Self-force: foundations and formalism

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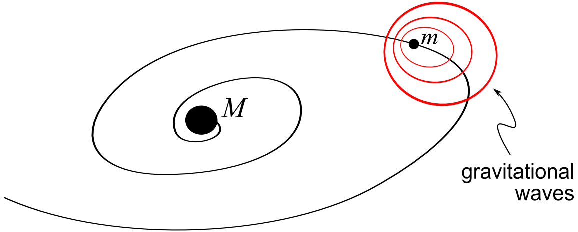
Motivation

Extreme-mass-ratio inspirals

- solar-mass neutron star or black hole orbits supermassive black hole
- m emits gravitational radiation, loses energy, spirals into M
- waveforms carry information about strong-field dynamics and structure of spacetime near black hole
- need to model motion of small body



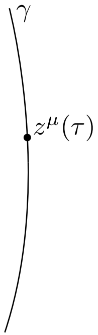
Point particle picture

Linearized theory

- treat m as point particle in background $g_{\mu\nu}$
  $\Rightarrow T^{\mu\nu}_{(1)} = \int_\gamma m u^\mu u^\nu \frac{\delta^4(x^\rho - z^\rho(\tau))}{\sqrt{-g}} d\tau$
- linearized EFE $\delta G^{\mu\nu}[h^{(1)}_{\rho\sigma}] = 8\pi T^{\mu\nu}_{(1)}$
  $\Rightarrow h^{(1)}_{\mu\nu} = m \int_\gamma G_{\mu\nu\mu'\nu'} u^{\mu'} u^{\nu'} d\tau$

Tails

- perturbation propagates within light cone
- also, caustics develop—light "cone" intersects itself
  $\Rightarrow h^{(1)}_{\mu\nu}$ depends on entire past history of γ



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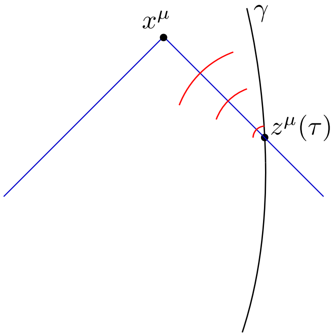

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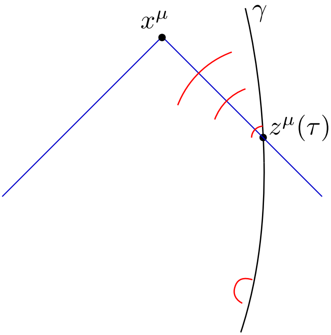

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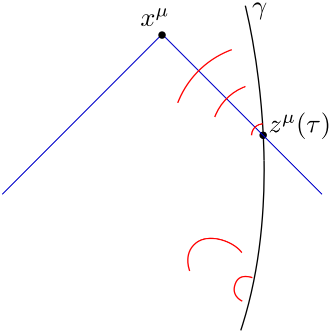

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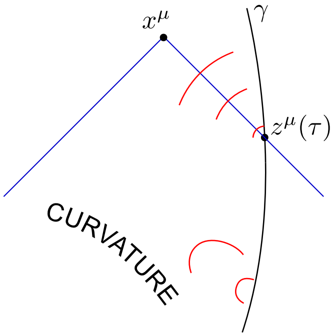

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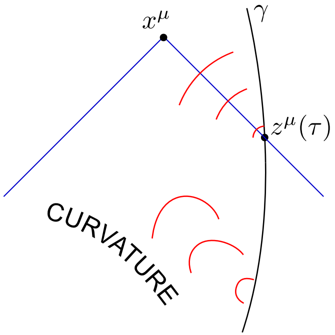

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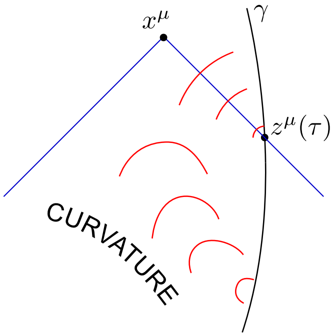

