Effective source approach to self-force: The (3+1) effort

Ian Vega[1]

Peter Diener[2] Barry Wardell[3]

[1]University of Guelph
[2]Louisiana State University
[3]Albert Einstein Institute

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Outline

- General idea
- 3+1 implementation
- Preliminary results
- Summary

General idea

Capra mandate

(a) Understand the two-body problem in general relativity in the extreme-mass-ratio regime
(b) Develop methods for modeling such systems.

Main issue

- Multiple scales: spatial scale of large black hole vs scale of the small compact object + short dynamical time scale vs long radiation reaction time scale

General line of attack
Map the EMRI problem onto that of the motion of a point mass in black hole spacetime, where the motion needs to incorporate the effects of the self-force.

Point sources → delta functions → locally divergent fields requiring regularization

Numerical self-force: Mode sum method

- Use delta-function source

$$\Box\psi = \delta$$

The physical solution, ψ^{ret}, diverges at the particle.

- Break into spherical harmonic modes that are all finite.

- Regularize l-modes.

$$\mathscr{R}_\alpha{}^\ell = \left(\nabla_\alpha \psi^{\text{ret}}\right)^\ell - (\ell + \tfrac{1}{2})A_\alpha - B_\alpha - \frac{D_\alpha}{\left(\ell - \frac{1}{2}\right)\left(\ell + \frac{3}{2}\right)},$$

- Sum the remainders.

$$F_\alpha = \sum_{\ell=0}^{L} \mathscr{R}_\alpha{}^\ell$$

Effective source

What if we avoid using a delta function altogether?
Perhaps replace it by a less singular source?
$\rightarrow$ Fields will be finite.

- Choose $\bar{\psi}$ st $\Box\bar{\psi} = \delta + O(\rho^n),\ n \geq -1$
- Then regularize the delta function source

$$\begin{aligned} \Box(\psi^{\mathsf{R}} + \bar{\psi}) &= \delta \\ \Box\psi^{\mathsf{R}} &= -\Box\bar{\psi} + \delta \\ \Box\psi^{\mathsf{R}} &= S = O(\rho^n) \end{aligned}$$

- Effective source:

$$S := -\Box\bar{\psi} + \delta = O(\rho^n)$$

Effective source

Capra 10, (Alabama '07): The idea was independently proposed by two groups at the same time.

- Barack and Golbourn (PRD 2007): "puncture" scheme

$$\bar{\psi} = q/\epsilon, \quad \epsilon^2 = (g_{\alpha\beta} + u_\alpha u_\beta)\delta x^\alpha \delta x^\beta$$

$$\Box\bar{\psi} = \delta + O(\rho^{-1})$$

$\psi^{\mathsf{R}} = C^0 \rightarrow$ enough to recover retarded field and fluxes

- Vega and Detweiler (PRD 2008): "smeared-out" sources

$$\bar{\psi} = \tilde{\psi}^{\mathsf{S}} \text{ (DW singular field)}$$

$$\Box\tilde{\psi}^{\mathsf{S}} = \delta + O(\rho^{1})$$

$\psi^{\mathsf{R}} = C^2 \rightarrow$ enough to compute for self force

Choice of $\bar{\psi}$

A convenient choice of $\bar{\psi}$ is the Detweiler-Whiting singular field

$$\psi^{\mathsf{S}}(x) := q \int G^{\mathsf{S}}(x, z(\tau)) d\tau$$

$$G^{\mathsf{S}}(x, x') = \frac{1}{2} U(x, x') \delta(\sigma) - \frac{1}{2} V(x, x') \theta(\sigma)$$

This choice has the following neat features:

- $\Box \psi^{\mathsf{R}}(x) = -\Box \psi^{\mathsf{S}}(x) + \delta = 0$ when $x \in \mathcal{N}(z)$
- $F_\alpha = \nabla_\alpha \psi^{\mathsf{R}}$, where $\psi^{\mathsf{R}} := \psi^{\mathsf{ret}} - \psi^{\mathsf{S}}$

So not only do we produce a wave equation with a regular source, the resulting physical solution of this wave equation immediately gives the self-force.

