Calculation of the Effective Source

Barry Wardell
Albert Einstein Institute

Capra 14, Southampton

Effective Source Method

- Subtract 'puncture' field at the level of the wave equation.

$$\boxed{\Box\Phi = \Box\Phi_{\mathrm{S}} + \Box\Phi_{\mathrm{R}}}$$

- Solve for residual field with an effective source which is regular everywhere

$$\boxed{\Box\Phi_{\mathrm{R}} = S_{\mathrm{eff}} \qquad S_{\mathrm{eff}} \equiv \int_\gamma \delta(x, z(\tau'))d\tau' - \Box\Phi_{\mathrm{S}}}$$

SINGULAR FIELD

- Need

$$\boxed{S_{\text{eff}} \equiv \int_\gamma \delta(x, z(\tau'))d\tau' - \Box\Phi_{\text{S}}}$$

- Given the Detweiler-Whiting Green function,

$$\boxed{G_{\text{DW}}(x, x') = \frac{1}{2}\left\{U(x, x')\delta\left(\sigma(x, x')\right) + V(x, x')\theta\left(\sigma(x, x')\right)\right\}}$$

Φ_S is given by

$$\boxed{\Phi_{\text{S}}(x) = q\int_\gamma G_{\text{DW}}(x, x')d\tau'}$$

Singular Field

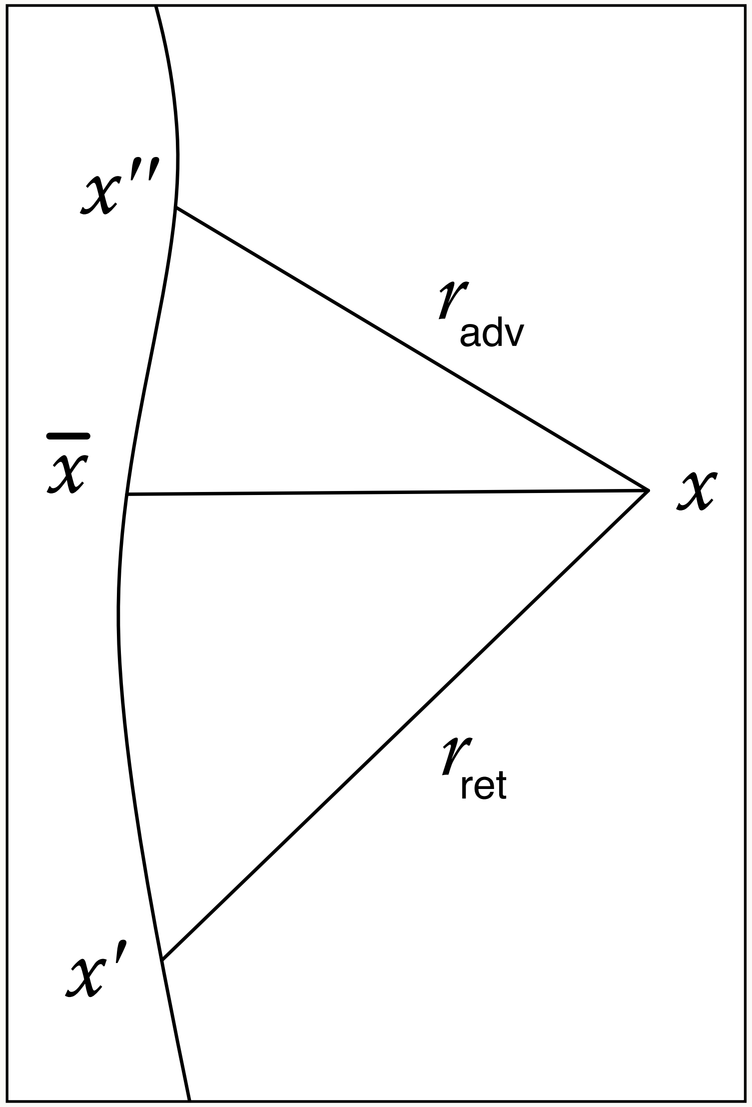
- So

$$\Phi_{\rm S}(x) = \frac{q}{2}\left[\frac{U(x,x')}{r_{\rm ret}} + \frac{U(x,x'')}{r_{\rm adv}} - \int_{x'}^{x''} V(x,z(\tau'))d\tau'\right]$$

- For x close to the world-line,

$$\begin{aligned} U(x,x') = U(x,x'') &= 1 + \mathcal{O}(\delta x^4) \\ V(x,z(\tau')) &= \mathcal{O}(\delta x^4) \end{aligned}$$

and we expand $r_{\rm ret}$ and $r_{\rm adv}$ in the geodesic distance between x and the world-line (Haas and Poisson)