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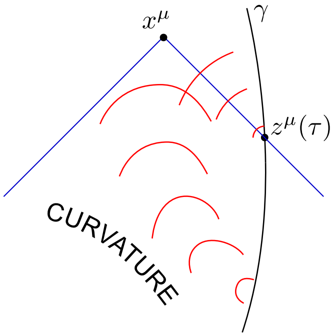
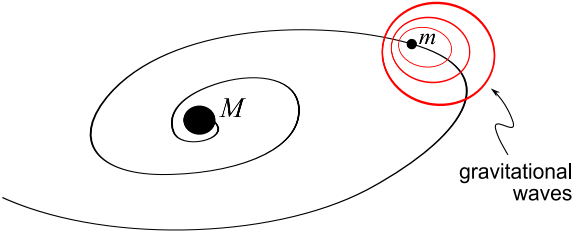


Extreme-mass-ratio inspirals



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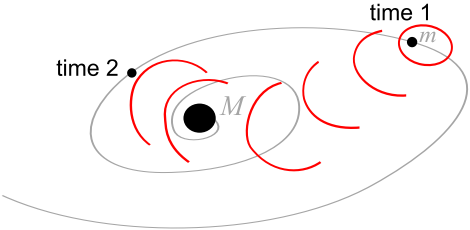


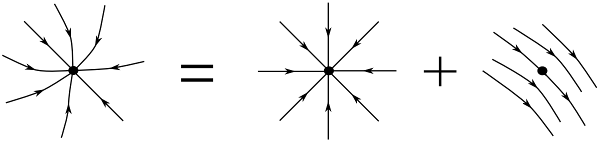
Geodesic motion in an effective metric

MiSaTaQuWa (Mino,Sasaki,Tanaka, & Quinn,Wald) equation

- nonlocal tail acts as potential, exerts force $F^\mu \sim m\nabla^\mu \text{tail}$
- tail isn't nice: non-differentiable, not a solution to a field equation

Detweiler-Whiting decomposition

- local field near particle split into two: $h^{(1)}_{\mu\nu} = h^{\mathrm{S}(1)}_{\mu\nu} + h^{\mathrm{R}(1)}_{\mu\nu}$
- $h^{\mathrm{S}(1)}_{\mu\nu} \sim \frac{m}{r} + O(r^0)$; local bound field of particle
- $h^{\mathrm{R}(1)}_{\mu\nu} \sim \text{tail} + \text{local terms}$; smooth solution to source-free EFE
- motion is geodesic in effective metric $g_{\mu\nu} + h^{\mathrm{R}(1)}_{\mu\nu}$



particle's field $h^{(1)}_{\mu\nu}$ = singular field $h^{\mathrm{S}(1)}_{\mu\nu}$ + regular field $h^{\mathrm{R}(1)}_{\mu\nu}$

Outline

Outline

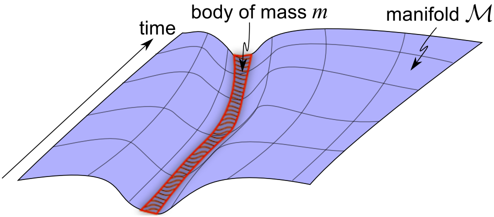
A small extended body moving through spacetime

Fundamental question

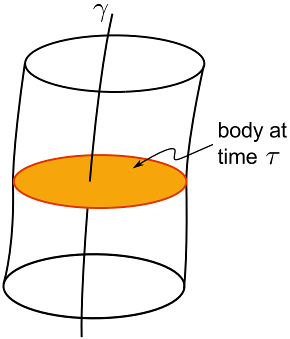
- how does a body's gravitational field affect its own motion?

