Second Order Gravitational Self-Force

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Motion of Small Bodies

Consider a body that is small compared to the scale of variation of the external universe. Imagine expanding in the size/mass M of the body.

<u>At least to some finite order in M</u>, one would expect to be able to describe the body as following a worldline in a background spacetime.

What is the acceleration of the worldline?

M^0: zero (geodesic motion). ~100 years old; many derivations; no controversy.

M^1: MiSaTaQuWa force. ~15 years old; several derivations; some controversy.

M^2: no standard expression; much controversy

"second order gravitational self-force"

M^n: ???

Difficulties with Point Particles

Point particle sources don't make sense in GR (Geroch and Traschen 1987)

Full GR: $$\cancel{G_{ab}[g] = 8\pi m \int_Z u_a u_b \delta_4(x, Z)}$$ no mathematical meaning

We could try to fix things by taking M small,

M^1: $$G^{(1)}_{ab}[g^{(1)}] = 8\pi m \int_{Z^{(0)}} u^{(0)}_a u^{(0)}_b \delta_4(x, Z^{(0)})$$ meaningful

Now the equation is linear and makes sense. What about going to order M^2?

M^2: $$\cancel{\overset{\text{okay}}{G^{(1)}_{ab}[g^{(2)}]} + \underbrace{G^{(2)}_{ab}[g^{(1)}]}_{\text{not a distribution}} = 8\pi m \overset{\text{okay}}{\left(\int_Z u_a u_b \delta_4(x, Z)\right)^{(1)}}}$$

no mathematical meaning

Involves products of the distribution $g^{(1)}$;
Off Z, diverges as $(x-Z)^{-4}$ → not locally integrable

What equation gives the metric of a small body to O(M^2)???

We must return to a finite size body and consider a limit of small size/mass M. One way to do this is with the formalism of SEG & Wald 2008.

To motivate our assumptions, consider approximating the Schwarzschild deSitter metric by using a parameter lambda,

$$ds^2(\lambda) = -\left(1 - \frac{2M_0\lambda}{r} - C_0 r^2\right) dt^2 + \left(1 - \frac{2M_0\lambda}{r} - C_0 r^2\right)^{-1} dr^2 + r^2 d\Omega^2$$

As λ→0 we recover deSitter (the "background metric"),

$$ds^2(\lambda = 0) = (1 - C_0 r^2)dt^2 + (1 - C_0 r^2)^{-1} dr^2 + r^2 d\Omega^2$$

The body has shrunk to zero size and disappeared altogether.

1-param family: $ds^2(\lambda) = -\left(1 - \frac{2M_0\lambda}{r} - C_0 r^2\right) dt^2 + \left(1 - \frac{2M_0\lambda}{r} - C_0 r^2\right)^{-1} dr^2 + r^2 d\Omega^2$

But there is another interesting limit. Introduce "scaled coordinates" $\bar{t} \equiv (t - t_0)/\lambda$ and $\bar{r} \equiv r/\lambda$ and the family becomes,

$$ds^2(\lambda) = -\lambda^2 \left(1 - \frac{2M_0}{\bar{r}} - C_0\lambda^2\bar{r}^2\right) d\bar{t}^2 + \lambda^2 \left(1 - \frac{2M_0}{\bar{r}} - C_0\lambda^2\bar{r}^2\right)^{-1} d\bar{r}^2 + \lambda^2\bar{r}^2 d\Omega^2$$

Also introduce a new, scaled *metric* $\bar{g}_{\bar{\mu}\bar{\nu}} \equiv \lambda^{-2} g_{\bar{\mu}\bar{\nu}}$ and you get

$$d\bar{s}^2(\lambda) = \left(1 - \frac{2M_0}{\bar{r}} - C_0\lambda^2\bar{r}^2\right) d\bar{t}^2 + \left(1 - \frac{2M_0}{\bar{r}} - C_0\lambda^2\bar{r}^2\right)^{-1} d\bar{r}^2 + \bar{r}^2 d\Omega^2$$

Now the limit as $\lambda \to 0$ yields the "body metric" of Schwarzschild,

$$d\bar{s}^2|_{\lambda=0} = -\left(1 - \frac{2M_0}{\bar{r}}\right) d\bar{t}^2 + \left(1 - \frac{2M_0}{\bar{r}}\right)^{-1} d\bar{r}^2 + \bar{r}^2 d\Omega^2$$

This procedure has "zoomed in" on the body, because the coordinates scale at the same rate as the body.

We assume a one-parameter-family g(λ) where ordinary and scaled limits of this sort exist and are smoothly related to each other in a certain sense. The main output is a form of the perturbative metric,

$$\begin{aligned}
g^{(0)} &= \eta && + \; 0 && + \; a_{20}r^2 && + \; a_{30}r^3 && + \; O(r^4)\\
g^{(1)} &= a_{01}r^{-1} && + \; a_{11} && + \; a_{21}r && + \; a_{31}r^2 && + \; O(r^3)\\
g^{(2)} &= a_{02}r^{-2} && + \; a_{12}r^{-1} && + \; a_{22} && + \; a_{32}r && + \; O(r^2)\\
g^{(3)} &= a_{03}r^{-3} && + \; a_{13}r^{-2} && + \; a_{23}r^{-1} && + \; a_{33} && + \; O(r),
\end{aligned}$$

