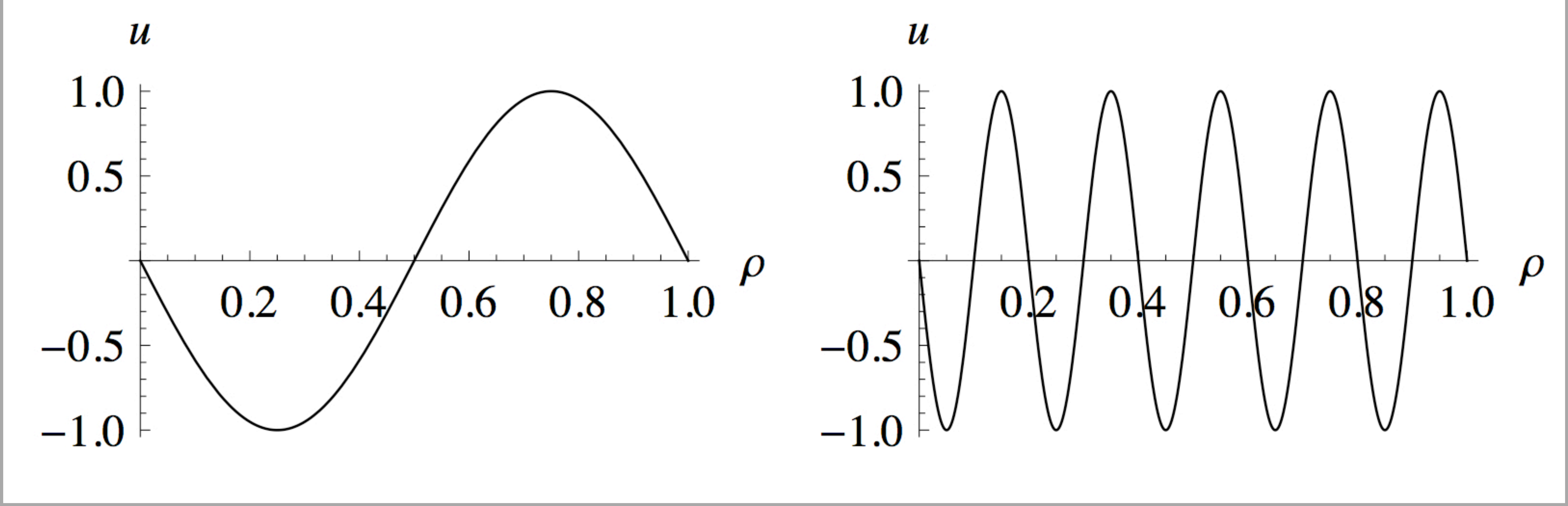

C = 1 C = 5

Hyperboloidal Compactification

- Hyperboloidal Compactification for our wave equations:

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- The change of time slicing and spatial coordinate are:

$$t \longmapsto \tau = t - h(r^*)\,,$$

$$r^* \in [R, +\infty) \longrightarrow \rho \in (R, S)$$

$$r^* \qquad \longmapsto \qquad \rho = r^*\,\Omega\,,$$

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- The representation of outgoing null rays is left invariant

$$t - r^* = \tau - \rho$$

if we choose:

$$h(r^*) = r^* - \rho(r^*)$$

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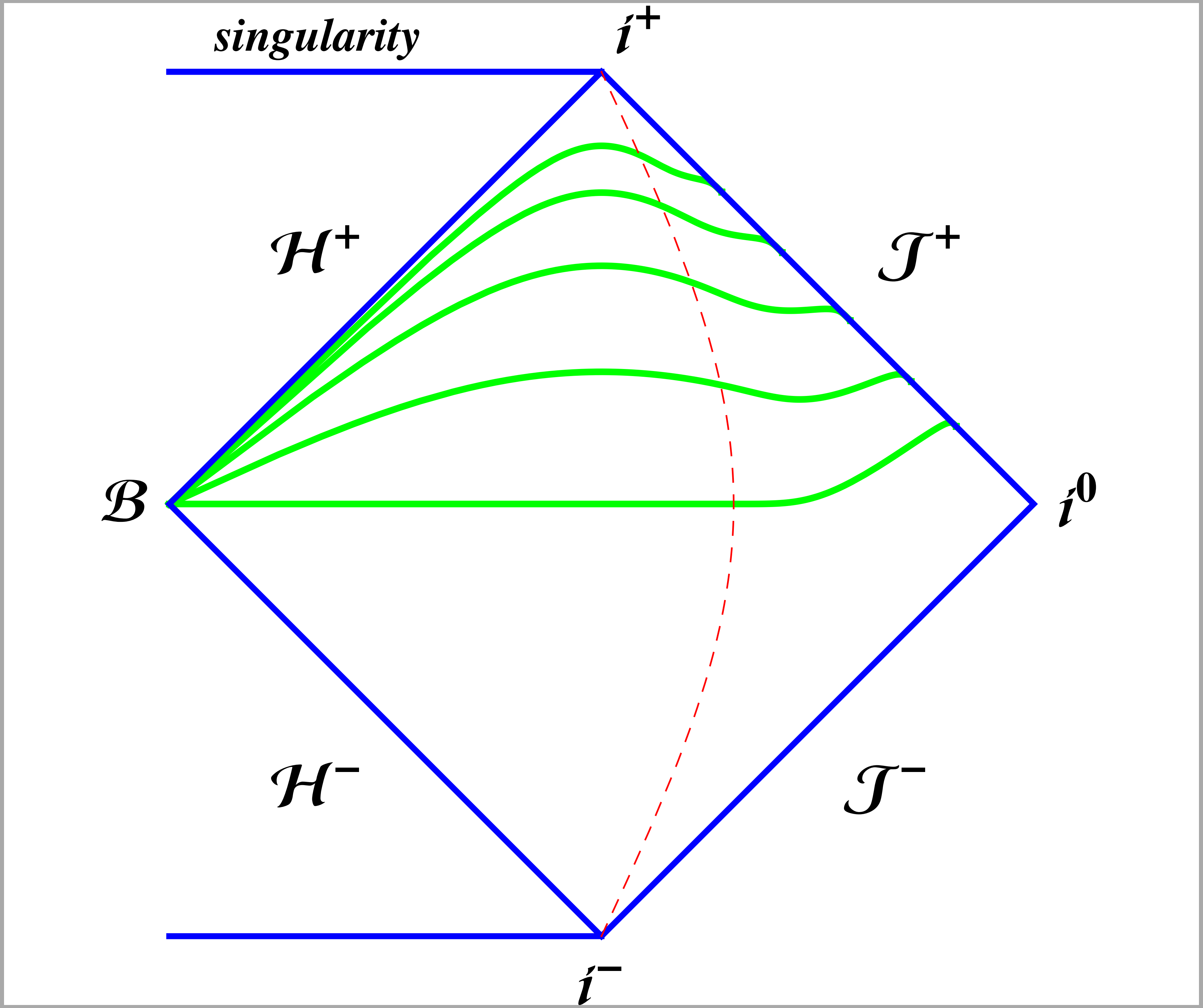