Regime: small body

- examine spacetime $(\mathcal{M}, g_{\mu\nu})$ containing body of mass m and external lengthscales $\mathcal{R}$
- seek representation of body's motion when its mass and size are $\ll \mathcal{R}$



Non-perturbative approach [Harte '11]



Momentum

- assume the body is material, not a black hole
- give body stress-energy $T^{\mu\nu}$
- define momentum $P \sim \int_{body} T^{\mu\nu}$

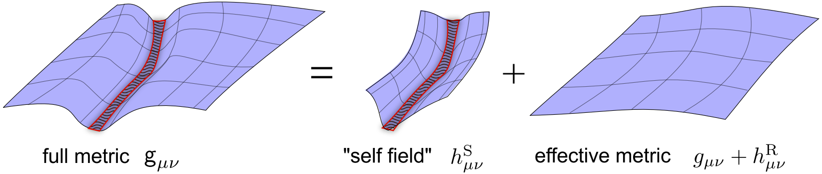
Motion

- choose representative worldline γ with coordinates $z^\mu(\tau)$ inside body
- relate $u^\mu = \frac{dz^\mu}{d\tau}$ to P
  $\Rightarrow \frac{DP}{d\tau}$ determines acceleration of γ

Motion of a test body in an effective metric

Non-perturbative decomposition

- split metric into "self-field" generated by body and slowly varying remainder



Equation of motion

- define multipole moments $I \sim \int_{body} T^{\mu\nu}$
- body moves as test body in effective metric $g_{\mu\nu} + h^{\mathrm{R}}_{\mu\nu}$:
  motion is geodesic except for coupling of multipole moments to curvature of effective metric

However...

Material body

- integrals over body's interior preclude description of black hole

Field

- describing motion in terms of metric isn't sufficient: we need a means of solving the EFE to obtain the metric (and isolating the piece of it that determines the motion)

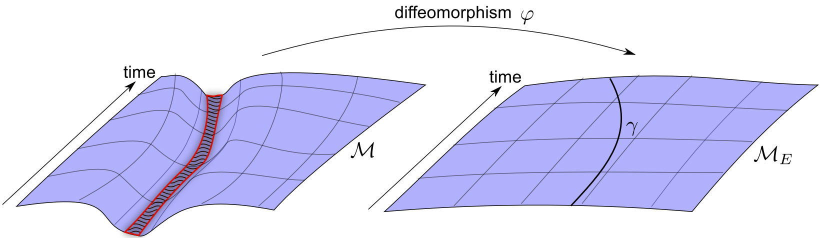
Outline

Perturbation theory

- treat body as source of perturbation of external background spacetime $(\mathcal{M}_E, g_{\mu\nu})$:

$$\mathsf{g}_{\mu\nu} = g_{\mu\nu} + \epsilon h^{(1)}_{\mu\nu} + \epsilon^2 h^{(2)}_{\mu\nu} + \ldots$$

- ϵ counts powers of m
- assume body is compact, so as $m \to 0$, linear size $\to 0$ at same rate
- seek representation of motion in $(\mathcal{M}_E, g_{\mu\nu})$



Body in exact spacetime

Approximation of motion in external spacetime

Approach I [Gralla & Wald '08]: power series

Expansion of EFE

- expand metric in Taylor series:

$$\mathrm{g}_{\mu\nu}(x,\epsilon) = g_{\mu\nu}(x) + \epsilon h^{(1)}_{\mu\nu}(x) + \epsilon^2 h^{(2)}_{\mu\nu}(x) + \dots$$

- solve EFE order by order *outside body*:

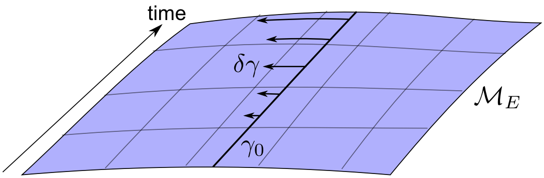
$$\begin{aligned} \delta G_{\mu\nu}[h^{(1)}] &= 0 \\ \delta G_{\mu\nu}[h^{(2)}] &= -\delta^2 G_{\mu\nu}[h^{(1)}] \\ &\vdots \end{aligned}$$

- motion determined by Bianchi identity

Representation of motion in power series

Expanded worldline

- worldline γ_0 identified as remnant of body left at $\epsilon = 0$
- γ_0 is geodesic
- corrections accounted for by deviation vector $\delta\gamma$



Problem

- as body drifts away from γ_0, $\delta\gamma$ grows large
- representation of motion only meaningful and accurate for short time