Singular Field

$$\Phi_{\rm S} = q\Bigg\{\frac{1}{s} + \left[\frac{\bar r^2 - s^2}{6s^3}R_{u\sigma u\sigma} + \frac{1}{12s}\left(2\bar r R_{u\sigma} - R_{\sigma\sigma} + R_{uu}(\bar r^2 + s^2)\right) + \frac{1}{2}\left(\xi - \frac{1}{6}\right)\bar R s\right]$$
$$+ \left[\frac{1}{24s}\big(- R_{\sigma\sigma|\sigma} + (R_{\sigma\sigma|u} - 2R_{u\sigma|\sigma})\bar r + (2R_{u\sigma|u} - R_{uu|\sigma})(\bar r^2 + s^2) + 2R_{uu|u}(\bar r^3 + 3\bar r s^2)\big)\right.$$
$$\left. + \frac{1}{4}(\xi - \frac{1}{6})(\bar R_{|u}\bar r s - \bar R_\sigma s) + \frac{1}{24s^3}\Big(\left(\bar r^2 - 3s^2\right)\bar r R_{u\sigma u\sigma|u} - \left(\bar r^2 - s^2\right)R_{u\sigma u\sigma|\sigma}\Big)\right]\Bigg\}.$$

$$s^2 = (g^{\bar a\bar b} + u^{\bar a}u^{\bar b})\sigma_{\bar a}\sigma_{\bar b}$$
$$\bar r = \sigma_{\bar a}u^{\bar a}$$

and $\sigma(x, \bar x)$ is the Synge world-function.

Singular Field

- Expand all functions of *x* in coordinate series

$$s^2 = B_{\bar{a}\bar{b}}\Delta x^{\bar{a}}\Delta x^{\bar{b}} + B_{\bar{a}\bar{b}\bar{c}}\Delta x^{\bar{a}}\Delta x^{\bar{b}}x^{\bar{c}} + \cdots$$

$$\bar{r} = C_{\bar{a}}\Delta x^{\bar{a}} + C_{\bar{a}\bar{b}}\Delta x^{\bar{a}}\Delta x^{\bar{b}} + \cdots$$

$$\sigma^{\bar{\alpha}} = D^{\bar{\alpha}}{}_{\bar{a}}\Delta x^{\bar{a}} + D^{\bar{\alpha}}{}_{\bar{a}\bar{b}}\Delta x^{\bar{a}}\Delta x^{\bar{b}} + \cdots$$

$$\Phi^{\mathrm{S}} = \frac{A_{\bar{a}\bar{b}}\Delta x^{\bar{a}}\Delta x^{\bar{b}} + A_{\bar{a}\bar{b}\bar{c}}\Delta x^{\bar{a}}\Delta x^{\bar{b}}\Delta x^{\bar{c}}}{(B_{\bar{a}\bar{b}}\Delta x^{\bar{a}}\Delta x^{\bar{b}} + B_{\bar{a}\bar{b}\bar{c}}\Delta x^{\bar{a}}\Delta x^{\bar{b}}\Delta x^{\bar{c}})^{3/2}} + \mathcal{O}(\Delta x)$$

ISSUES - DIVERGENCES

- Singular field written as a series expansion about particle's location
- Will in general diverge far from the particle
- Window function/world tube can be employed to kill off singular field before divergences are reached
- Location of divergences depends on orbital configuration (unpredictable) => desirable to eliminate/control them

Issues - Divergences

- Eliminate problems (divergences and discontinuities) in azimuthal direction by re-summation

$$\boxed{\Delta\phi^n \to f_n(\Delta\phi) = \Delta\phi^n + \mathcal{O}\left(\Delta\phi^N\right)}$$

- Typically choose trigonometric functions

$$\Delta\phi^2 \to \frac{5}{2} - \frac{8}{3}\cos\Delta\phi + \frac{1}{6}\cos 2\Delta\phi = \Delta\phi^2 + O(\Delta\phi^6)$$
$$\Delta\phi^4 \to 6 - 8\cos\Delta\phi + 2\cos 2\Delta\phi = \Delta\phi^4 + O(\Delta\phi^6)$$

- Other choices are sometimes better, e.g., for avoiding numerical roundoff issues at small $\Delta\phi$

Issues - Divergences

- Eliminate singularities in effective source by re-expansion of the singular field such that the denominator is positive definite everywhere (except at the particle)

$$
\begin{aligned}
\Phi^{\mathrm{S}} &= \frac{A_{\bar{a}\bar{b}}\Delta x^{\bar{a}}\Delta x^{\bar{b}} + A_{\bar{a}\bar{b}\bar{c}}\Delta x^{\bar{a}}\Delta x^{\bar{b}}\Delta x^{\bar{c}}}{(B_{\bar{a}\bar{b}}\Delta x^{\bar{a}}\Delta x^{\bar{b}} + B_{\bar{a}\bar{b}\bar{c}}\Delta x^{\bar{a}}\Delta x^{\bar{b}}\Delta x^{\bar{c}})^{3/2}} + \mathcal{O}(\Delta x) \\
&= \frac{1}{((g_{\bar{a}\bar{b}} + u_{\bar{a}}u_{\bar{b}})\Delta x^{\bar{a}}\Delta x^{\bar{b}})^{1/2}} + \frac{C_{\bar{a}\bar{b}\bar{c}}\Delta x^{\bar{a}}\Delta x^{\bar{b}}\Delta x^{\bar{c}}}{((g_{\bar{a}\bar{b}} + u_{\bar{a}}u_{\bar{b}})\Delta x^{\bar{a}}\Delta x^{\bar{b}})^{3/2}} + \mathcal{O}(\Delta x)
\end{aligned}
$$

- For $\Delta t = 0$, new denominator is positive everywhere

Issues - Computational Efficiency

- Singular field and effective source are a complex function of the particle's position and the distance from it
- In *m*-mode scheme additionally have integration over azimuthal direction
- Computationally expensive source to a computationally cheap wave equation
- Highly desirable to make calculation as efficient as possible

Issues - Computational Efficiency

- Singular field - power series in Δx
- Coefficients only depend on particle position => change from one iteration to the next but not from point to point
- Optimisation (50x): precompute coefficients at the start of each iteration and then compute relatively cheap power series at each point

Issues - Computational Efficiency

- In 2+1D m-mode scheme, numerical integration over azimuthal angle is quite expensive and negates some of the benefits of the reduction from 3+1D
- Optimization: get rid of numerical integration by manipulating the singular field into a form where the integration can be done analytically in terms of elliptic functions.