These coordinates have the background worldline of the particle (the place where it "disappeared to") at r=0.

a_{nm} are functions of time and angles

The n^th order perturbation diverges as 1/r^n near the background worldline

(Notice how this behavior was present in the example family,

$$ds^2(\lambda) = -\left(1 - \frac{2M_0\lambda}{r} - C_0 r^2\right) dt^2 + \left(1 - \frac{2M_0\lambda}{r} - C_0 r^2\right)^{-1} dr^2 + r^2 d\Omega^2 \text{)}$$

Einstein's equation

The perturbed Einstein equations…

$$G^{(1)}_{\mu\nu}[h] = 0 \qquad (\text{for } r > 0)$$
$$G^{(1)}_{\mu\nu}[j] + G^{(2)}_{\mu\nu}[h] = 0 \qquad (\text{for } r > 0)$$

New notation:

g: background (smooth)
h: first perutrbation (1/r)
j: second perturbation (1/r^2)

…together with the assumed (singular) metric form contain the complete information about the metric perturbations.

Okay, great. How do you find h and j in practice?

In SEG&Wald we proved that, at first order, the above description is equivalent to the linearized Einstein equation sourced by a point paticle.
(This <u>derives</u> the point particle description from extended bodies! See also Pound's work.)

But what do we do at second order, which doesn't play nice with point particles?

Answer: Effective Source Method!

Barack and Golbourn and Detweiler and Vega introduced a technique for determining the field of a point particle by considering smooth sources.

We can recast their method in our language *without ever mentioning point particles*. Things then generalize to second order.

(The only new wrinkle at first order is the gauge freedom; previous effective source work has considered Lorenz gauge only.)

Effective source at first order in our approach:

We know that $\boxed{G^{(1)}_{\mu\nu}[h] = 0 \qquad (\text{for } r > 0) \quad \text{and} \quad h \sim 1/r \text{ near } r=0}$

At some level we have a "singular boundary condition". How to remove it? Solve analytically for h in series in r. Find the *general solution* in a *particular gauge*.

$$h^P = \mathcal{M}^{(1)} r^{-1} + a_{21} r + a_{31} r^2 + O(r^3)$$

(explicit expressions given)

The solution contains free functions. But note by inspection that we may isolate off a "singular piece" h^S such that

1) h^S has no free functions (depends only on M and background curvature)

2) $h^P - h^S$ is C^2 at r=0 (or some desired smoothness)

Pick this h^S and call it the "singular field".

Our choice of "singular field":

$$h^S_{00} = \frac{2}{r} + 2r\mathcal{E}_{ij}n^in^j + \frac{2}{3}r^2\mathcal{E}_{ijk}n^in^jn^k + O(r^3)$$

$$h^S_{i0} = \frac{2}{3}r\epsilon_{ikl}n^jn^k\mathcal{B}_j{}^l + \frac{2}{9}r^2\Big(2\epsilon_{ij}{}^m n^jn^kn^l\mathcal{B}_{klm} + n^j\dot{\mathcal{E}}_{ij} - n_in^jn^k\dot{\mathcal{E}}_{jk}\Big) + O(r^3)$$

(expressed in a local inertial coordinate system of the background metric about the background worldline)

$$h^S_{ij} = \frac{2}{r}\delta_{ij} - 2r\left(2\mathcal{E}_{ij} + \delta_{ij}\mathcal{E}_{kl}n^kn^l\right) + \frac{1}{3}r^2\Big(-4\epsilon_{kl(i}\dot{\mathcal{B}}_{j)}{}^l n^k + 2n_{(i}\epsilon_{j)lm}n^kn^l\dot{\mathcal{B}}_k{}^m - 6n^k\mathcal{E}_{ijk} - 2\delta_{ij}\mathcal{E}_{klm}n^kn^ln^m\Big) + O(r^3).$$

The claim is that the general solution has h-h^S sufficiently regular *when h is expressed in a particular gauge* ("P gauge").

But now consider any *smoothly related* gauge ("P-smooth gauges"),

$$h = h^P - \mathcal{L}_\xi g + O(r^3)$$ (Xi smooth)

It is of course still true that h-h^S is sufficiently regular.

So we have a “singular field” and a corresponding class of gauges such that h-h^S is always sufficiently regular.

Choose an arbitrary extension of h^S to the entire manifold and define

$$\hat{h}^R = h - \hat{h}^S \qquad \text{(hats denote extended quantities)}$$

Then Einstein’s equation,

$$G^{(1)}_{\mu\nu}[h] = 0 \qquad (\text{for } r > 0)$$

becomes

Can drop!

$$\boxed{G^{(1)}[\hat{h}^R] = -G^{(1)}[\hat{h}^S]} \qquad (\text{for } \cancel{r} > 0)$$

The right-hand-side is the “effective source” and is C^0. No more “singular boundary condition”. Numerical integrators happy.