Approach II [Pound '10]: self-consistent expansion

Expansion of EFE

- allow γ to depend on ϵ and assume expansion of form

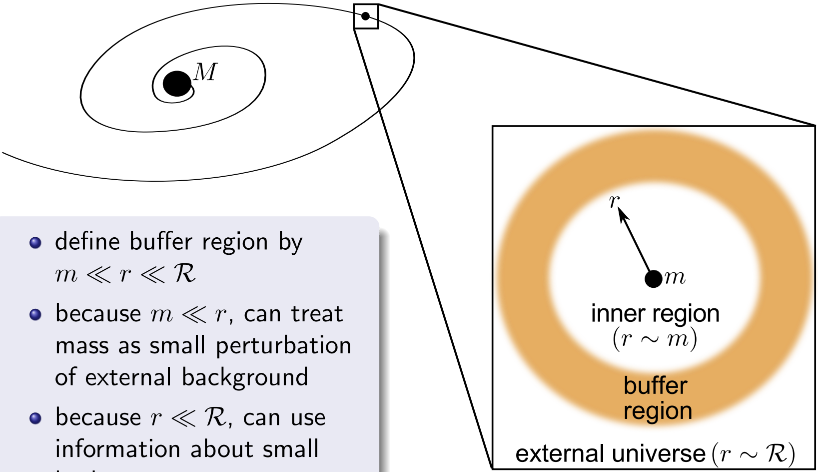
$$\begin{aligned} \mathsf{g}_{\mu\nu}(x,\epsilon) &= g_{\mu\nu}(x) + h_{\mu\nu}(x;\gamma_\epsilon) \\ &= g_{\mu\nu}(x) + \epsilon h^{(1)}_{\mu\nu}(x;\gamma_\epsilon) + \epsilon^2 h^{(2)}_{\mu\nu}(x;\gamma_\epsilon) + \ldots \end{aligned}$$

- need a method of systematically solving for each $h^{(n)}_{\mu\nu}$
  $\Rightarrow$ impose Lorenz gauge (or other wave gauge) on the total perturbation: $\nabla_\mu \bar{h}^{\mu\nu} = 0$
- $\delta G_{\mu\nu}$ becomes a wave operator and EFE outside body becomes weakly nonlinear wave equation:

$$\Box \bar{h}_{\mu\nu} + 2R_\mu{}^\rho{}_\nu{}^\sigma \bar{h}_{\rho\sigma} = 2\delta^2 G_{\mu\nu}[h] + \ldots$$

- can be split into wave equations for each subsequent $h^{(n)}_{\mu\nu}[\gamma]$ and exactly solved for arbitrary γ
- gauge condition will then constrain γ

How to determine motion? Buffer region

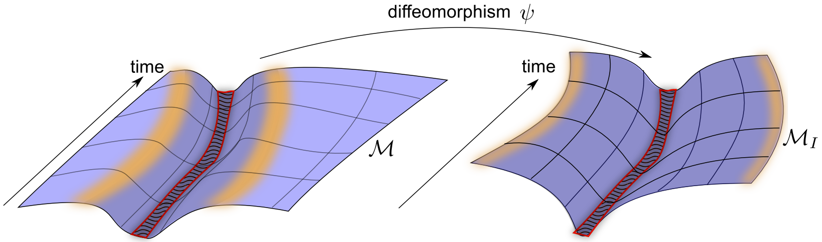


- define buffer region by $m \ll r \ll \mathcal{R}$
- because $m \ll r$, can treat mass as small perturbation of external background
- because $r \ll \mathcal{R}$, can use information about small body