Choice of $\bar{\psi}$

While by definition the DW singular field, ψ^{S}, ought to give

$$\Box\psi^{\mathsf{R}}(x) = 0 \;\text{ when }\; x \in \mathcal{N}(z)$$

in practice, one can find an explicit (coordinate) expression only for an approximation to this singular field, $\tilde{\psi}^{\mathsf{S}}$, so that

$$\Box\psi^{\mathsf{R}} = -\Box\tilde{\psi}^{\mathsf{S}} + \delta = O(\rho^n)$$

Two methods to get a coordinate expression

- Locally inertial coordinates (THZ) in which the singular field looks like an ordinary Coulomb field (Detweiler et al, 2003)
- Covariant expansion + coordinate expansion (Haas and Poisson, 2006)

(Spatially) compactifying the source

The coordinate expression for the singular field typically results in an effective source that misbehaves away from the particle. Moreover, we would typically want a source that is spatially compact.

This is easily done by choosing a window function, W, whose role it is to force the effective source to zero outside some specified region. The effective source is then constructed as

$$S = \begin{cases} -\Box(W\tilde{\psi}^{\mathsf{S}}), & x \neq z \\ 0, & x = z \end{cases}$$

This implies that $\psi^{\mathsf{R}} = \psi^{\mathsf{ret}} - W\tilde{\psi}^{\mathsf{S}}$. W should be picked such that it does not affect the condition that $F_\alpha := \nabla_\alpha \psi^{\mathsf{R}}$

(Spatially) compactifying the source

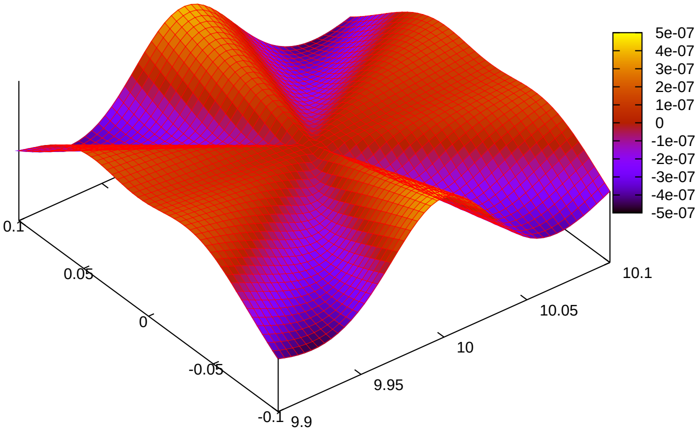
The window function needs to satisfy the following conditions:

1. $W \to 1$ sufficiently fast as one approaches the particle,
2. $\nabla_\alpha W \to 0$ sufficiently fast as one approaches the particle, and
3. $W = 0$ outside a compact region R surrounding the particle.
4. Smoothness (optional)

These conditions guarantee that

(a) $\nabla_\alpha \psi^{\mathsf{R}}|_{\text{point charge}}$ gives the self-force.
(b) ψ^{R} gives fluxes in the wavezone.

Example



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3e-07
2e-07
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-5e-07
0.1
0.05
0
-0.05
-0.1
9.9
9.95
10
10.05
10.1

Prescription

Self-force calculation

- Prescribe worldline, $z^\alpha(\tau)$.
- Solve
$$\Box\psi^{\mathsf{R}} = S(x^a, z^a(\tau), u^a(\tau))$$
- And evaluate $\nabla_\alpha\psi^{\mathsf{R}}$ along the worldline.

Self-consistent evolution

- Simultaneously integrate
$$\Box\psi^{\mathsf{R}} = S(x^a, z^a(\tau), u^a(\tau))$$
$$\frac{\mathrm{d}^2 z^a}{\mathrm{d}\tau^2} = \frac{q}{m}(g^{ab} + u^a u^b)(\nabla_b\psi^{\mathsf{R}})|_{z^a}.$$

Choice of integration method

Why (3+1)?

- Non-post-processing (therefore straightforward) approach to self-consistent evolution.
- Does not rely on the underlying symmetries of the spacetime.
- Difficulty of the calculation (IN PRINCIPLE) should not depend on the orbit (except that the effective source tends to be more expensive to compute for generic orbits)

Why not (3+1)?

