# Frequency-domain approach to self-force calculations

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# TD vs. FD

| Time Domain | Frequency domain |
|---|---|
| PDEs | ODEs |
| Same speed regardless of orbit type | Fast for low eccentricity orbits |
| Self-consistent evolution | Osculating orbit method |

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**Scalar Self Force (SSF) in Kerr spacetime**

Generic circular and eccentric equatorial orbits

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**Scalar Self Force (SSF) in Kerr spacetime**

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**Gravitational Self Force (GSF) in Schwarzschild spacetime**

Eccentric orbits
See also Akcay's talk

# SSF in Kerr

## Wave equation

The minimally coupled Klein-Gordon equation with source $T$

$$\Box\Phi \equiv \Phi_{;\alpha}^{\ \ \alpha} = -4\pi T, \qquad T = q\int \delta^4(x^\mu - x_p^\mu(\tau))[-g(x)]^{-1/2}\,d\tau$$

## Method

- Decompose field $\Phi$ into spheroidal harmonic and frequency modes

$$\Phi = \sum_{\hat{l}m} R_{\hat{l}m}(r) S_{\hat{l}m}(\theta;\sigma^2) e^{im\phi} e^{-i\omega t}$$

  where $\sigma^2 = -a^2\omega^2$

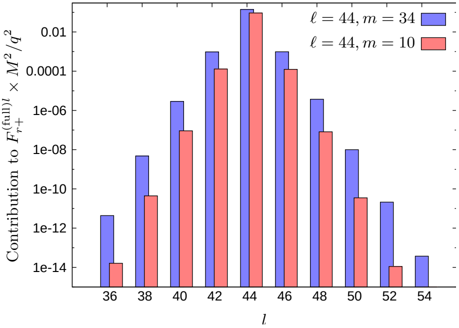
- Numerically solve for the radial equation $R_{\hat{l}m}(r)$ for each mode
- Mode-sum regularization: $F_\alpha^{\text{self}} = \sum_l \left[F_\alpha^{(\text{full})l}(x) - A_\alpha(l+1/2) - B_\alpha\right]$

# Spheroidal to spherical decomposition

## Spherical decomposition

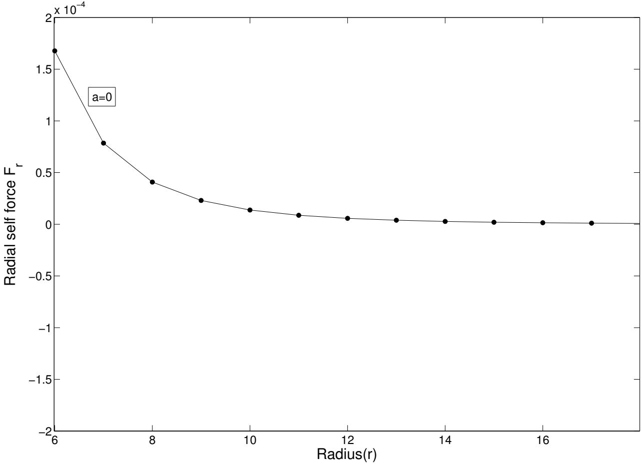
Mode-sum scheme requires spherical harmonic modes as input

$$S_{\hat{l}m}(\theta;\sigma^2)e^{im\phi} = \sum_{l=0}^{\infty} b^{\hat{l}}_{lm}(\sigma^2)Y_{lm}(\theta)$$

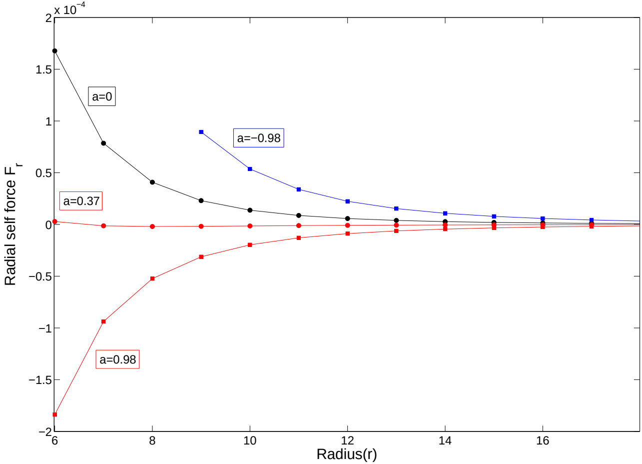



$$F^{(\mathrm{full})l}_{\alpha}(x) = q\nabla_\alpha \sum_{\hat{l}=0}^{\infty} \sum_{m=-\hat{l}}^{\hat{l}} b^{\hat{l}}_{lm} R_{\hat{l}m}(r) Y_{lm}(\theta,\phi) e^{-i\omega_m t}$$