Matched asymptotic expansions: *inner expansion*

Zoom in on body

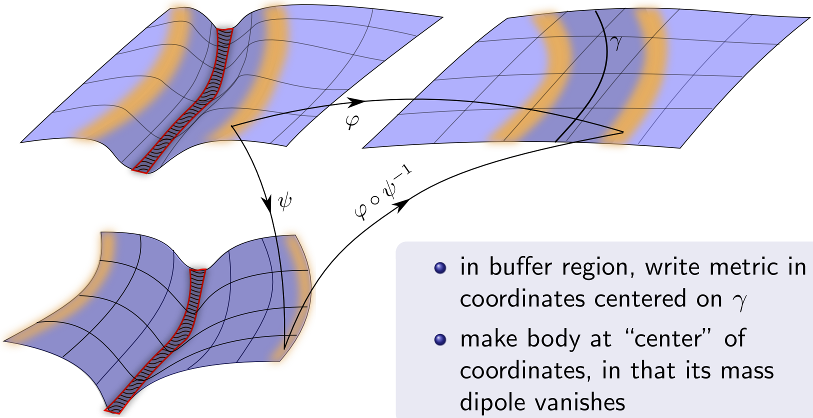
- use scaled coords $\tilde{r} \sim r/\epsilon$ to keep size of body fixed, send other distances to infinity as $\epsilon \to 0$
- unperturbed body defines background spacetime $g_{I\mu\nu}$ in inner expansion
- buffer region at asymptotic infinity $r \gg m$
  $\Rightarrow$ can define multipole moments without integrals over body



Representation of motion in self-consistent approximation

Enforce a relationship between the expansions

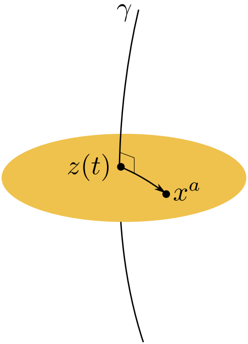
...to define a worldline for all time, even for black hole



- in buffer region, write metric in coordinates centered on γ
- make body at "center" of coordinates, in that its mass dipole vanishes

Outline

Local coordinates



Fermi-Walker coordinates

- spatial coordinates x^a span surface intersecting $z^\mu(\tau)$ orthogonally
- time t on that surface $=$ proper time τ
- radial distance $r^2 = \delta_{ab}x^a x^b$ is geodesic distance from γ

Solving the EFE in buffer region

Expansion for small r

- allow all negative powers of r in $h^{(n)}_{\mu\nu}$
- but inner expansion must not have negative powers of ϵ
  $\Rightarrow$ most singular power of r in $\epsilon^n h^{(n)}_{\mu\nu}$ is $\frac{\epsilon^n}{r^n} = \frac{\epsilon^n}{\epsilon^n \tilde{r}^n} = \frac{1}{\tilde{r}^n}$

Therefore

$$h^{(n)}_{\mu\nu} = \frac{1}{r^n} h^{(n,-n)}_{\mu\nu} + r^{-n+1} h^{(n,-n+1)}_{\mu\nu} + r^{-n+2} h^{(n,-n+2)}_{\mu\nu} + \ldots$$

Information from inner expansion

- $1/\tilde{r}^n$ terms arise from asymptotic expansion of zeroth-order background in inner expansion
  $\Rightarrow$ $h^{(n,-n)}_{\mu\nu}$ is determined by multipole moments of isolated body

Form of solution in buffer region

What appears in the solution?

- throw expansion into nth-order wave equation, solve order by order in r
- expand each $h^{(n,p)}_{\mu\nu}$ in spherical harmonics
- given a worldline γ, the solution at all orders is fully characterized by
  1. body's multipole moments (and corrections thereto): $\sim \frac{Y^{\ell m}}{r^{\ell+1}}$
  2. smooth solutions to vacuum wave equation: $\sim r^\ell Y^{\ell m}$
- everything else made of (linear or nonlinear) combinations of the above

Self field and regular field

- multipole moments define $h^{\mathrm{S}(n)}_{\mu\nu}$; interpret as bound field of body
- smooth homogeneous solutions define $h^{\mathrm{R}(n)}_{\mu\nu}$; free radiation, determined by global boundary conditions

First and second order solutions

First order

- $h_{\mu\nu}^{(1)} = h_{\mu\nu}^{S(1)} + h_{\mu\nu}^{R(1)}$
- $h_{\mu\nu}^{S(1)} \sim 1/r + O(r^0)$ defined by mass monopole m
- $h_{\mu\nu}^{R(1)}$ is undetermined homogenous solution regular at $r = 0$
- evolution equations (from gauge condition): $\dot{m} = 0$ and $a^\mu_{(0)} = 0$ (assuming $a^\mu = a^\mu_{(0)} + \epsilon a^\mu_{(1)} + \ldots$)