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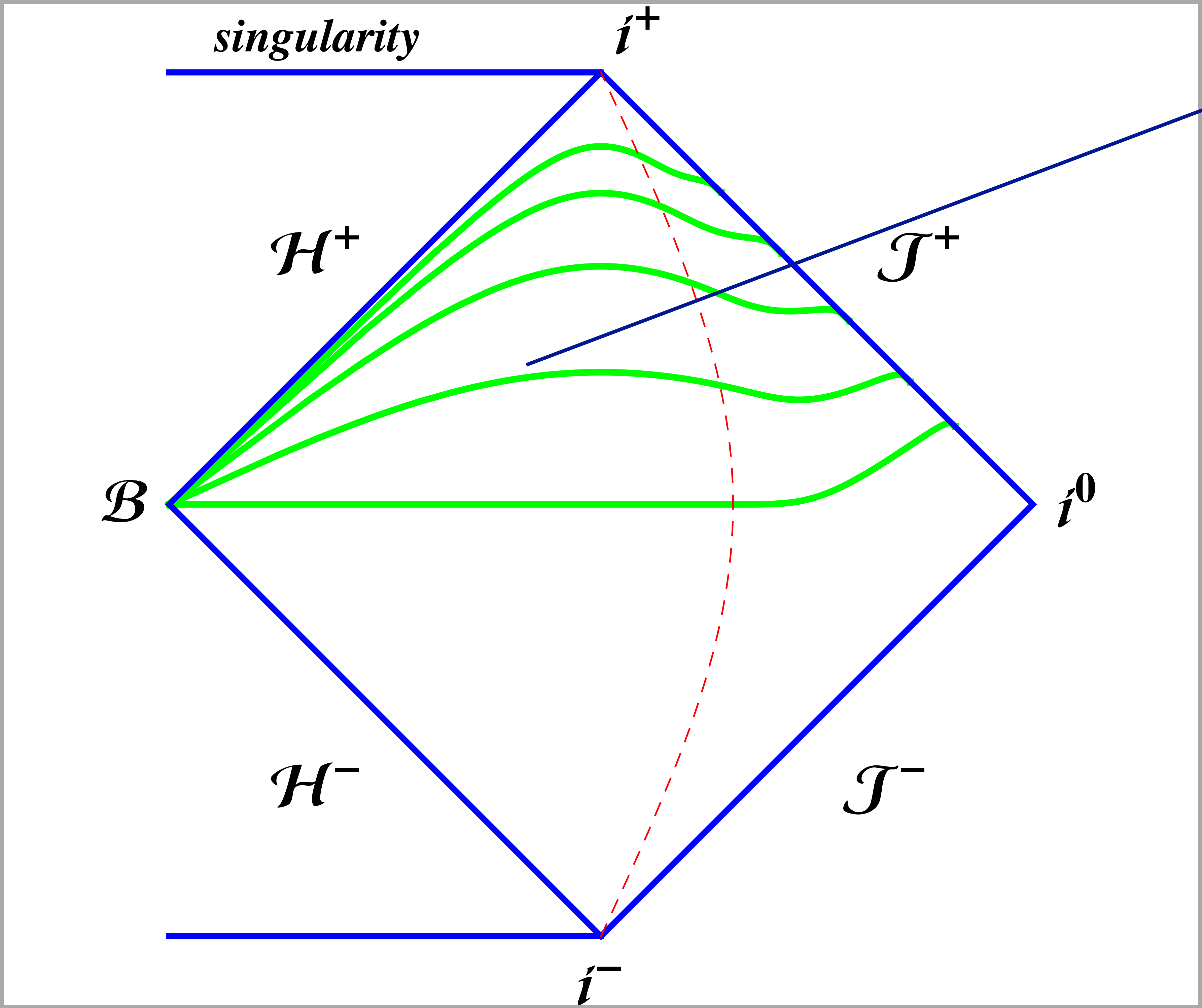

