Self-force on a scalar particle on a generic orbit in Kerr spacetime

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in collaboration with

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Outline

The need for high accuracy

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The need for high accuracy ⟦already discussed by Evans⟧

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Goals

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Overall plan

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Some details of puncture-function computations:

- time-domain, AMR (vs. frequency-domain)
- *m*-mode decomposition and 2-D evolution (vs. 3-D evolution)
- worldtube scheme (vs. window function)
- computing the puncture function and effective source
  - efficiently computing the ϕ integrals
  - efficiently evaluating Wardell's 4th-order puncture
- finite differencing near the puncture

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(Very) Preliminary results

Conclusions, Directions for Further Research

Goals

- Kerr
- scalar field for now, gravitational field in the future
- compute self-force very accurately
- equatorial circular orbit $\rightarrow$ equatorial eccentric orbit $\rightarrow$ generic orbit
- as efficient as possible (orbital evolution in the future)

This is work in progress!

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Puncture-function regularization ⟦as per Vega, Wardell, Dolan talks⟧

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m-mode decomposition $\Rightarrow$ 2+1D numerical evolution for each m

Worldtube scheme to treat far-from-the-particle region

m-mode Decomposition and 2-D Evolution

Vega, Detweiler, Diener, *et al.* numerically solve for the residual field φ_r in 3+1D,

$$\Box\varphi_\mathrm{r} = S_\mathrm{eff}$$

We (Barack, Dolan, Wardell, Thornburg) prefer to Fourier-decompose in ϕ [actually in $\tilde{\phi} := \phi + f(r)$ to avoid infinite-twisting at horizon], then numerically solve for each individual Fourier mode of the residual field $\varphi_\mathrm{r}^m := (\varphi - \varphi_\mathrm{punct})^m$ in 2+1D,

$$\Box^m\varphi_\mathrm{r}^m = S_\mathrm{eff}^m$$

where the 2-D puncture function and effective source are given by

$$\begin{aligned}
\varphi_\mathrm{punct}^m(t,r,\theta) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi_\mathrm{punct}(t,r,\theta,\phi) e^{-im\tilde{\phi}}\, d\tilde{\phi} \\
S_\mathrm{eff}^m(t,r,\theta) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} S_\mathrm{eff}(t,r,\theta,\phi) e^{-im\tilde{\phi}}\, d\tilde{\phi}
\end{aligned}$$

The Worldtube (Idea)

Problem: puncture function and effective source aren't well-defined far from the particle.

Solutions:

- window function ⟦Vega, Detweiler, Diener, *et al.*⟧

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Solutions:

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- world tube

World tube: Introduce 2-D world tube (of macroscopic size) containing particle. Then the "numerical field"

$$\varphi^m_{\text{num}} := \begin{cases} (\varphi - \varphi_{\text{punct}})^m & \text{inside worldtube} \\ \varphi^m & \text{outside worldtube} \end{cases}$$

satisfies

$$\Box_m \varphi^m_{\text{num}} = \begin{cases} S^m_{\text{eff}} & \text{inside worldtube} \\ 0 & \text{outside worldtube} \end{cases}$$

The Worldtube (Implementation)

The worldtube is very easy to implement at the finite-differencing level:

- store φ^m_{num} as a grid function in the usual manner
- for each finite difference operation at each grid point
    - if the finite difference molecule crosses the worldtube boundary, then copy all the input data for the molecule to a small scratch array, adjusting by $\pm\varphi^m_{\mathsf{punct}}$ as appropriate, and apply the standard finite difference molecule to the scratch array
    - if the finite difference molecule doesn't cross the worldtube boundary, then use the standard finite difference operation

Only a small fraction of grid points are within a molecule radius of the worldtube boundary & hence need the [slow] adjustment; most grid points can use the [fast] ordinary finite difference operation. Thus this scheme is quite fast on average.

Efficiently Computing the ϕ Integrals (1)

For the 4th order puncture we haven't been able to do these Fourier integrals analytically, so we have to do them numerically. Since S_{eff}^m appears as the source term in the 2+1D evolution equation, we have to compute the S_{eff}^m Fourier integral at each 2+1D grid point (actually, at each 2+1D grid point in the worldtube) at each time step.

That's a lot of integrals!

[For an equatorial-circular orbit we can just compute the Fourier integrals at each 2+1D grid point in the worldtube at the first time step, then apply a phase factor to get S_{eff}^m cheaply at each later time step. But this "rotation trick" doesn't work for more general particle orbits.]