Gravitational Case

- First order metric perturbation

$$\mathsf{g}_{ab} = g_{ab} + h_{ab}$$

- Trace-reversed form

$$\gamma^{ab} = h^{ab} - \frac{1}{2} g^{ab} h^{\alpha}{}_{\alpha}$$

- Lorenz gauge

$$\gamma^{ab}{}_{;b} = 0$$

Gravitational Case

- First order Lorenz-gauge perturbation equation

$$\boxed{\Box\gamma_{ab} + 2R_a{}^\alpha{}_b{}^\beta\gamma_{\alpha\beta} = -16\pi m \int_\gamma g_a{}^{a'} g_b{}^{b'} u_{a'} u_{b'} \delta_4(x, z(\tau'))d\tau'}$$

- Split into singular and regular parts

$$\boxed{\gamma_{ab} = \gamma^{\mathrm{S}}_{ab} + \gamma^{\mathrm{R}}_{ab}}$$

- Solve regularized equation with an effective source

$$\boxed{\Box\gamma^{\mathrm{R}}_{ab} + 2R_a{}^\alpha{}_b{}^\beta\gamma^{\mathrm{R}}_{\alpha\beta} = S^{\mathrm{eff}}_{ab}}$$

Gravitational Singular Field

- Analogous to scalar case, the gravitational DW Green function is

$$\boxed{G^{\mathrm{S}}_{aba'b'}(x,x') = \frac{1}{2}\Big[U(x,x')_{aa'bb'}\delta\left(\sigma(x,x')\right) + V(x,x')_{aa'bb'}\theta\left(\sigma(x,x')\right)\Big]}$$

- Singular field given by integral of this Green function

$$\boxed{\gamma^{\mathrm{S}}_{ab} = \int_\gamma G^{\mathrm{S}}_{aa'bb'}(x, z(\tau'))u^{a'}u^{b'}d\tau'}$$

Gravitational Singular Field

- Exact expression for the singular field

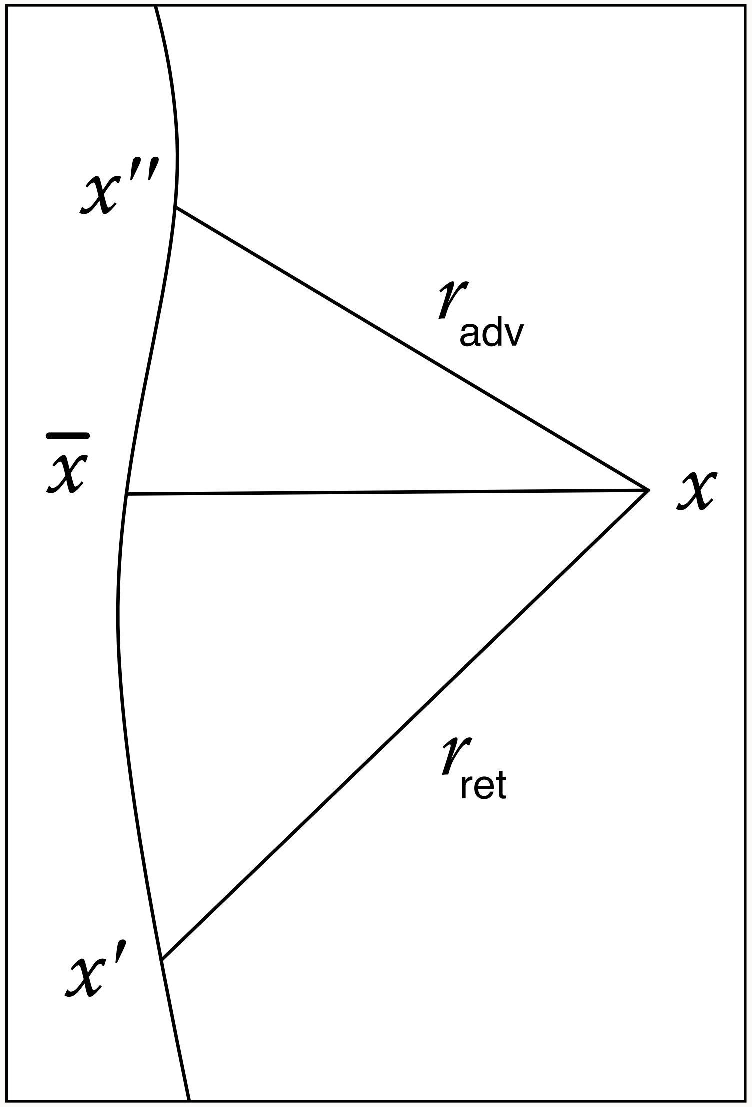
$$
\begin{aligned}
\gamma^{\mathrm{S}}_{ab} =& \frac{\Delta^{1/2}(x,x')}{2\sigma_{\alpha'}u^{\alpha'}} g_{(a}{}^{a'} g_{b)}{}^{b'} u_{a'}u_{b'} \\
&+ \frac{\Delta^{1/2}(x,x'')}{2\sigma_{\alpha''}u^{\alpha''}} g_{(a}{}^{a''} g_{b)}{}^{b''} u_{a''}u_{b''} \\
&+ \frac{1}{2}\int_u^v V_{aa'bb'}(x,z(\tau'))u^{a'}u^{b'}d\tau'
\end{aligned}
$$