Pick initial/boundary conditions representing the physics of interest and pick any gauge condition (such as Lorenz on h^R) such that h^R is C^2. **Then h = h^R+h^S is the physical metric perturbation expressed in a P-smooth gauge.**

Effective source at second order:

We have $\boxed{G^{(1)}_{\mu\nu}[j] + G^{(2)}_{\mu\nu}[h] = 0 \qquad (\text{for } r > 0) \quad \text{and} \quad j \sim 1/r^2 \text{ near } r=0}$

To remove the "singular boundary condition" find *the general solution* in a *particular gauge* in series in r.

$$j^P = \mathcal{M}^{(2)} r^{-2} + a_{22} + a_{32} r + O(r^2)$$

(explicit expressions given)

New subtlety: a smooth gauge transformation changes j by a singular amount!

singular also singular!

$$j = j^P - \mathcal{L}_\xi h^P + \frac{1}{2}\left(\mathcal{L}_\xi \mathcal{L}_\xi g - \mathcal{L}_\Xi g\right).$$

(xi, Xi smooth)

We must include the second term in the singular field j^S.
We need to determine xi!

Determining xi

We gave a prescription for computing h in a P-smooth gauge,

$$h = h^P - \mathcal{L}_\xi g + O(r^3) \qquad \text{(Xi smooth)}$$

Now that we know h we need to "invert" this equation and solve for xi.

Recall that h^P contains free functions. It turns out these are determined uniquely by h and xi. Then we have an equation just for xi. After some work we find a complicated expression for the general solution, which depends on

1) Background curvature

2) The regular field

3) A choice of "initial data" for the value and derivative of xi on the background worldline.

(A and B are value and derivative of xi on the background worldline)

$$A_\mu = \xi_\mu|_\gamma$$
$$B_{\mu\nu} = (\nabla_\nu \xi_\mu)|_\gamma$$

(A and B obey transport equations)

$$\dot{A}_0 = -\frac{1}{2} h^R_{00}$$
$$\dot{A}_i = B_{i0}$$
$$\dot{B}_{i0} = -\mathcal{E}_{ij} A^j - \partial_0 h^R_{0i} + \frac{1}{2}\partial_i h^R_{00}$$
$$\dot{B}_{[ij]} = \epsilon_{ijl} \mathcal{B}^l{}_k A^k + \partial_{[i} h^R_{j]0},$$

Xi

$$\xi_0 = A_0 - B_{i0}x^i - h^R_{0i}x^i + \left(-\frac{1}{2}\partial_j h^R_{0i} + \frac{1}{4}\partial_0 h^R_{ij} + A_0\mathcal{E}_{ij} + A^k\epsilon_{kil}\mathcal{B}_j{}^l\right)x^ix^j + \left(-\frac{1}{6}B_{j0}\mathcal{E}_i{}^j + \frac{5}{18}A_j\dot{\mathcal{E}}_i{}^j\right)x^ir^2$$
$$+ \left(-\frac{1}{6}\partial_j\partial_k h^R_{0i} + \frac{1}{12}\partial_0\partial_k h^R_{ij} + \frac{2}{3}B_{lj}\epsilon^l{}_{im}\mathcal{B}_k{}^m + \frac{8}{9}A^l\epsilon_{lj}{}^m\mathcal{B}_{kim} - \frac{1}{3}B_{i0}\mathcal{E}_{jk} - \frac{2}{3}h^R_{0i}\mathcal{E}_{jk} - \frac{4}{9}A_i\dot{\mathcal{E}}_{jk} + \frac{1}{3}A_0\mathcal{E}_{ijk}\right)x^ix^jx^k$$
$$+ O(r^4) \tag{84}$$

$$\xi_i = A_i + B_{ij}x^j - \left(\frac{1}{2}\partial_k h^R_{ij} + \frac{1}{4}\partial_i h^R_{jk} - A_i\mathcal{E}_{jk} - \frac{2}{3}A_0\epsilon_{ikl}\mathcal{B}_j{}^l\right)x^jx^k + \left(n_iA_jn_k\mathcal{E}^{jk} - \frac{1}{2}A_j\mathcal{E}_i{}^j\right)r^2$$
$$+ \left(\frac{1}{12}\partial_j\partial_0 h^R_{0i} + \frac{1}{12}\partial_i\partial_0 h^R_{0j} - \frac{1}{12}\partial_i\partial_j h^R_{00} - \frac{1}{12}\partial_0\partial_0 h^R_{ij} + \frac{1}{6}h^R_{00}\mathcal{E}_{ij} + \frac{1}{6}h^R_{ik}\mathcal{E}_j{}^k - \frac{1}{6}h^R_{ik}\mathcal{E}_j{}^k\right)r^2x^j$$
$$+ \left(-\frac{1}{6}B_{k0}\epsilon^k{}_{jl}\mathcal{B}_i{}^l + \frac{1}{6}B_{k0}\epsilon_i{}^k{}_l\mathcal{B}_j{}^l - \frac{1}{12}A_k\epsilon^k{}_{jl}\dot{\mathcal{B}}_i{}^l - \frac{1}{12}A_k\epsilon_{ijl}\dot{\mathcal{B}}^{kl}\right)r^2x^j$$
$$+ \left(-\frac{1}{3}\partial_0\partial_k h^R_{0j} + \frac{1}{6}\partial_j\partial_k h^R_{00} + \frac{1}{6}\partial_0\partial_0 h^R_{jk} + \frac{2}{3}A_0\dot{\mathcal{E}}_{jk} - \frac{1}{3}h^R_{00}\mathcal{E}_{jk} + \frac{2}{3}B_{l0}\epsilon^l{}_{jm}\mathcal{B}_k{}^m + \frac{5}{12}A_l\epsilon^l{}_{km}\dot{\mathcal{B}}_j{}^m\right)x_ix^jx^k$$
$$+ \left(-\frac{1}{6}\partial_k\partial_l h^R_{ij} + \frac{1}{12}\partial_i\partial_l h^R_{jk} - \frac{1}{3}h^R_{ij}\mathcal{E}_{kl} + \frac{1}{3}\epsilon_{ilm}h^R_{0j}\mathcal{B}_k{}^m\right)x^jx^kx^l$$
$$+ \left(\frac{1}{3}B_{j0}\epsilon_{ilm}\mathcal{B}_k{}^m + \frac{1}{12}A_j\epsilon_{ilm}\dot{\mathcal{B}}_k{}^m - \frac{2}{3}A_0\epsilon_{ij}{}^m\mathcal{B}_{klm} - \frac{1}{3}B_{jk}\mathcal{E}_{il} - B_{ij}\mathcal{E}_{kl} - \frac{1}{3}A_i\mathcal{E}_{jkl}\right)x^jx^kx^l + O(r^4), \tag{85}$$