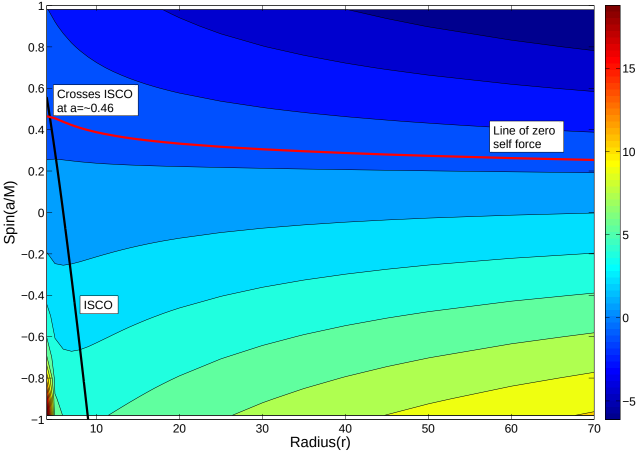
# Circular, equatorial orbits: Zero spin



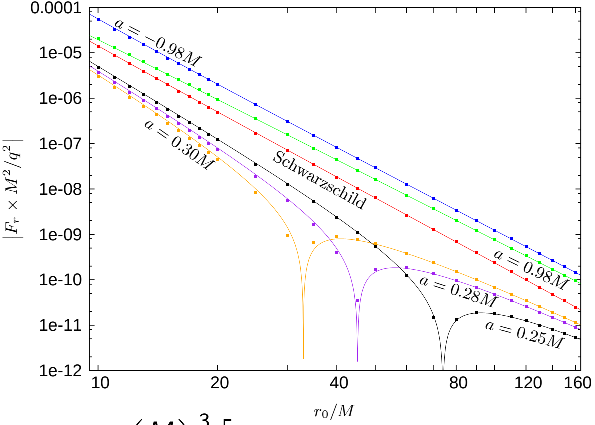
# Circular, equatorial orbits: effect of spin



# Circular, equatorial orbits: $r^5 F_r$

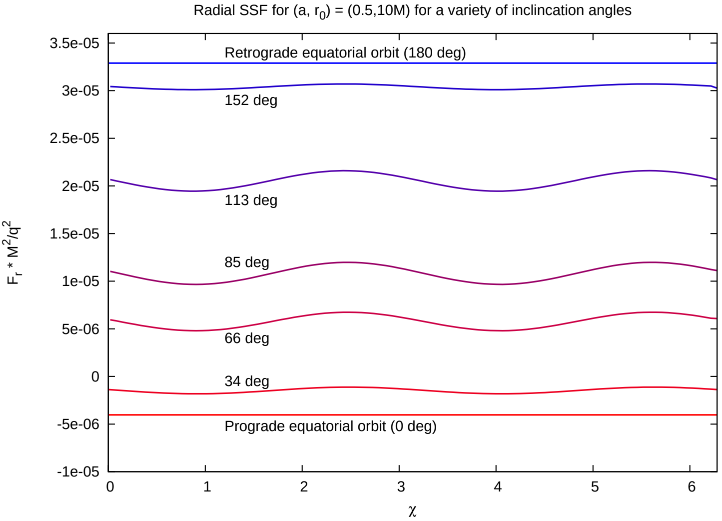


# Circular, equatorial orbits: PN Fit




$$F_r^{\text{Kerr}} = q^2/r_0^2 \left(\frac{M}{r_0}\right)^3 \left[\underbrace{\left(c_1 + c_2 \log\left(\frac{r_0}{M}\right)\right)}_{\text{Hikida } etal.} -1.009 a\mathcal{L}\right] + \dots$$