- Not feasible to use in this form for practical calculations - proceed by expansion

Gravitational Singular Field

$$\gamma^{\mathrm{S}}_{ab} = \frac{\Delta^{1/2}(x,x')}{2\sigma_{\alpha'}u^{\alpha'}} g_{(a}{}^{a'} g_{b)}{}^{b'} u_{a'} u_{b'} + \frac{\Delta^{1/2}(x,x'')}{2\sigma_{\alpha''}u^{\alpha''}} g_{(a}{}^{a''} g_{b)}{}^{b''} u_{a''} u_{b''} + \frac{1}{2}\int_u^v V_{aa'bb'}(x,z(\tau'))u^{a'}u^{b'}\,d\tau'$$

$$f(x') = f(\bar{x}) + f'(\bar{x})(x' - \bar{x}) + \cdots$$



Gravitational Singular Field

$$
\begin{aligned}
\gamma^S_{ab} = g_{(a}{}^{\bar{a}} g_{b)}{}^{\bar{b}} \Big[& \frac{u_{\bar{a}} u_{\bar{b}}}{s} \\
& - \frac{1}{6s^3} \Big(6s^4 R_{\bar{\alpha}\bar{a}\bar{\beta}\bar{b}} u^{\bar{\alpha}} u^{\bar{\beta}} + 6\bar{r} s^2 R_{\bar{\alpha}\bar{a}\bar{\beta}\bar{\gamma}} u^{\bar{\alpha}} u^{\bar{\beta}} \sigma^{\bar{\gamma}} u_{\bar{b}} \\
& \quad + (s^2 - \bar{r}^2) R_{\bar{\alpha}\bar{\beta}\bar{\gamma}\bar{\delta}} u^{\bar{\alpha}} u^{\bar{\gamma}} \sigma^{\bar{\beta}} \sigma^{\bar{\delta}} u_{\bar{a}} u_{\bar{b}} \Big) \\
& + \frac{1}{24 s^3} \Big(12 s^4 (R_{\bar{\alpha}\bar{a}\bar{\beta}\bar{b};\bar{\gamma}} u^{\bar{\alpha}} u^{\bar{\beta}} \sigma^{\bar{\gamma}} - \bar{r} R_{\bar{\alpha}\bar{a}\bar{\beta}\bar{b};\bar{\gamma}} u^{\bar{\alpha}} u^{\bar{\beta}} u^{\bar{\gamma}}) \\
& \quad - 8 s^2 \big((\bar{r}^2 + s^2) R_{\bar{\alpha}\bar{a}\bar{\beta}\bar{\gamma};\bar{\delta}} u^{\bar{\alpha}} u^{\bar{\beta}} \sigma^{\bar{\gamma}} u^{\bar{\delta}} - \bar{r} R_{\bar{\alpha}\bar{a}\bar{\beta}\bar{\gamma};\bar{\delta}} u^{\bar{\alpha}} u^{\bar{\beta}} \sigma^{\bar{\gamma}} \sigma^{\bar{\delta}} \big) \\
& \quad + \bar{r} (\bar{r}^2 - 3s^2) R_{\bar{\alpha}\bar{\beta}\bar{\gamma}\bar{\delta};\bar{\epsilon}} u^{\bar{\alpha}} u^{\bar{\gamma}} u^{\bar{\epsilon}} \sigma^{\bar{\beta}} \sigma^{\bar{\delta}} u_{\bar{a}} u_{\bar{b}} \\
& \quad + (s^2 - \bar{r}^2) R_{\bar{\alpha}\bar{\beta}\bar{\gamma}\bar{\delta};\bar{\epsilon}} u^{\bar{\alpha}} u^{\bar{\gamma}} \sigma^{\bar{\epsilon}} \sigma^{\bar{\beta}} \sigma^{\bar{\delta}} u_{\bar{a}} u_{\bar{b}} \Big) \Big]
\end{aligned}
$$

Gravitational Singular Field

- Expand all functions of *x* in coordinate series

$$g^{\bar{b}}{}_{a} = g^{\bar{b}}{}_{\bar{a}} + A^{\bar{b}}{}_{\bar{a}\bar{c}}\Delta x^{\bar{c}} + \cdots$$

$$s^2 = B_{\bar{a}\bar{b}}\Delta x^{\bar{a}}\Delta x^{\bar{b}} + B_{\bar{a}\bar{b}\bar{c}}\Delta x^{\bar{a}}\Delta x^{\bar{b}}x^{\bar{c}} + \cdots$$

$$\bar{r} = C_{\bar{a}}\Delta x^{\bar{a}} + C_{\bar{a}\bar{b}}\Delta x^{\bar{a}}\Delta x^{\bar{b}} + \cdots$$

$$\sigma^{\bar{\alpha}} = D^{\bar{\alpha}}{}_{\bar{a}}\Delta x^{\bar{a}} + D^{\bar{\alpha}}{}_{\bar{a}\bar{b}}\Delta x^{\bar{a}}\Delta x^{\bar{b}} + \cdots$$