Spacelike Hyperboloidal Slices

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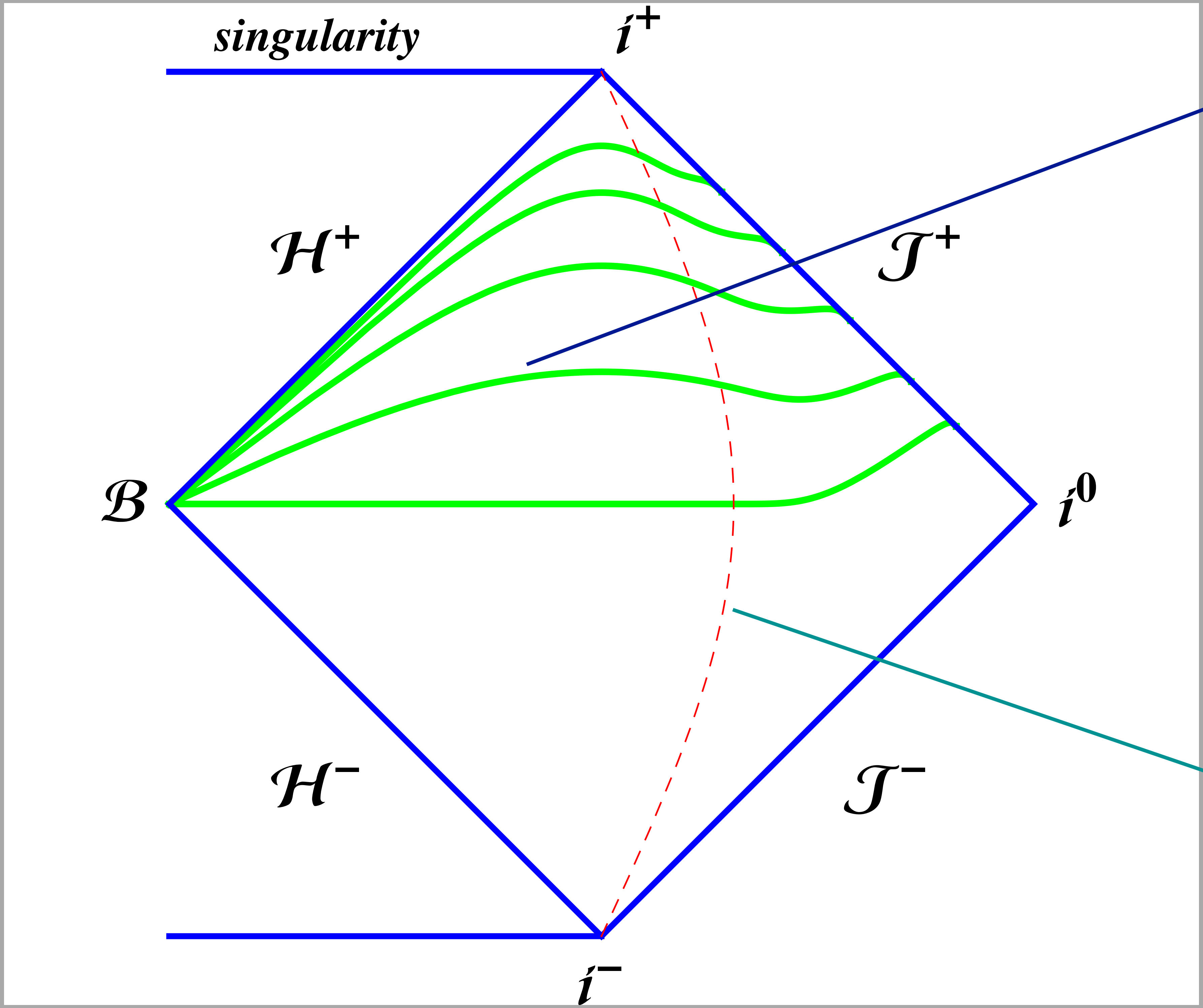

Spacelike Hyperboloidal Slices

Radial location of the interface (r*=R)

Hyperboloidal Compactification

- The equations transform from:

$$\left(-\partial_t^2 + \partial_{r*}^2 - V_\ell(r)\right)\psi_\pm^{\ell m} = 0\,,$$

to:

$$\left[-(1+H)\partial_\tau^2 - 2H\partial_\tau\partial_\rho + (1-H)\partial_\rho^2 - (\partial_\rho H)(\partial_\tau + \partial_\rho) - \frac{V_\ell}{1-H}\right]\psi_+^{\ell m} = 0\,,$$

where

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where

$$H = \frac{dh}{dr^*} = 1 - \frac{d\rho}{dr^*}$$

We can construct a 1st-order hyperbolic reduction introducing the following variables:

$$\phi := (1+H)\partial_\tau\psi + H\partial_\rho\psi, \qquad \varphi := \partial_\rho\psi.$$

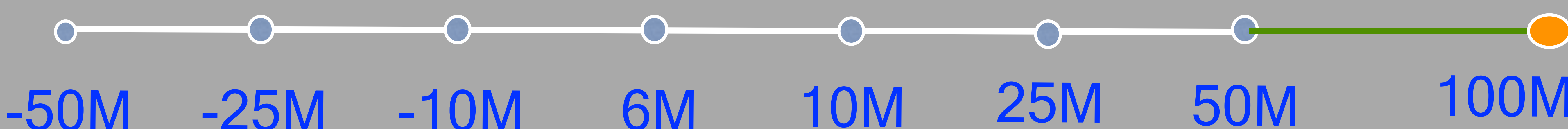
Hyperboloidal Compactification

- We use the following spatial compactification:

$$r^*(\rho) = \frac{\rho}{\Omega(\rho)}\,, \qquad \Omega = 1 - \left(\frac{\rho - R}{S - R}\right)^4 \Theta(\rho - R)\,.$$

but we have tried others.