Singular Field $j^S = j^P - \mathcal{L}_\xi h^P$

$$
\begin{aligned}
j^S_{00} = {} & \frac{-2}{r^2}\Big[1 - A^i n_i\Big] - \frac{1}{r}\Big[2h^R_{00} + h^R_{ij}n^i n^j\Big] + \frac{1}{2}\Big[-4\partial^i h^R_{00} n_i - \partial_k h^R_{ij} n^i n^j n^k + 2A^i n^j \mathcal{E}_{ij} + n^i n^j \mathcal{E}_{ij} + 4A^i n_i n^j n^k \mathcal{E}_{jk}\Big] \\
& + \frac{1}{12} r\Big[-4\partial_0 \partial^j h^R_{0i} n_i n_j - 10\partial_i \partial_j h^R_{00} n^i n^j + 2\partial_0\partial_0 h^R_{ij} n^i n^j - 2\partial_k \partial_l h^R_{ij} n^i n^j n^k n^l - 40\dot{A}^i \epsilon_{ikl} n^j n^k \mathcal{B}_j{}^l + 4A^i \epsilon_{ikl} n^j n^k \dot{\mathcal{B}}_j{}^l \\
& - 4h^R_{00} n^i n^j \mathcal{E}_{ij} - 16 h^R_{ij} n^i n^j n^k n^l \mathcal{E}_{kl} + n^i n^j n^k \mathcal{E}_{ijk} + 8A^i n_i n^j n^k n^l \mathcal{E}_{jkl}\Big] + O(r^2) \qquad \text{(B4)}
\end{aligned}
$$

$$
\begin{aligned}
j^S_{i0} = {} & \frac{-2}{r}\Big[h^R_{0i} + 2\dot{A}_i\Big] + \frac{1}{6}\Big[-12\partial_i h^R_{0j} n^j + 12\partial_0 h^R_{ij} n^j + 4A^j \epsilon_{jkl} n^k \mathcal{B}_i{}^l + 8A^j \epsilon_{ikl} n^k \mathcal{B}_j{}^l - \epsilon_{ikl} n^j n^k \mathcal{B}_j{}^l - 8A^j \epsilon_{ijl} n^k \mathcal{B}_k{}^l \\
& + 4A^j \epsilon_{ilm} n_j n^k n^l \mathcal{B}_k{}^m\Big] + \frac{1}{9} r\Big[-9\partial_i \partial_k h^R_{0j} n^j n^k + 9\partial_0 \partial_k h^R_{ij} n^j n^k + 6\epsilon_{ikm} B^{[lm]} n^j n^k \mathcal{B}_{jl} + 3\epsilon_i{}^{km} h^R_{lm} n^j n^k \mathcal{B}_{jl} \\
& - 6\epsilon_k{}^{lm} B_{[il]} n^j n^k \mathcal{B}_{jm} - 3\epsilon_k{}^{lm} h^R_{il} n^j n^k \mathcal{B}_{jm} - 3\epsilon_i{}^{kp} \delta^{lm} h^R_{lm} n^j n_k \mathcal{B}_{jp} + 6\epsilon_i{}^{lm} B_{[jl]} n^j n^k \mathcal{B}_{km} + 3\epsilon_i^{lm} h^R_{jl} n^j n^k \mathcal{B}_{km} \\
& - 3\epsilon_{imp} h^R_{jk} n^j n^k n^l n^m \mathcal{B}_l{}^p + 8A^j \epsilon_{jk}{}^m n^k n^l \mathcal{B}_{ilm} + 8A^j \epsilon_{ik}{}^m n^k n^l \mathcal{B}_{jlm} - 8A^j \epsilon_{ij}{}^m n^k n^l \mathcal{B}_{klm} - 2\epsilon_{ij}{}^m n^j n^k n^l \mathcal{B}_{klm} \\
& + 4A^j \epsilon_{ik}{}^p n_j n^k n^l n^m \mathcal{B}_{lmp} + 36\dot{A}^j \mathcal{E}_{ij} + 36\dot{A}^j n_j n^k \mathcal{E}_{ik} + 18h^R_{0j} n^j n^k \mathcal{E}_{ik} - 18\dot{A}^j n_i n^k \mathcal{E}_{jk} - 18h^R_{0i} n^j n^k \mathcal{E}_{jk} + 10A^j \dot{\mathcal{E}}_{ij} \\
& - 76n^j \dot{\mathcal{E}}_{ij} + 10A^j n_j n^k \dot{\mathcal{E}}_{ik} - 2A^j n_i n^k \dot{\mathcal{E}}_{jk} + 2A_i n^j n^k \dot{\mathcal{E}}_{jk} - 17n_i n^j n^k \dot{\mathcal{E}}_{jk} - 2A^j n_i n_j n^k n^l \dot{\mathcal{E}}_{kl}\Big] + O(r^2) \qquad \text{(B5)}
\end{aligned}
$$