Gravitational Singular Field

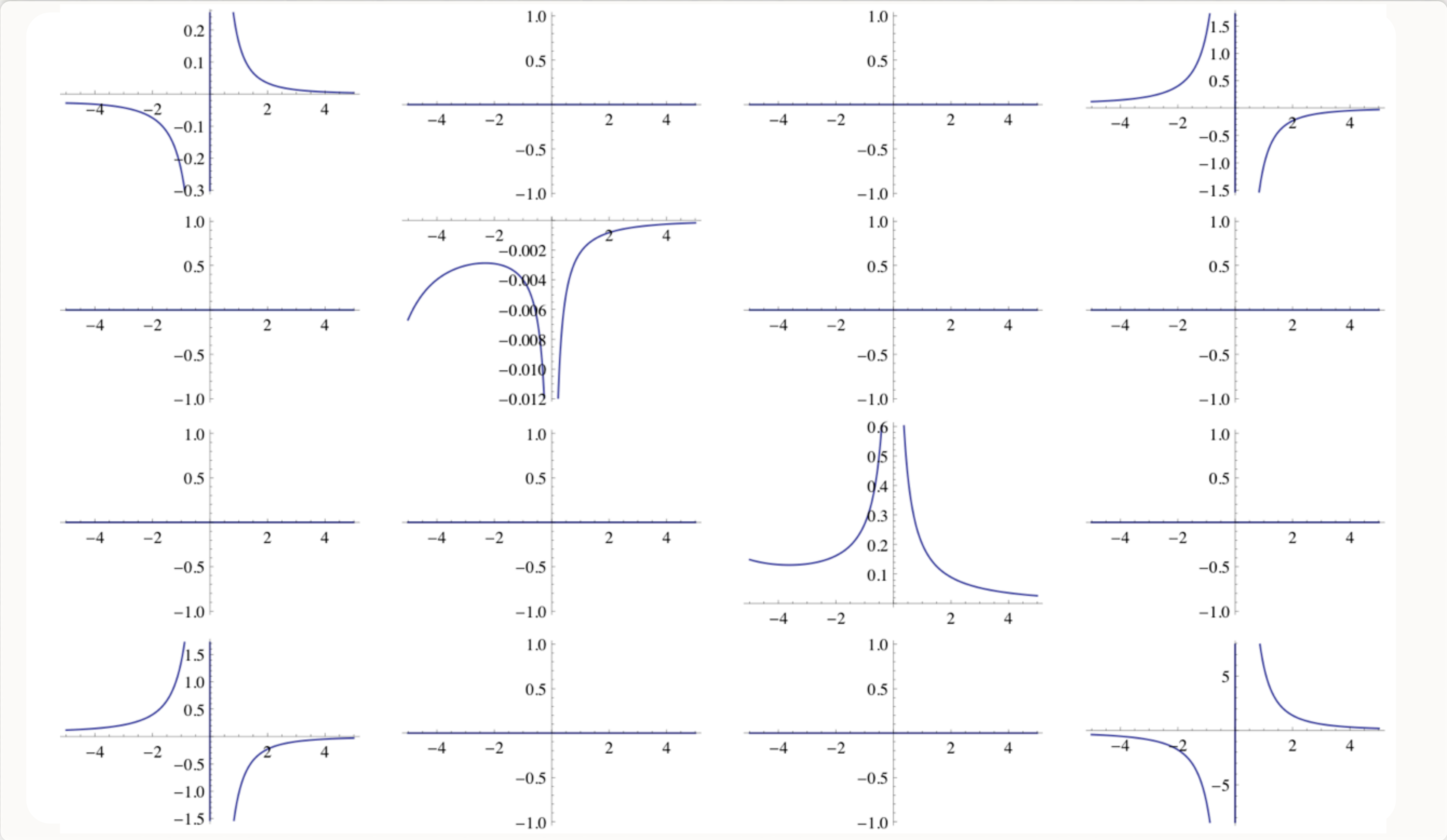
- Re-expand, make periodic, etc.
- Final result - singular field written as expansion in powers of coordinate distance from the world-line

$$\gamma^{\mathrm{S}}_{ab} = \frac{X_{\bar{a}\bar{b}}}{(B_{\bar{\alpha}\bar{\beta}}\Delta x^{\bar{\alpha}}\Delta x^{\bar{\beta}})^{1/2}} + \frac{Y_{\bar{a}\bar{b}\bar{c}\bar{d}\bar{e}}\Delta x^{\bar{c}}\Delta x^{\bar{d}}\Delta x^{\bar{e}}}{(B_{\bar{\alpha}\bar{\beta}}\Delta x^{\bar{\alpha}}\Delta x^{\bar{\beta}})^{3/2}} + \cdots$$