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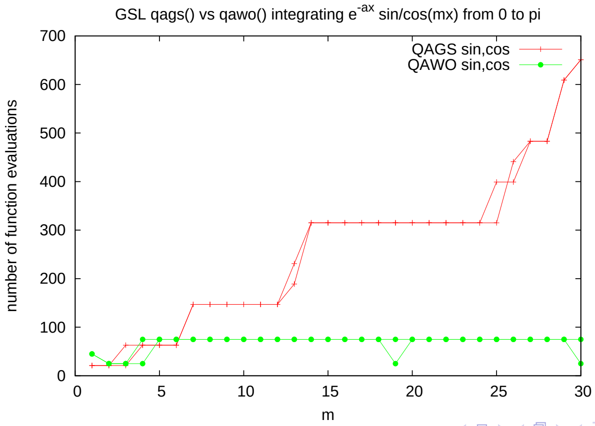
$\Rightarrow$ We don't yet know what fraction of the total computational cost will be consumed by these integrals.

Efficiently Computing the ϕ Integrals (2)

To actually compute the Fourier integrals, we use an adaptive numerical quadrature routine (GSL's `QAWO`) designed for oscillatory integrands.

$\Rightarrow$ cost of computing a Fourier integral is $\approx$ independent of m

(with a general-purpose numerical quadrature routine, cost $\propto m$)



The 3-D Puncture Function and Effective Source

Fix a $t =$ constant slice. Supose the particle is at x_p^i and we want to evaluate the 3-D puncture fn φ_{punct} at the position $x^i := x_p^i + \delta x^i$. Then

$$\varphi_{\text{punct}}(\delta r, \delta\theta, \delta\phi) = \frac{\sum_{ijk} N_{ijk} (\delta r)^i (\delta\theta)^j (\delta\phi)^{(k)}}{\left(\sum_{ijk} D_{ijk} (\delta r)^i (\delta\theta)^j (\delta\phi)^{(k)}\right)^{3/2}}$$

where the 18 N_{ijk} and 18 D_{ijk} coefficients depend on M, a, and x_p^i, but do **not** depend on δx^i, $k \in \{0, 2, 4\}$, and where

$$\begin{aligned}
\delta\phi^{(0)} &= 1 \\
\delta\phi^{(2)} &= \frac{5}{2} - \frac{8}{3}\cos\delta\phi + \frac{1}{6}\cos 2\delta\phi \quad \sim \quad (\delta\phi)^2 \ \text{ for } |\delta\phi| \ll 1 \\
\delta\phi^{(4)} &= 6 - 8\cos\delta\phi + 2\cos 2\delta\phi \quad \sim \quad (\delta\phi)^4 \ \text{ for } |\delta\phi| \ll 1
\end{aligned}$$

The 3-D effective source $S_{\text{eff}}(\delta r, \delta\theta, \delta\phi)$ is similar, but messier.

Efficiently Computing the 2-D Puncture Fn & Effective Src

To compute the 2-D effective source S^m_{eff} at a given $(\delta r, \delta\theta)$ (corresponding to a given 2-D (r, θ) grid point), the numerical integration of the Fourier integral requires computing $S_{\text{eff}}(\delta r, \delta\theta, \delta\phi)$ for many (~ 100) different $\delta\phi$ values.

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Thus, we define the coefficients

$$N_k(\delta r, \delta\theta) := \sum_{ij} N_{ijk}(\delta r)^i (\delta\theta)^j \qquad D_k(\delta r, \delta\theta) := \sum_{ij} D_{ijk}(\delta r)^i (\delta\theta)^j$$

so that

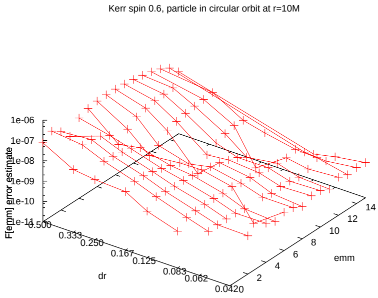
$$\varphi_{\text{punct}}(\delta r, \delta\theta, \delta\phi) = \frac{\sum_k N_k(\delta r, \delta\theta) \delta\phi^{(k)}}{\left(\sum_k D_k(\delta r, \delta\theta) \delta\phi^{(k)}\right)^{3/2}}$$