Second order

- $h_{\mu\nu}^{(2)} = h_{\mu\nu}^{S(2)} + h_{\mu\nu}^{R(2)}$
- $h_{\mu\nu}^{S(2)} \sim 1/r^2 + O(1/r)$ defined by
  1. mass correction δm
  2. mass dipole M^μ
  3. spin dipole S^μ
- evolution equations: $\dot{S}^\mu = 0$, $\delta\dot{m} = \ldots$, and $\dot{M}^\mu = \ldots$

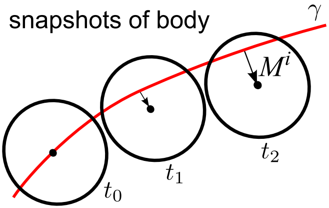
A master equation of motion

Evolution of mass dipole

$$\ddot{M}^\alpha - R^\alpha{}_{\beta\gamma\delta}u^\beta u^\gamma M^\delta = -m a^\alpha_{(1)} + \tfrac{1}{2}R^\alpha{}_{\beta\gamma\delta}u^\beta S^{\gamma\delta} - \tfrac{1}{2}m\left(g^{\alpha\delta}+u^\alpha u^\delta\right)\left(2h^{R(1)}_{\delta\beta;\gamma} - h^{R(1)}_{\beta\gamma;\delta}\right)u^\beta u^\gamma$$

Includes

- geodesic deviation
- first-order term in acceleration of γ
- Mathisson-Papapetrou spin force
- self-force (force due to regular field)
- this relationship between a^α and M^α is valid for *any* γ



Equations of motion

Self-force in self-consistent expansion

- γ defined by $M_\alpha(t) \equiv 0$. Therefore

$$a^\alpha_{(1)} = -\tfrac{1}{2}\left(g^{\alpha\delta} + u^\alpha u^\delta\right)\left(2h^{R(1)}_{\delta\beta;\gamma} - h^{R(1)}_{\beta\gamma;\delta}\right)u^\beta u^\gamma$$

- through order ϵ, small body moves on a geodesic of $g_{\mu\nu} + h^R_{\mu\nu}$

Self-force in power series expansion

- γ is geodesic, so $a^\mu_{(n)} = 0$. Therefore

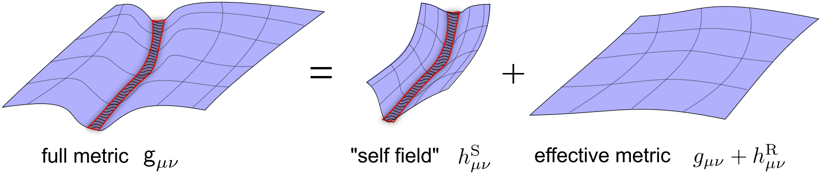
$$\partial_t^2 M^\alpha = R^\alpha{}_{\beta\gamma\delta}u^\beta u^\gamma M^\delta - \tfrac{1}{2}m\left(g^{\alpha\delta} + u^\alpha u^\delta\right)\left(2h^{R(1)}_{\delta\beta;\gamma} - h^{R(1)}_{\beta\gamma;\delta}\right)u^\beta u^\gamma$$

Outline

Effective interior metric

From self-field to singular field

- $h^{\mathrm{S}}_{\mu\nu}$ and $h^{\mathrm{R}}_{\mu\nu}$ derived only in buffer region
- simply extend them to all $r > 0$ (and $r = 0$, for $h^{\mathrm{R}}_{\mu\nu}$)
- does not change field in buffer region or beyond



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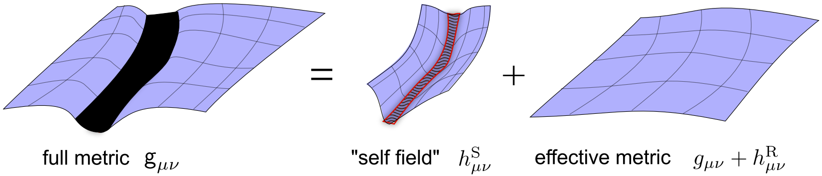