# Circular, inclined orbits



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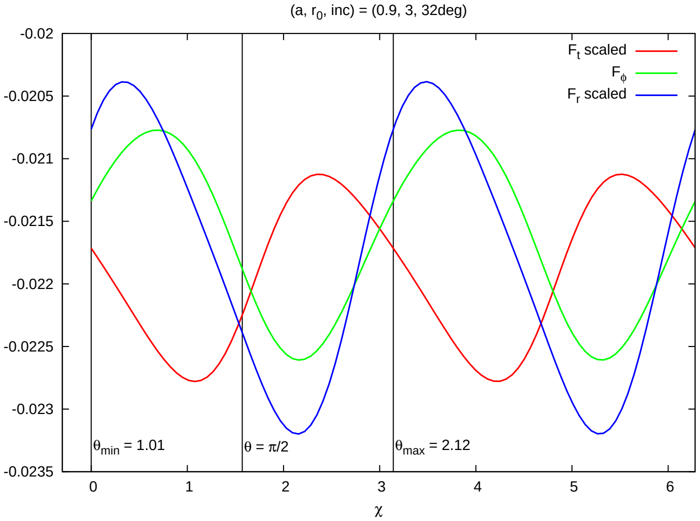
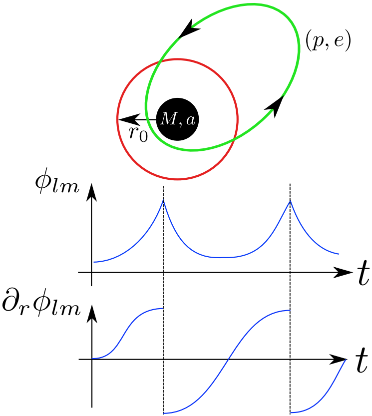


# Method of extended homogeneous solutions

## Eccentric orbits

Spectrum now bi-periodic

$$\omega_{mn} = m\Omega_\phi + n\Omega_r$$
$$\Phi_{lm}(t,r) = \sum_n \phi_{lmn}(r) e^{-i\omega_{mn}t}$$



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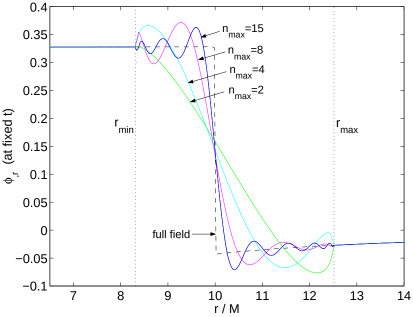


Credit: Barack, Ori and Sago

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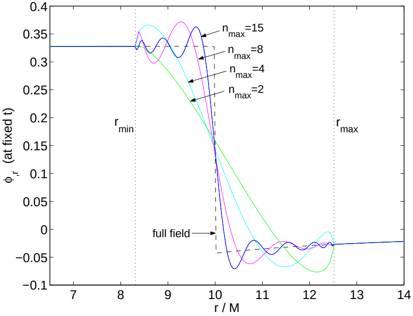
## Extended homogeneous solutions

- Avoids Gibbs phenomenon
- Exponential convergence with $n$

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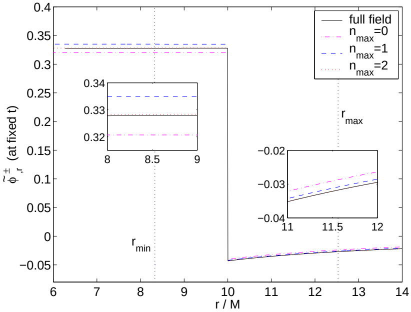


Credit: Barack, Ori and Sago

# Method of extended homogeneous solutions

1. With suitable BCs solve radial equation to find homogeneous solutions: $\psi^{\pm}_{\hat{l}mn}(r)$
2. (Naïve method) Use the standard method of variation of parameters to find the inhomogeneous solutions

$$\psi^{\text{inh}}_{\hat{l}mn}(r) = \psi^{+}_{\hat{l}mn}(r)\int_{r_{\text{min}}}^{r} \frac{\psi^{-}_{\hat{l}mn}(r')Z_{\hat{l}mn}(r')r'^2}{\Delta(r')W}\,dr' + \psi^{-}_{\hat{l}mn}(r)\int_{r}^{r_{\text{max}}} \frac{\psi^{+}_{\hat{l}mn}(r')Z_{\hat{l}mn}(r')r'^2}{\Delta(r')W}\,dr'$$