$$
\begin{aligned}
j^S_{ij} = {} & \frac{1}{2r^2}\Big[3\delta_{ij} + 4A^k \delta_{ij} n_k\Big] + \frac{1}{r}\Big[2h^R_{ij} - \delta_{ij} h^R_{kl} n^k n^l\Big] + 2\partial_k h^R_{ij} n^k - \frac{1}{2}\delta_{ij}\partial_m h^R_{kl} n^k n^l n^m - 8\mathcal{E}_{ij} + 4A^k n_k \mathcal{E}_{ij} + A^k \delta_{ij} n^l \mathcal{E}_{kl} \\
& + \delta_{ij} n^k n^l \mathcal{E}_{kl} - 2A^k \delta_{ij} n_k n^l n^m \mathcal{E}_{lm} + \frac{1}{24} r\Big[8\partial_i \partial_j h^R_{00} - 8\partial_0 \partial_j h^R_{0i} - 8\partial_0 \partial_i h^R_{0j} + 8\partial_0 \partial_0 h^R_{ij} + 8\partial_0 \partial^k h^R_{0j} n_i n_k \\
& + 8\partial_0 \partial_j h^R_{0k} n_i n^k + 8\partial_0 \partial^k h^R_{0i} n_j n_k + 8\partial_0 \partial_i h^R_{0k} n_j n^k - 8\partial_j \partial_k h^R_{00} n_i n^k - 8\partial_0 \partial_0 h^R_{jk} n_i n^k - 8\partial_i \partial_k h^R_{00} n_j n^k \\
& - 8\partial_0 \partial_0 h^R_{ik} n_j n^k - 72\delta_{ij} \partial_0 \partial^l h^R_{0k} n_k n_l + 48\partial_0 \partial^l h^R_{0k} n_i n_j n_k n_l + 36\delta_{ij} \partial_k \partial_l h^R_{00} n^k n^l + 36\delta_{ij} \partial_0 \partial_0 h^R_{kl} n^k n^l \\
& + 16\partial_k \partial_l h^R_{ij} n^k n^l + 8\partial_j \partial_l h^R_{ik} n^k n^l + 8\partial_i \partial_l h^R_{jk} n^k n^l - 8\partial_i \partial_j h^R_{kl} n^k n^l - 24\partial_k \partial_l h^R_{00} n_i n_j n^k n^l - 24\partial_0 \partial_0 h^R_{kl} n_i n_j n^k n^l \\
& - 4\delta_{ij} \partial_m \partial_p h^R_{kl} n^k n^l n^m n^p - 16\dot{A}^k \epsilon_{jkl} \mathcal{B}_i{}^l - 16\dot{A}^k \epsilon_{klm} n_j n^l \mathcal{B}_i{}^m + 16\dot{A}^k \epsilon_{jlm} n_k n^l \mathcal{B}_i{}^m + 16\epsilon_{jlm} h^R_{0k} n^k n^l \mathcal{B}_i{}^m \\
& - 16\dot{A}^k \epsilon_{ikl} \mathcal{B}_j{}^l - 16\dot{A}^k \epsilon_{klm} n_i n^l \mathcal{B}_j{}^m + 16\dot{A}^k \epsilon_{ilm} n_k n^l \mathcal{B}_j{}^m + 16\epsilon_{ilm} h^R_{0k} n^k n^l \mathcal{B}_j{}^m + 16\dot{A}^k \epsilon_{jkm} n_i n^l \mathcal{B}_l{}^m \\
& + 16\dot{A}^k \epsilon_{ikm} n_j n^l \mathcal{B}_l{}^m + 144\dot{A}^k \epsilon_{kmp} \delta_{ij} n^l n^m \mathcal{B}_l{}^p - \dot{A}^k \epsilon_{kmp} n_i n_j n^l n^m \mathcal{B}_l{}^p + 12A^k \epsilon_{jkl} \dot{\mathcal{B}}_i{}^l - 32\epsilon_{jkl} n^k \dot{\mathcal{B}}_i{}^l \\
& - 4A^k \epsilon_{klm} n_j n^l \dot{\mathcal{B}}_i{}^m + 20A^k \epsilon_{jlm} n_k n^l \dot{\mathcal{B}}_i{}^m + 12A^k \epsilon_{ikl} \dot{\mathcal{B}}_j{}^l - 32\epsilon_{ikl} n^k \dot{\mathcal{B}}_j{}^l - 4A^k \epsilon_{klm} n_i n^l \dot{\mathcal{B}}_j{}^m + 20A^k \epsilon_{ilm} n_k n^l \dot{\mathcal{B}}_j{}^m \\
& - 4A_j \epsilon_{ilm} n^k n^l \dot{\mathcal{B}}_k{}^m - 4A_i \epsilon_{jlm} n^k n^l \dot{\mathcal{B}}_k{}^m + 35\epsilon_{jlm} n_i n^k n^l \dot{\mathcal{B}}_k{}^m + 35\epsilon_{ilm} n_j n^k n^l \dot{\mathcal{B}}_k{}^m + 4A^k \epsilon_{jkm} n_i n^l \dot{\mathcal{B}}_l{}^m \\
& + 4A^k \epsilon_{ikm} n_j n^l \dot{\mathcal{B}}_l{}^m + 48A^k \epsilon_{kmp} \delta_{ij} n^l n^m \dot{\mathcal{B}}_l{}^p - 32A^k \epsilon_{kmp} n_i n_j n^l n^m \dot{\mathcal{B}}_l{}^p + 8A^k \epsilon_{jmp} n_i n_k n^l n^m \dot{\mathcal{B}}_l{}^p \\
& + 8A^k \epsilon_{imp} n_j n_k n^l n^m \dot{\mathcal{B}}_l{}^p - 16h^R_{00} \mathcal{E}_{ij} - 64h^R_{kl} n^k n^l \mathcal{E}_{ij} - 96B_{[jk]} \mathcal{E}_i{}^k - 48h^R_{jk} \mathcal{E}_i{}^k + 16h^R_{00} n_j n^k \mathcal{E}_{ik} + 16h^R_{jk} n^k n^l \mathcal{E}_{il} \\
& - 96B_{[ik]} \mathcal{E}_j{}^k - 48h^R_{ik} \mathcal{E}_j{}^k + 16h^R_{00} n_i n^k \mathcal{E}_{jk} + 16h^R_{ik} n^k n^l \mathcal{E}_{jl} - 72\delta_{ij} h^R_{00} n^k n^l \mathcal{E}_{kl} - 16h^R_{ij} n^k n^l \mathcal{E}_{kl} + 48h^R_{00} n_i n_j n^k n^l \mathcal{E}_{kl} \\
& + 96\delta_{ij} B_{[km]} n^k n^l \mathcal{E}_l^m + 48\delta_{ij} h^R_{km} n^k n^l \mathcal{E}_l{}^m - 96B_{[km]} n_i n_j n^k n^l \mathcal{E}_l{}^m - 48h^R_{km} n_i n_j n^k n^l \mathcal{E}_l{}^m \\
& + 16\delta_{ij} h^R_{kl} n^k n^l n^m n^p \mathcal{E}_{mp} - 96A_0 \dot{\mathcal{E}}_{ij} + 48A_0 \delta_{ij} n^k n^l \dot{\mathcal{E}}_{kl} - 48A_0 n_i n_j n^k n^l \dot{\mathcal{E}}_{kl} + 48A^k \mathcal{E}_{ijk} - 96n^k \mathcal{E}_{ijk} + 48A^k n_k n^l \mathcal{E}_{ijl} \\
& - 48A^k \delta_{ij} n^l n^m \mathcal{E}_{klm} + 48A^k n_i n_j n^l n^m \mathcal{E}_{klm} + 6\delta_{ij} n^k n^l n^m \mathcal{E}_{klm} - 16A^k \delta_{ij} n_k n^l n^m n^p \mathcal{E}_{lmp}\Big] + O(r^2) \qquad \text{(B6)}
\end{aligned}
$$