- Apply wave operator to singular field to get effective source

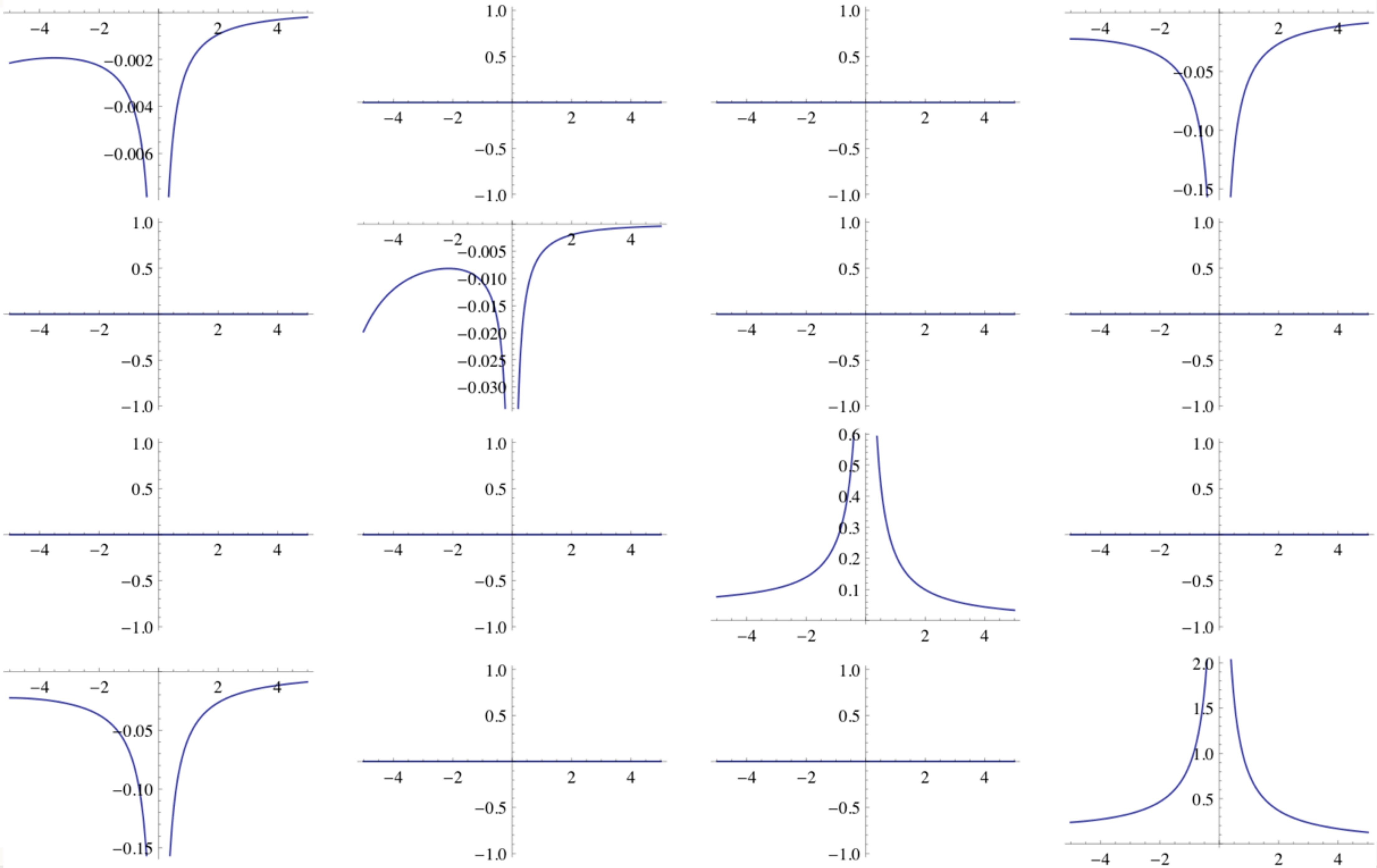
$$S^{\mathrm{eff}}_{ab} = -(\Box\gamma^{\mathrm{S}}_{ab} + 2R_a{}^\alpha{}_b{}^\beta\gamma^{\mathrm{S}}_{\alpha\beta}) - 16\pi m\int_\gamma g_a{}^{a'} g_b{}^{b'} u_{a'}u_{b'}\delta_4(x, z(\tau'))d\tau'$$