# Method of extended homogeneous solutions

1. With suitable BCs solve radial equation to find homogeneous solutions: $\psi^{\pm}_{\hat{l}mn}(r) \qquad r > 2M$
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2. (Method of EHS) Define

$$\tilde{\psi}^{\pm}_{\hat{l}mn}(r) = C^{\pm}_{\hat{l}mn}\psi^{\pm}_{\hat{l}mn}(r)\,, \qquad C^{\pm}_{\hat{l}mn} = \int_{r_{\text{min}}}^{r_{\text{max}}} \frac{\psi^{-}_{\hat{l}mn}(r')Z_{\hat{l}mn}(r')r'^2}{\Delta(r')W}\,dr'$$

$$\tilde{\phi}^{\pm}_{lm}(t,r) = \sum_{n=0}^{\infty}\sum_{\hat{l}=0}^{\infty} b^{\hat{l}}_{lm}\psi^{\pm}_{\hat{l}mn}(r)e^{-i\omega_{mn}t}$$

Then the correct TD field is given by

$$\phi_{lm}(t,r) = \begin{cases} \tilde{\phi}^{+}_{lm}(t,r) & r \geq r_p(t) \\ \tilde{\phi}^{-}_{lm}(t,r) & r \leq r_p(t) \end{cases}$$

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$$\psi^{\rm inh}_{\hat{l}mn}(r) = \psi^{+}_{\hat{l}mn}(r)\int_{r_{\rm min}}^{r}\frac{\psi^{-}_{\hat{l}mn}(r')Z_{\hat{l}mn}(r')r'^2}{\Delta(r')W}dr' + \psi^{-}_{\hat{l}mn}(r)\int_{r}^{r_{\rm max}}\frac{\psi^{+}_{\hat{l}mn}(r')Z_{\hat{l}mn}(r')r'^2}{\Delta(r')W}dr'$$

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$$\tilde{\phi}^{\pm}_{lm}(t,r) = \sum_{n=0}^{\infty}\sum_{\hat{l}=0}^{\infty} b^{\hat{l}}_{lm}\psi^{\pm}_{\hat{l}mn}(r)e^{-i\omega_{mn}t}$$

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$$\tilde{\psi}^{\pm}_{\hat{l}mn}(r) = C^{\pm}_{\hat{l}mn} \psi^{\pm}_{\hat{l}mn}(r) \,, \qquad C^{\pm}_{\hat{l}mn} = -\frac{8\pi q S_{\hat{l}mn}(\pi/2)}{T_r W} \int_0^{T_r/2} \frac{\psi^{\mp}_{\hat{l}m\omega}(r_p(t)) \cos(\omega_{mn} t - m\varphi_p(t))}{r_p(t) u^t(r_p(t))} \, dt$$

$$\tilde{\phi}^{\pm}_{lm}(t,r) = \sum_{n=0}^{\infty} \sum_{\hat{l}=0}^{\infty} b^{\hat{l}}_{lm} \psi^{\pm}_{\hat{l}mn}(r) e^{-i\omega_{mn} t}$$

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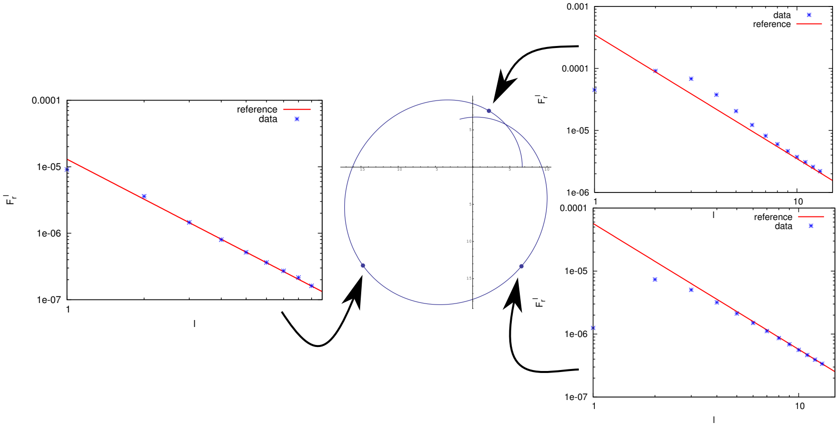
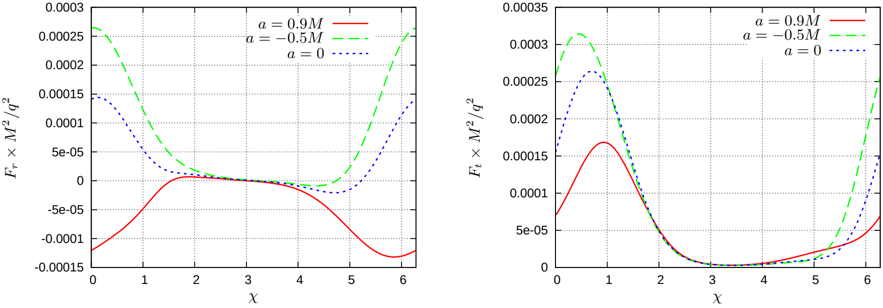
# Validation: regularization