Choose the second-order singular field to be

$$j^S = j^P - \mathcal{L}_\xi h^P$$

Choose an arbitrary extension of jS to the entire manifold and define

$$\hat{j}^R = j - \hat{j}^S, \qquad \text{(hats denote extended quantities)}$$

Then Einstein's equation,

$$G^{(1)}_{\mu\nu}[j] + G^{(2)}_{\mu\nu}[h] = 0 \qquad (\text{for } r > 0)$$

becomes

Can drop!

$$\boxed{G^{(1)}[\hat{j}^R] = -G^{(1)}[\hat{j}^S] - G^{(2)}[h]} \qquad (\text{for } \cancel{r} > 0).$$

The right-hand-side is the "effective source" and is bounded. No more "singular boundary condition". Numerical integrators happy.

Pick initial/boundary conditions representing the physics of interest and pick any gauge condition (such as Lorenz on j^R) such that j^R is C^1. **Then $j = j^R + j^S$ is the physical metric perturbation expressed in a P-smooth gauge.**

This provides a prescription for computing the metric of a small body through second order in its size/mass. You can do a lot with just this: fluxes, snapshot waveforms, etc.

But what about the motion? Actually, with all this hard work done, it's trivial.

The secret is that we chose this P gauge to be "mass centered":

If you take the near-zone limit of the P-gauge metric perturbation, then the near-zone metric is just the ordinary Schwarzschild metric in isotropic coordinates.