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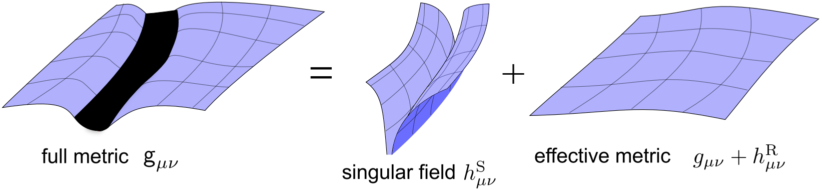
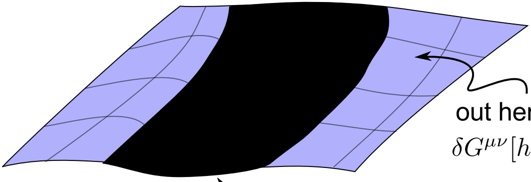


Obtaining global solution

Puncture/effective source scheme

- define $h^{\mathcal{P}}_{\mu\nu}$ as small-r expansion of $h^{\mathrm{S}}_{\mu\nu}$ truncated at order r or higher
- define $h^{\mathcal{R}}_{\mu\nu} = h_{\mu\nu} - h^{\mathcal{P}}_{\mu\nu} \simeq h^{\mathrm{R}}_{\mu\nu}$



out here, solve

$$\delta G^{\mu\nu}[h_{\rho\sigma}] = -(\delta^2 G^{\mu\nu}[h_{\rho\sigma}] + \ldots)$$

in here, solve

$$\delta G^{\mu\nu}[h^{\mathcal{R}}_{\rho\sigma}] = 8\pi T^{\mu\nu}[\gamma] - (\delta^2 G^{\mu\nu}[h_{\rho\sigma}] + \ldots) - \delta G^{\mu\nu}[h^{\mathcal{P}}_{\rho\sigma}]$$

The point...

- $h^{\mathrm{S}}_{\mu\nu}$ found in buffer region suffices to determine both $h^{\mathrm{R}}_{\mu\nu}$ and global solution outside body

Effective stress-energy tensor

What looks like the source of the perturbation?

- all terms in $h^{\mathrm{S}}_{\mu\nu}$ are (linear and nonlinear) combinations of multipole moment terms $\sim Y^{\ell m}/r^{\ell+1}$
- using $\partial^i\partial_i 1/r = -4\pi\delta^3(x^a)$, can show moments are *effectively* sourced by
  $T^{\mu\nu}[\gamma] = \sum_\ell \int_\gamma I^{\mu\nu\alpha_1\ldots\alpha_\ell}\nabla_{\alpha_1}\cdots\nabla_{\alpha_1}\frac{\delta^4(x^\rho - z^\rho(\tau))}{\sqrt{-g}}d\tau$
- in buffer region and outside it, body looks like a skeleton of multipole moments on γ

Point particle picture recovered

- at first order, there is only the mass monopole
  $\Rightarrow T^{\mu\nu}_{(1)}[\gamma] = \int_\gamma m u^\mu u^\nu \frac{\delta^4(x^\rho - z^\rho(\tau))}{\sqrt{-g}}d\tau$
- all the early point-particle results hold true

Conclusion

Determining the motion of a small body

- a self-gravitating material body moves as a test body in an effective geometry $g_{\mu\nu} + h^{\mathrm{R}}_{\mu\nu}$
- EFE solved perturbatively to find full field $h_{\mu\nu}$ outside body and the piece $h^{\mathrm{R}}_{\mu\nu}$ that determines the motion
- singular field $h^{\mathrm{S}}_{\mu\nu}$ calculated in buffer region outside body suffices to determine both $h^{\mathrm{R}}_{\mu\nu}$ and $h_{\mu\nu}$

Current status

- point particle picture and MiSaTaQuWa equation have been justified
- for spherical body, analytical portion of problem now also complete at second order
- for more general body, we will require some model for evolution of body's multipole moments