$$(a, p, e) = (0.9M, 10M, 0.5)$$

# Sample results




$$(p, e) = (10M, 0.5)$$

# Schwarzschild ISCO shift

- Self-force corrections shift the location of the ISCO
- Formula for the radial shift given by

$$\Delta r_{\text{isco}} = 216 F_0^r - 72 F_1^r + 6\sqrt{2} F_t^1 + \frac{4}{\sqrt{3}} F_\phi^1$$

where e.g.

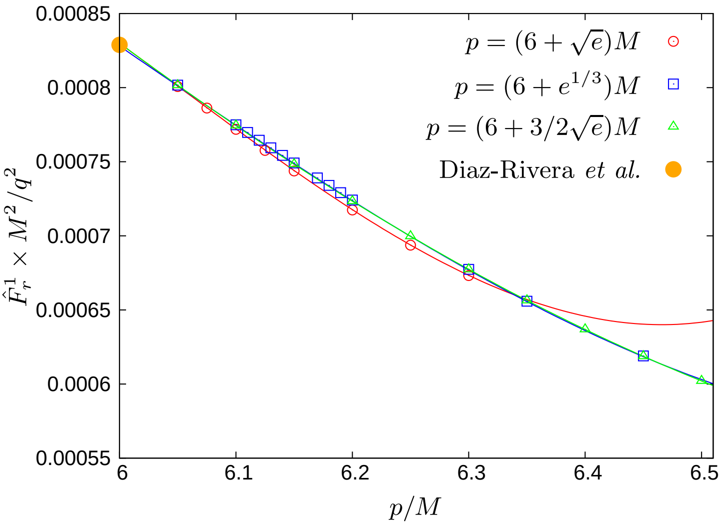
$$F^r = F_0^r + e\hat{F}_1^r \cos(\omega\tau)$$

and

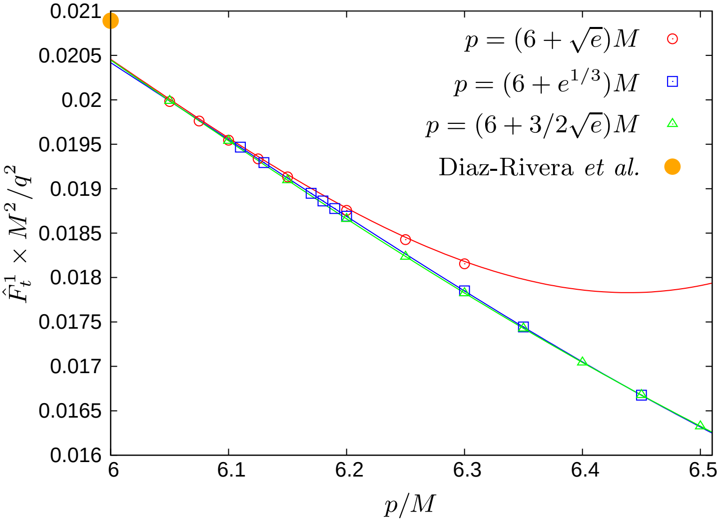
$$F_1^r = \lim_{p\to 6} \lim_{e\to 0} \hat{F}_1^r$$

- Calculate the (conservative) self-force and $\hat{F}_1^r$, $\hat{F}_\phi^1$ and $\hat{F}_t^1$ for slightly eccentric orbits and then extrapolate to the ISCO

# $F^r_1$


0.00085
0.0008
0.00075
0.0007
0.00065
0.0006
0.00055
6
6.1
6.2
6.3
6.4
6.5
$p/M$
$\hat{F}^1_r \times M^2/q^2$
$p = (6 + \sqrt{e})M$
$p = (6 + e^{1/3})M$
$p = (6 + 3/2\sqrt{e})M$
Diaz-Rivera *et al.*