So, we say that the perturbed position of the particle **vanishes** in P gauge**.**
But we worked in P-smooth gauges. What is the description there? Well, how does a point on the manifold "change" under a gauge transformation…

$$x'^{\mu} = x^{\mu} + \lambda\xi^{\mu} + \frac{1}{2}\lambda^2\left(\Xi^{\mu} + \xi^{\nu}\partial_{\nu}\xi^{\mu}\right) + O(\lambda^3)$$

New perturbed position:

$$Z^{(1)\mu} = \xi^{\mu}|_{\gamma}$$
$$Z^{(2)\mu} = (\Xi^{\mu} + \xi^{\nu}\partial_{\nu}\xi^{\mu})|_{\gamma}.$$

So, we need to find the gauge vectors. Or do we? Here's a trick:

Let $g^{BG}_{\mu\nu}(\lambda) \equiv g^{(0)}_{\mu\nu}$ where this equation holds <u>only in the P gauge</u>.

In a P smooth gauge we have $g^{BG}(\lambda) = g - \lambda h^{BG} + \lambda^2 j^{BG} + O(\lambda^3),$

$$
\begin{aligned}
h^{BG} &= -\mathcal{L}_\xi g \\
j^{BG} &= \frac{1}{2}(\mathcal{L}_\xi \mathcal{L}_\xi g - \mathcal{L}_\Xi g),
\end{aligned}
$$

Since the background motion is geodesic, vanishing perturbed motion means that the motion is geodesic in $g^{BG}_{\mu\nu}(\lambda) + O(\lambda^3)$.

This is an invariant statement and holds in any gauge! The motion is **geodesic in the BG fields.** This can be simply related to the regular fields that arise in practice, completing the prescription for determining the metric and motion.

Recall $h = h^P \underbrace{- \mathcal{L}_\xi g}_{h^{BG}}$ $\qquad j = j^P - \mathcal{L}_\xi h^P + \frac{1}{2}\underbrace{(\mathcal{L}_\xi \mathcal{L}_\xi g - \mathcal{L}_\Xi g)}_{j^{BG}}.$

Second order Motion

$$
\begin{aligned}
\ddot{Z}^{(2)}{}_0 &= -\frac{1}{2}\partial_0 j^R_{00} - h^R_{0\nu}\ddot{Z}^{(1)\nu} \\
&+ \dot{Z}^{(1)\gamma}\partial_\gamma h^R_{00} + \frac{1}{2}Z^{(1)\gamma}\partial_\gamma\partial_0 h^R_{00} \\
&- 2\mathcal{E}_{ij}\dot{Z}^{(1)i}Z^{(1)j} - \frac{1}{2}\dot{\mathcal{E}}_{ij}Z^{(1)i}Z^{(1)j}, \\
\ddot{Z}^{(2)}{}_i &= -\partial_0 j^R_{0i} + \frac{1}{2}\partial_i j^R_{00} - \mathcal{E}_{ij}Z^{(2)j} + \delta\mathcal{E}_{ij}Z^{(1)j} \\
&- h^R_{i\nu}\ddot{Z}^{(1)\nu} + Z^{(1)\gamma}\partial_\gamma(-\partial_0 h^R_{0i} + \frac{1}{2}\partial_i h^R_{00}) \\
&+ 2\dot{Z}^{(1)0}\ddot{Z}^{(1)i} - \dot{Z}^{(1)j}\left(\partial_0 h^R_{ij} + \partial_j h^R_{i0} - \partial_i h^R_{j0}\right) \\
&- 2\dot{Z}^{(1)j}Z^{(1)k}\epsilon_{ijl}\mathcal{B}_k{}^l + \frac{2}{3}Z^{(1)k}Z^{(1)l}\epsilon_{ipk}\dot{\mathcal{B}}_l{}^p \\
&- \frac{1}{2}\mathcal{E}_{ijk}Z^{(1)j}Z^{(1)k} - \dot{\mathcal{E}}_{ij}Z^{(1)0}Z^{(1)j}, \qquad (104)
\end{aligned}
$$

“self-force”

The Prescription

1) Choose a vacuum background spacetime and geodesic.

2) Find a coordinate transformation between your favorite global coordinate system and my favorite local coordinate system ("RWZ coordinates").

3) Compute h^S from the RWZ formula I give, choose an extension and compute the effective source, and solve for h^R in some convenient gauge.

4) Integrate some transport equations along the worldline to determine A and B, choosing trivial initial data. (A is the first-order motion.)

5) Compute j^S from the RWZ formula I give (involving also h^R,A,B), choose an extension and compute the second-order effective source, and solve for j^R a a convenient gauge.

6) Integrate some more transport equations to get the second perturbed motion in your gauge.

What I have done…

Given a prescription for computing the second order metric and motion perturbation of a small body.

Good for local-in-time observables.

What I haven't done…

Told you how to compute a long-term inspiral waveform.

However, one should be able to apply adiabatic approaches (Mino; Hinderer and Flanagan) or self-consistent approaches, provided the role of gauge can be understood.

What I would like to do…

(or see others do!)