- We have done computations for circular orbits using domains like the following one:



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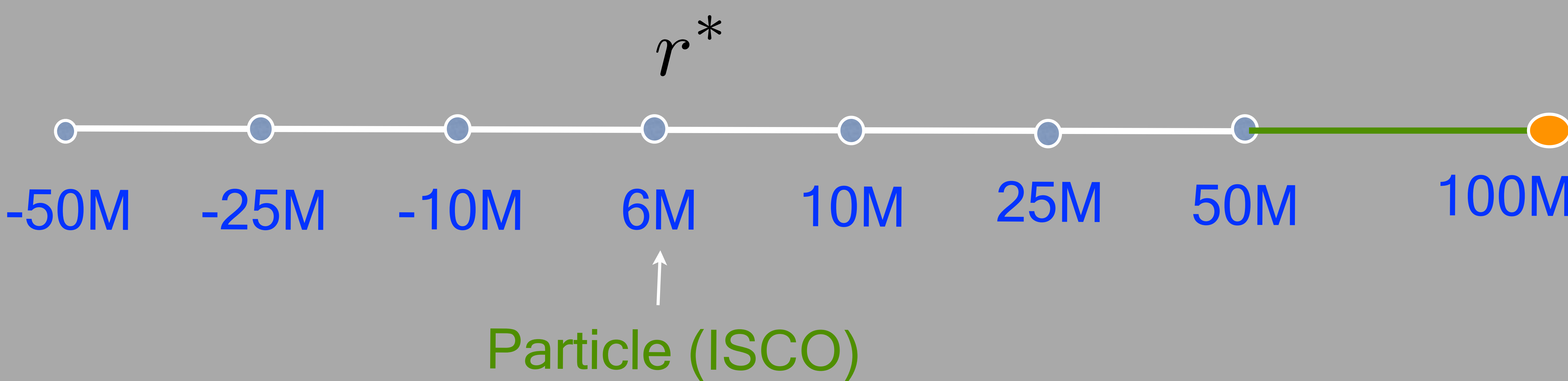

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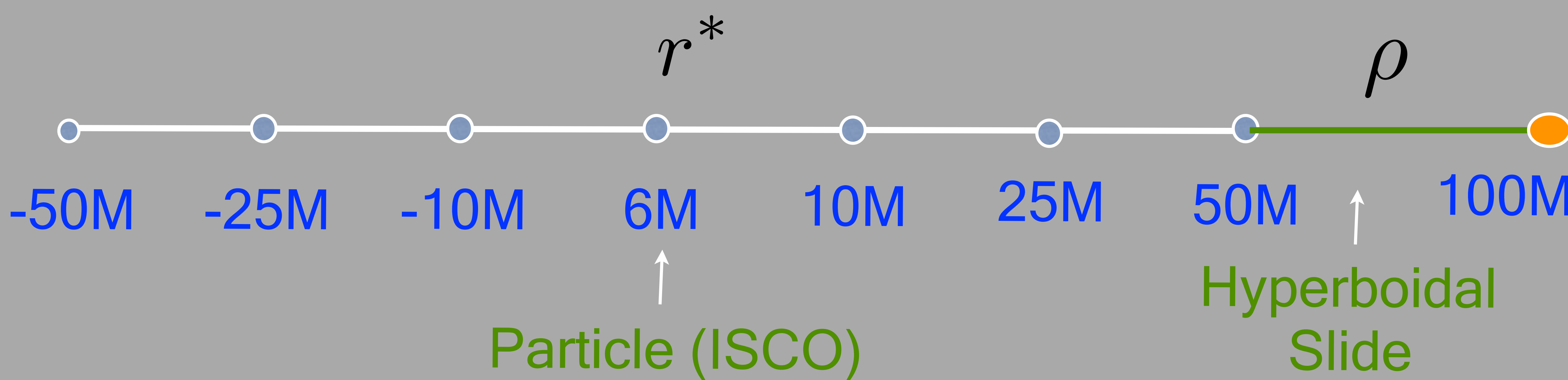

Some Preliminary Results

- Self-Force Calculations in the Circular Case:
  - The only component of the self-force (regularized gradient of the scalar field) that requires regularization is the radial component: $\Phi^R_r = 1.677282 \times 10^{-4}\, q/M^2\,,$

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  - However, at present, for long-term stability (hundreds of orbits) we need to apply a spectral filter to the compactified domain.

Remarks and Conclusions

- We have implemented the hyperboloidal compactification at spatial infinity for time-domain computations of the self-force.

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- Apply the same techniques at the Horizon

$$r^* \quad \longrightarrow \quad -\infty$$