# $F_t^1$


0.021
0.0205
0.02
0.0195
0.019
0.0185
0.018
0.0175
0.017
0.0165
0.016
$\hat{F}_t^1 \times M^2/q^2$
6
6.1
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6.3
6.4
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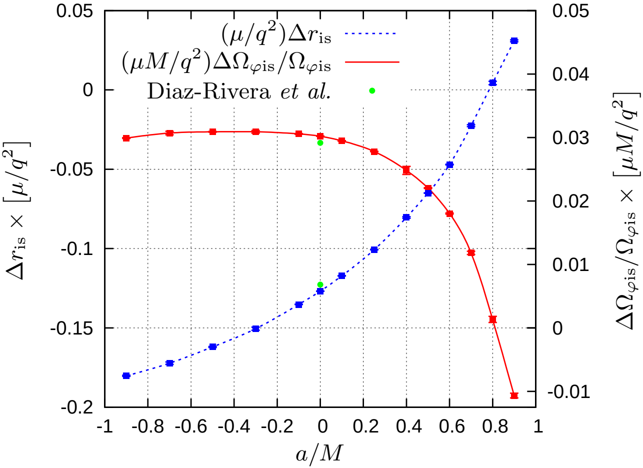
# Kerr ISCO shift

Similar procedure can be applied to Kerr



# Variation of rest mass

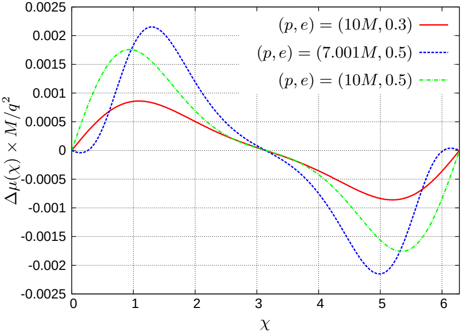
$$u^\beta \nabla_\beta (\mu u^\alpha) = q \nabla^\alpha \Phi^R = F^\alpha_{\text{self}} \qquad (1)$$

$$\mu \frac{du^\alpha}{d\tau} = (\delta^\alpha_\beta + u^\alpha u_\beta) F^\beta_{\text{self}}$$

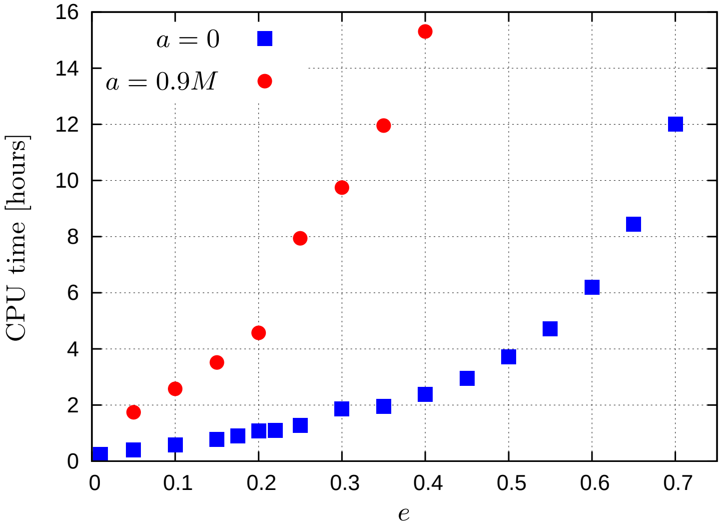
$$\frac{d\mu}{d\tau} = -u^\alpha F^{\text{self}}_\alpha \qquad (2)$$

Eq. (1) + Eq. (2) $\Longrightarrow$

$$\mu(\tau) = \mu_0 - q\Phi^R(\tau)$$



# Efficiency of the method



# GSF in Schwarzschild

## Overview

- Lorenz gauge calculation of the GSF for eccentric orbits about a Schwarzschild black hole
- Using the method of extended homogeneous solutions for coupled fields [See Akcay's talk]
- Fast computation of GSF for low eccentricity orbits
- Code works out to $p = 200M$