Gravitational Effective Source - First Order



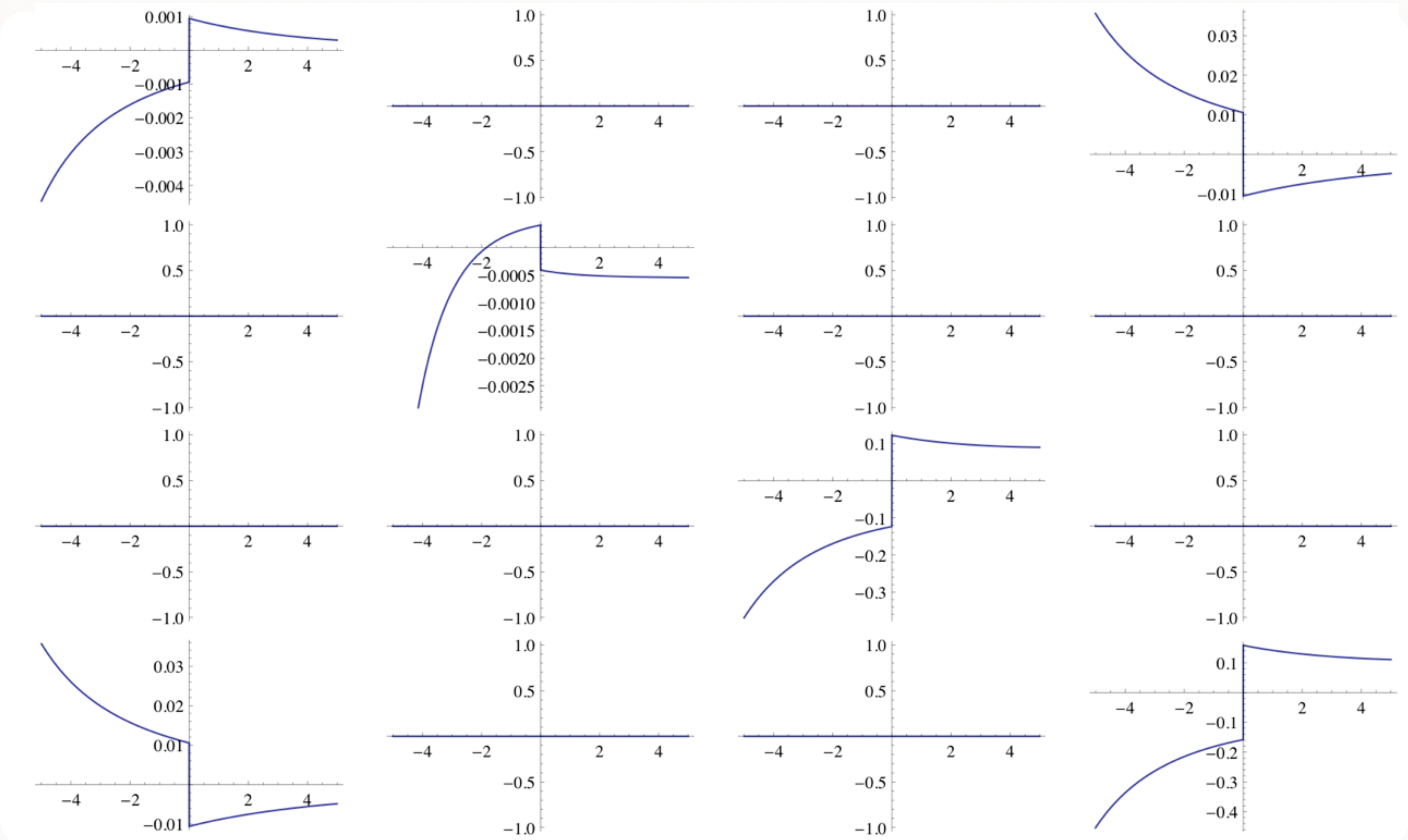
S^{eff}_{ab} for a circular equatorial orbit in Kerr - $r_0 = 10$, a=0.5, M=1

Gravitational Effective Source - Second Order



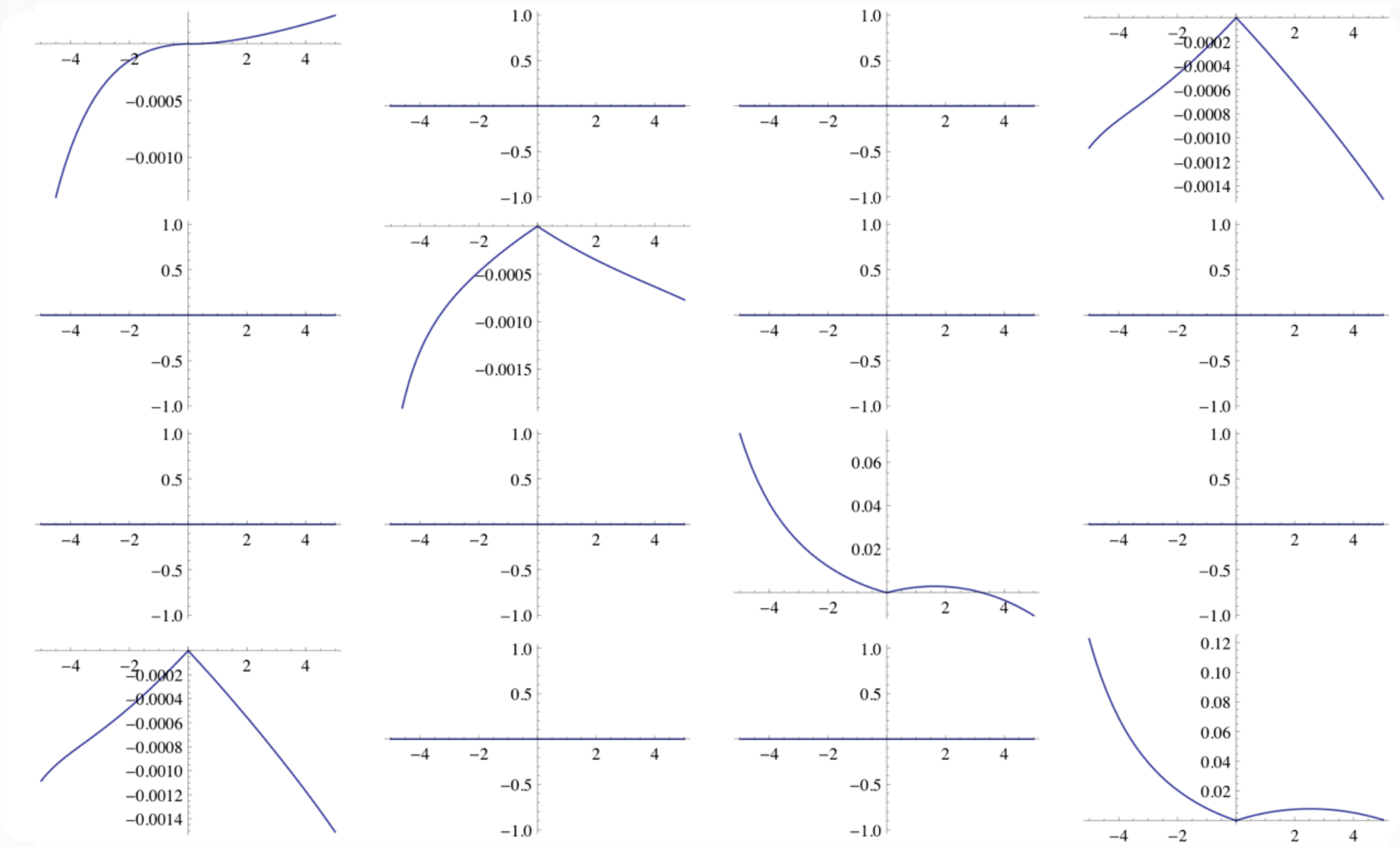
S_{ab}^{eff} for a circular equatorial orbit in Kerr - $r_0 = 10$, a=0.5, M=1