- Less accurate (though perhaps not necessarily)
- Much fewer checks as compared to methods based on some decomposition (e.g. no mode fall-off)

Past work

- L. Barack and D. Golbourn (PRD 2007): (2+1), puncture, compute retarded field, Schw
- IV and S. Detweiler (PRD 2008): (1+1), DW singular field, compute self-force, Schw
- IV, P. Diener, W. Tichy, S. Detweiler (PRD 2009): (3+1), circular orbit
- S. Dolan and L. Barack (PRD 2010): (2+1), higher-order puncture, compute self-force, Schw, generic orbits
- S. Dolan, L. Barack, B. Wardell (arxiv 2011): (2+1), Kerr, compute scalar self-force

See also recent review of the effective source approach by IV, B. Wardell, P. Diener (CQG, 2011).

(3+1) implementation

Evolution equations

Scalar wave equation in (3+1)

$$
\begin{aligned}
\alpha^2 \nabla_a \nabla^a \psi \quad = \quad & -\partial_t \partial_t \psi + \beta^i \partial_t \partial_i \psi \\
& + \frac{\alpha}{\sqrt{\gamma}} \partial_i \left(\frac{\sqrt{\gamma}}{\alpha} \beta^i \partial_t \psi \right) \\
& + \frac{\alpha}{\sqrt{\gamma}} \partial_i \left[\alpha \sqrt{\gamma} \left(\gamma^{ij} - \frac{\beta^i \beta^j}{\alpha^2} \right) \partial_j \psi \right] \\
& + \alpha^2 S
\end{aligned}
$$

where

$$H = 2m/r, \;\; \alpha^2 = (1+H)^{-1}, \;\; \beta^i = \alpha^2 H x^i / r$$

$$\gamma^{ij} = \eta^{ij} - \frac{H}{1+H} \frac{x^i x^j}{r^2}$$

Evolution equations

1st-order form

$$
\begin{aligned}
\partial_t \rho &= \beta^i \partial_i \rho + \frac{\alpha}{\sqrt{\gamma}} \partial_i \left[\alpha \sqrt{\gamma} \left(g^{ij} \phi_j + \frac{\beta^i \rho}{\alpha^2} \right) \right] - \alpha^2 S \\
\partial_t \phi_i &= \partial_i \rho \\
\partial_t \psi &= \rho
\end{aligned}
$$

$$
\rho := \partial_t \psi \\
\phi_i := \partial_i \psi
$$

$\{\rho, \phi_i\}$: evolved
ψ : solved as a simple ODE.

Self-force is simply ϕ_i interpolated to the particle location.

Hyperboloidal slicing

- Using standard spatial slices forces one to deal with the problem of imperfect outer boundary conditions.

- This is solved using hyperboloidal slicing, as proposed by A. Zenginoğlu and M. Tiglio (PRD 2009), that compactifies the spacetime, bringing $\mathscr{I}^+$ to some finite coordinate distance. (Of course the coordinate speed of the ingoing characteristic at $\mathscr{I}^+$ is zero).

- This amounts to the coordinate transformation

$$\begin{aligned} \tau &= t - h(r) \\ r &= \rho/\Omega(\rho), \text{ such that } \Omega(\rho) \to 0, \rho \to L \end{aligned}$$

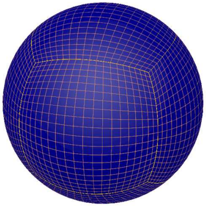
  and working with a conformally rescaled metric, $g = \Omega^2 \tilde{g}$, that is regular at $\rho = L$.

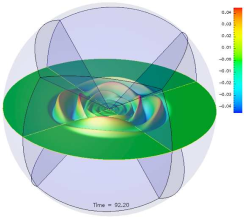
Hyperboloidal slicing

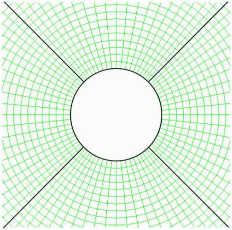
- To avoid having to worry about the source term, we choose to implement hyperboloidal slicing only in the source-free region, and retain the spatial Kerr-Schild time slices where the source is non-zero.
- This can be done by choosing $h(r)$ and $\Omega(\rho)$ appropriately, making sure that $h(r) = C$ and $\Omega(\rho) = 1$ in some $\rho < \rho_{\text{in}}$ while smoothly transitioning to the hyperboloidal slices of Zenginoğlu and Tiglio starting at $\rho = \rho_{\text{out}}$.
- This allows for very long-term evolution without any spurious reflection from the boundary.