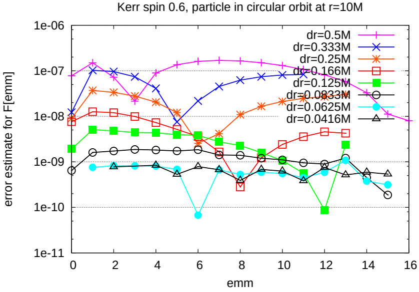
The coefficients N_{ijk} and D_{ijk} are computed once per time step (for the equatorial circular orbit case they're actually time-independent). The coefficients N_k and D_k are computed once per (worldtube) grid point per time step. Given N_k and D_k, $\varphi_{\text{punct}}(\delta r, \delta\theta, \delta\phi)$ is fairly cheap to compute.

(Very) Preliminary Results (1)

- Kerr BH (spin 0.6), particle in equatorial circular orbit at $r = 10M$
- unigrid, highest resolution $\Delta r_* = \frac{M}{16}$, $\Delta\theta = \frac{\pi}{640}$ radians ($\Rightarrow r\,\Delta\theta \approx \Delta r$ at particle)
- 4th order finite differencing (ignore non-smoothness at the particle)
- evolutions for $0 \le m \le 15$

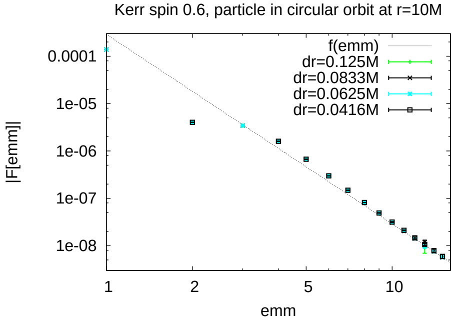
$\Rightarrow$ Between 2nd and 3rd order convergence with resolution:

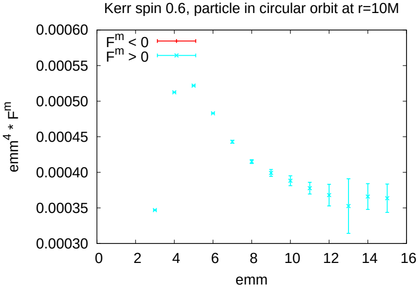




(Very) Preliminary Results (2)

Nice $\sim m^{-4}$ convergence of F^m at large m:

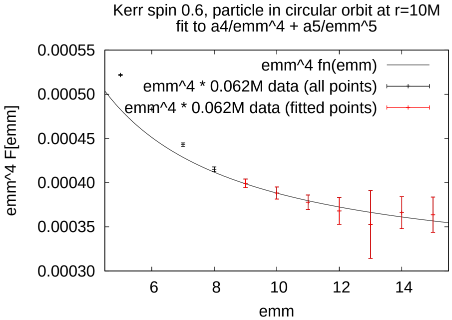




(Very) Preliminary Results (3)

Self-force in units of $10^{-6} q^2/M^2$:

$$
\begin{aligned}
\sum_{m=0}^{15} F^m &= -7.529 \pm 0.004 \\
\text{fitted } m^{-4} \text{ tail sum} &= 0.026 \pm 0.0008 \\
\text{fitted } m^{-5} \text{ tail sum} &= 0.004 \pm 0.0004 \\
\text{sum} &= -7.499 \pm 0.004 \quad (0.05\%) \\[4pt]
\text{true } F_{\text{self}} \text{ (Niels Warburton)} &= -7.491205 \\
\text{actual error} &= -0.008 \quad (0.11\%)
\end{aligned}
$$



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  - $\Rightarrow \delta F_{\text{self}} \sim 10^{-3}$
  - AMR should greatly improve this
  - don't yet know equal-efficiency point with frequency-domain

Directions for Further Research

Near-Term:

- don't integrate outside result domain of dependence
- angular boundary condition at equator
- better finite-differencing error estimates
- non-naive formula for $\delta\phi^{(2)}$ and $\delta\phi^{(4)}$
- interpolate singular field very close to particle to avoid cancellations in naive formula there
- more generic orbits (equatorial eccentric is in progress now)
- fancier finite differencing at particle (try to get $\geq$ 4th order)
- phase-align grid with particle?
- auto-adjust domain size

Medium-Term:

- AMR
- high-accuracy runs