Gravitational Effective Source - Third Order



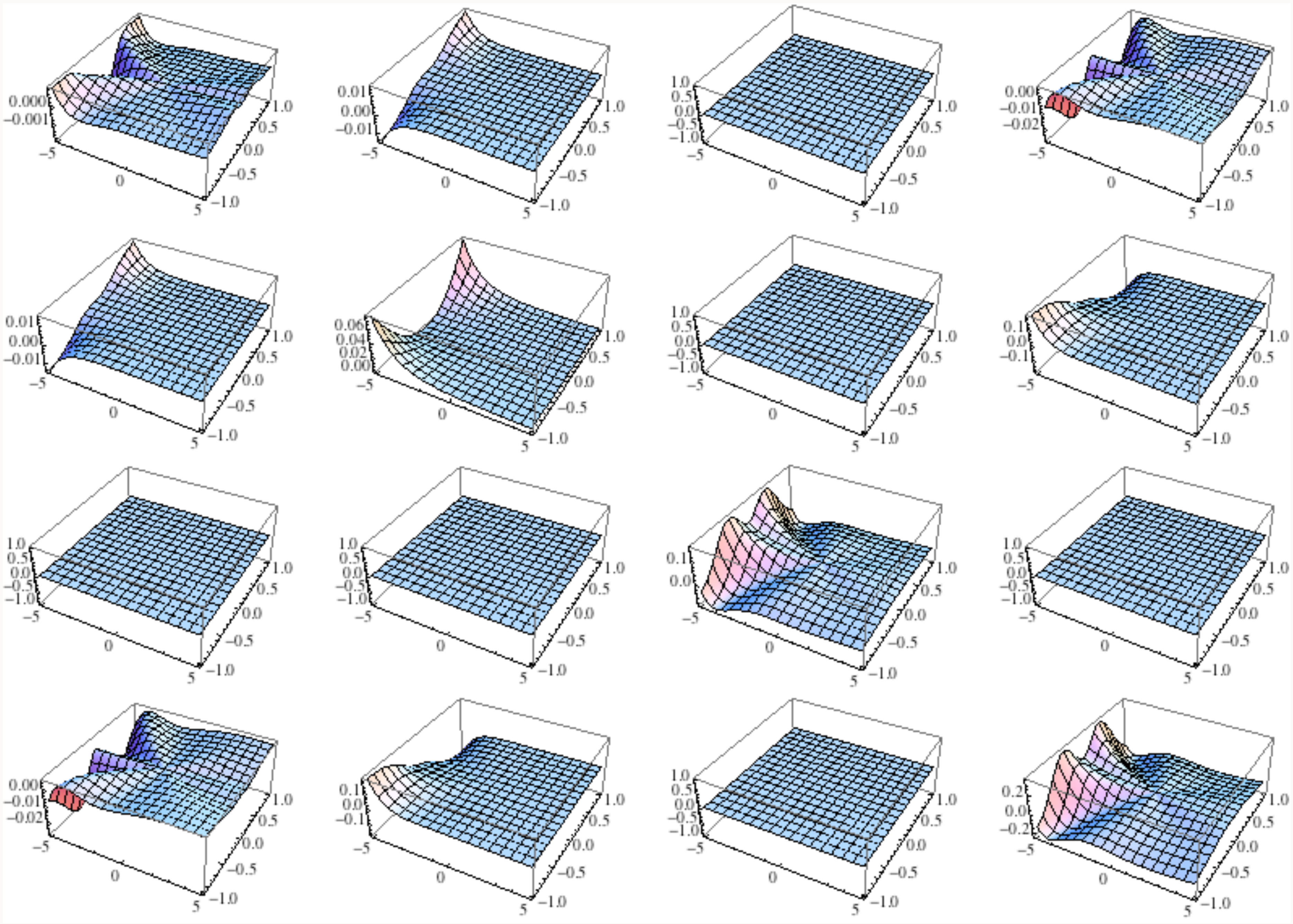
S^{eff}_{ab} for a circular equatorial orbit in Kerr - $r_0 = 10$, a=0.5, M=1

Gravitational Effective Source - Fourth Order



S_{ab}^{eff} for a circular equatorial orbit in Kerr - $r_0 = 10$, $a=0.5$, $M=1$

Gravitational Effective Source - Fourth Order



S_{ab}^{eff} for a circular equatorial orbit in Kerr - $r_0 = 10$, a=0.5, M=1

Conclusions

- Scalar and gravitational effective sources through fourth order are now at hand for Schwarzschild and Kerr
- Most technical issues related to series divergences solved by suitable re-expansion/re-summation
- Window function/worldtube resolves remaining technical problems.