Multi-block code

The evolution that's currently being used is the multi-block code described in [Schnetter, Diener, Dorband, Tiglio (2006)].







The code has also been used to compute quasinormal modes in Kerr via 3D scalar field evolutions. (Dorband et al, 2006)

It works very well for scalar fields!

Approximation to the singular field

$$\psi^{\mathsf{S}} = \frac{q}{2r} + \frac{q}{2r_{\mathsf{adv}}} + O(\epsilon^3)$$

$$\psi^{\mathsf{S}} = \frac{q}{s}\left(1 + \frac{\bar{r}^2 - s^2}{6s^2} R_{u\sigma u\sigma} + \frac{\bar{r}(\bar{r}^2 - 3s^2)}{24s^2} R_{u\sigma u\sigma|u} - \frac{(\bar{r}^2 - s^2)}{24s^2} R_{u\sigma u\sigma|\sigma}\right) + O(\epsilon^3)$$

Five scalar functions
$\{s, \bar{r}, R_{u\sigma u\sigma}, R_{u\sigma u\sigma|u}, R_{u\sigma u\sigma|\sigma}\}$.

$$\sigma_\alpha = \sigma_\alpha(x, \bar{x}) \rightarrow \text{requires coordinate expansion}$$

$$s^2 = (g^{\bar{\alpha}\bar{\beta}} + u^{\bar{\alpha}} u^{\bar{\beta}})\sigma_{\bar{\alpha}}\sigma_{\bar{\beta}}$$

$$\bar{r} = u^{\bar{\alpha}}\sigma_{\bar{\beta}}$$

$$R_{u\sigma u\sigma|u} = R_{\bar{\alpha}\bar{\beta}\bar{\gamma}\bar{\delta};\bar{\epsilon}} u^{\bar{\alpha}} \sigma^{\bar{\beta}} u^{\bar{\gamma}} \sigma^{\bar{\delta}} u^{\bar{\epsilon}}$$

Approximation to singular field

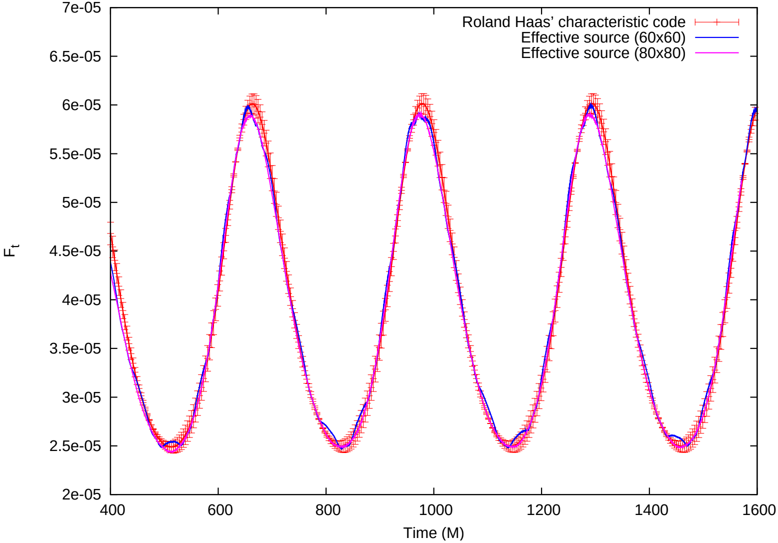
The straightforward coordinate expansion needs to be massaged a bit to produce an effective source that's amenable to a 3+1 code.

Currently, we do the following:

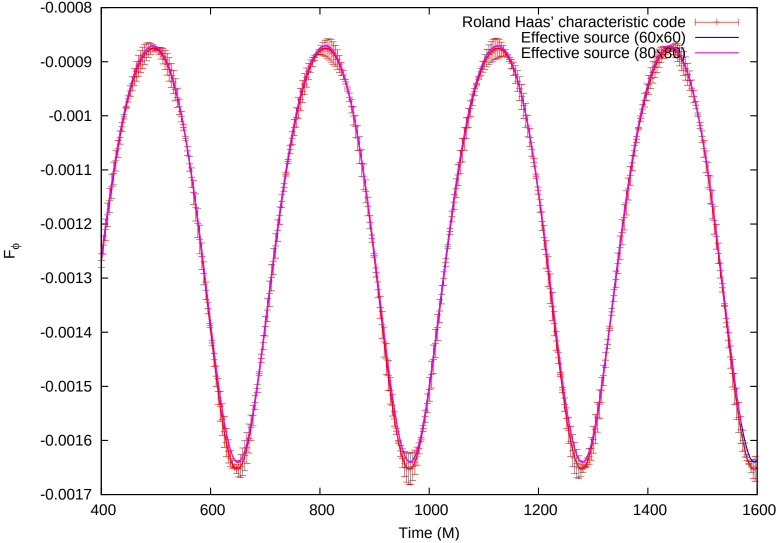
- Use a smooth window functions in θ and r.
- Choose window (and orbit) so that we never encounter divergences away from the particle.
- Periodicize the singular field.
- Employ some form of interpolation very close to the particle to take care of the round-off error there.

(These are described in more detail by Barry's talk).

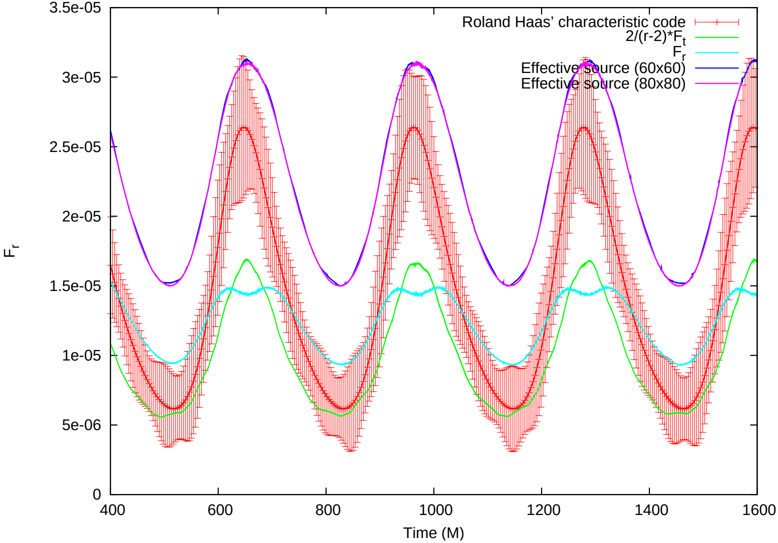
Results: F_t, eccentric orbit



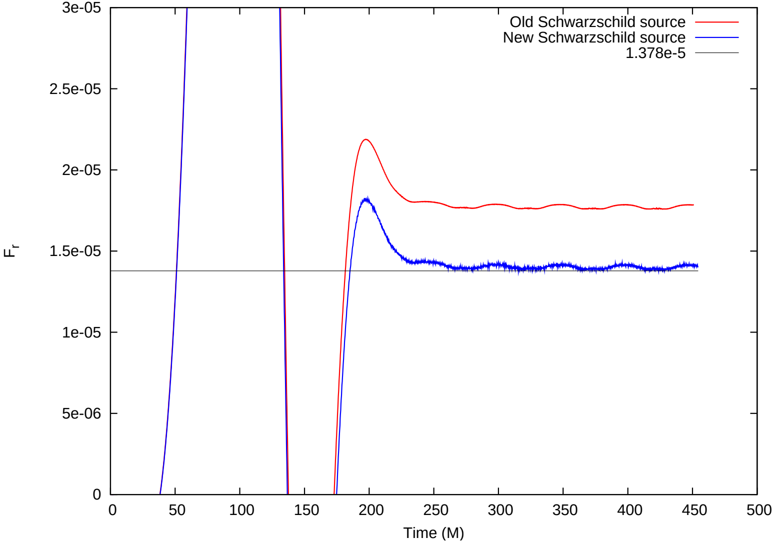
Results: F_ϕ, eccentric orbit



Results: F_r, eccentric orbit



Results: F_r, circular orbit



Summary

- There exists a (3+1) approach to the self-force programme.
- The (3+1) approach, in principle, is a robust method that (a) does not care about the symmetries of the background spacetime, and (b) should work just as well for any orbit.
- It can naturally be extended to provide self-consistent orbits.
- We still have quite a bit to do!