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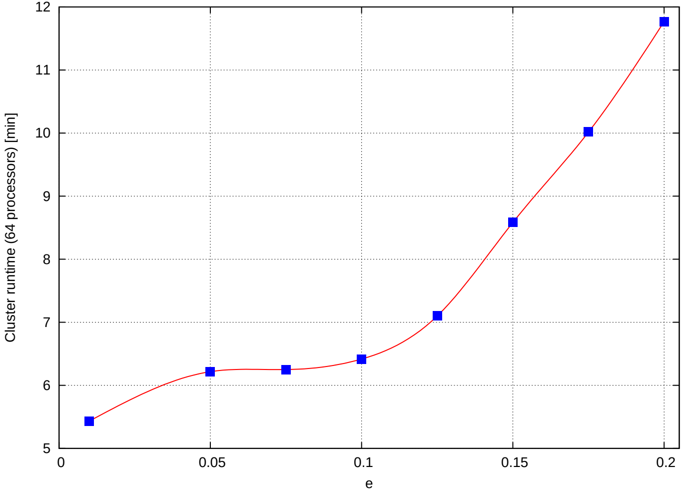
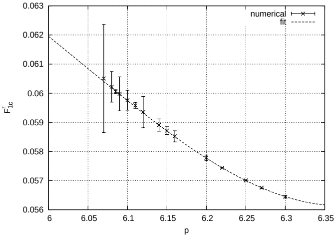
## Results

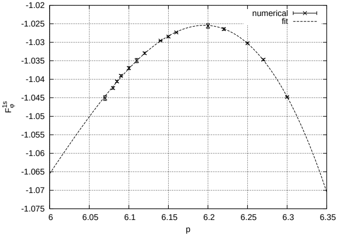
- Computing 15 l-modes at $(p, e) = (7M, 0.2)$ takes 1.5 hours on a dual core desktop machine
- Same code scales to a cluster (MPI implementation)
- e.g. Using 64 cores we can compute the same results as above in under 12 minutes
- Higher accuracy ISCO shift (work in progress)

# FD GSF efficiency



# GSF ISCO shift

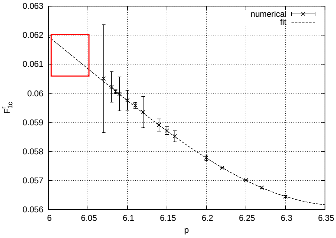


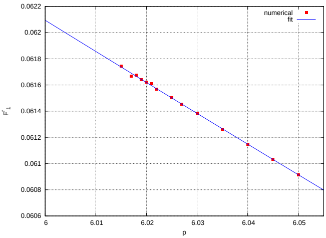



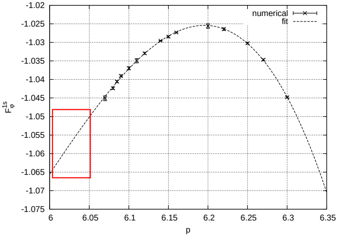

BS2010 (method I):

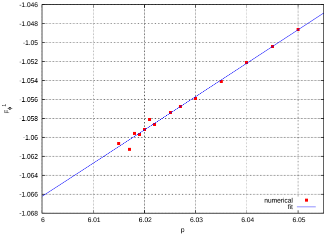
$$\frac{\Delta\Omega_{\text{isco}}}{\Omega_{\text{isco}}} = 0.484(2)\mu/M$$

# GSF ISCO shift




BS2010 (method I):

$$\frac{\Delta\Omega_{\rm isco}}{\Omega_{\rm isco}} = 0.484(2)\mu/M$$

This work (method I):

$$\frac{\Delta\Omega_{\rm isco}}{\Omega_{\rm isco}} = 0.4867(?)\mu/M$$

BS2010 (method II):

$$\frac{\Delta\Omega_{\rm isco}}{\Omega_{\rm isco}} = 0.4869(4)\mu/M$$

This work (method II):

$$\frac{\Delta\Omega_{\rm isco}}{\Omega_{\rm isco}} = ?$$

# Future Prospects

- SSF: generic orbits in Kerr

$$\omega = m\Omega_\phi + n\Omega_r + k\Omega_\theta$$

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- GSF: complete higher accuracy ISCO shift calculation
- GSF: eccentric orbit SF-PN comparison
- GSF: orbital evolution with osculating orbits

Scalar SF for eccentric orbits in Kerr: Phys. Rev. D. 83, 124038 (2011)
Scalar SF for circular orbits in Kerr: Phys. Rev. D. 81. 084039 (2010)