Understand the role of gauge in adiabatic and self-consistent approaches.

$$
\begin{aligned}
j^S_{00} = {} & \frac{-2}{r^2}\Big[1 - A^i n_i\Big] - \frac{1}{r}\Big[2h^R_{00} + h^R_{ij} n^i n^j\Big] + \frac{1}{2}\Big[-4\partial^i h^R_{00} n_i - \partial_k h^R_{ij} n^i n^j n^k + 2A^i n^j \mathcal{E}_{ij} + n^i n^j \mathcal{E}_{ij} + 4A^i n_i n^j n^k \mathcal{E}_{jk}\Big] \\
& + \frac{1}{12} r\Big[-4\partial_0 \partial^j h^R_{0i} n_i n_j - 10\partial_i \partial_j h^R_{00} n^i n^j + 2\partial_0 \partial_0 h^R_{ij} n^i n^j - 2\partial_k \partial_l h^R_{ij} n^i n^j n^k n^l - 40\dot{A}^i \epsilon_{ikl} n^j n^k \mathcal{B}_j{}^l + 4A^i \epsilon_{ikl} n^j n^k \dot{\mathcal{B}}_j{}^l \\
& - 4h^R_{00} n^i n^j \mathcal{E}_{ij} - 16h^R_{ij} n^i n^j n^k n^l \mathcal{E}_{kl} + n^i n^j n^k \mathcal{E}_{ijk} + 8A^i n_i n^j n^k n^l \mathcal{E}_{jkl}\Big] + O(r^2)
\end{aligned}
\tag{B4}
$$

$$
\begin{aligned}
j^S_{i0} = {} & \frac{-2}{r}\Big[h^R_{0i} + 2\dot{A}_i\Big] + \frac{1}{6}\Big[-12\partial_i h^R_{0j} n^j + 12\partial_0 h^R_{ij} n^j + 4A^j \epsilon_{jkl} n^k \mathcal{B}_i{}^l + 8A^j \epsilon_{ikl} n^k \mathcal{B}_j{}^l - \epsilon_{ikl} n^j n^k \mathcal{B}_j{}^l - 8A^j \epsilon_{ijl} n^k \mathcal{B}_k{}^l \\
& + 4A^j \epsilon_{ilm} n_j n^k n^l \mathcal{B}_k{}^m\Big] + \frac{1}{9} r\Big[-9\partial_i \partial_k h^R_{0j} n^j n^k + 9\partial_0 \partial_k h^R_{ij} n^j n^k + 6\epsilon_{ikm} B^{[lm]} n^j n^k \mathcal{B}_{jl} + 3\epsilon_i{}^{km} h^R_{lm} n^j n^k \mathcal{B}_{jl} \\
& - 6\epsilon_k{}^{lm} B_{[il]} n^j n^k \mathcal{B}_{jm} - 3\epsilon_k{}^{lm} h^R_{il} n^j n^k \mathcal{B}_{jm} - \text{[illegible]}\ \mathcal{B}_{jp} + 6\epsilon_i{}^{lm} B_{[jl]} n^j n^k \mathcal{B}_{km} + 3\epsilon_i{}^{lm} h^R_{jl} n^j n^k \mathcal{B}_{km} \\
& - 3\epsilon_{imp} h^R_{jk} n^j n^k n^l n^m \mathcal{B}_l{}^p + 8A^j \epsilon_{jk}{}^m n^k n^l \mathcal{B}_{il\ldots}\ \text{[illegible]}\ n^l \mathcal{B}_{jlm} - 8A^j \epsilon_{ij}{}^m n^k n^l \mathcal{B}_{klm} - 2\epsilon_{ij}{}^m n^j n^k n^l \mathcal{B}_{klm} \\
& + 4A^j \epsilon_{ik}{}^p n_j n^k n^l n^m \mathcal{B}_{lmp} + 36\dot{A}^j \mathcal{E}_{ij} + 36\dot{A}^j\ \text{[illegible]}\ n^j n^k \mathcal{E}_{ik} - 18\dot{A}^j n_i n^k \mathcal{E}_{jk} - 18h^R_{0i} n^j n^k \mathcal{E}_{jk} + 10A^j \dot{\mathcal{E}}_{ij} \\
& - 76n^j \dot{\mathcal{E}}_{ij} + 10A^j n_j n^k \dot{\mathcal{E}}_{ik} - 2A^j n_i n^k \dot{\mathcal{E}}_{jk} + 2A_i n^j n^k \dot{\mathcal{E}}_{jk} - 17n_i n^j n^k \dot{\mathcal{E}}_{jk} - 2A^j n_i n_j n^k n^l \dot{\mathcal{E}}_{kl}\Big] + O(r^2)
\end{aligned}
\tag{B5}
$$

Fine

$$
\begin{aligned}
j^S_{ij} = {} & \frac{1}{2r^2}\Big[3\delta_{ij} + 4A^k \delta_{ij} n_k\Big] + \frac{1}{r}\Big[2h^R_{ij} - \delta_{ij} h^R_{kl} n^k n^l\Big] + 2\partial_k h^R_{ij} n^k - \frac{1}{2}\delta_{ij} \partial_m h^R_{kl} n^k n^l n^m - 8\mathcal{E}_{ij} + 4A^k n_k \mathcal{E}_{ij} + A^k \delta_{ij} n^l \mathcal{E}_{kl} \\
& + \delta_{ij} n^k n^l \mathcal{E}_{kl} - 2A^k \delta_{ij} n_k n^l n^m \mathcal{E}_{lm} + \frac{1}{24} r\Big[8\partial_i \partial_j h^R_{00} - 8\partial_0 \partial_j h^R_{0i} - 8\partial_0 \partial_i h^R_{0j} + 8\partial_0 \partial_0 h^R_{ij} + 8\partial_0 \partial^k h^R_{0j} n_i n_k \\
& + 8\partial_0 \partial_j h^R_{0k} n_i n^k + 8\partial_0 \partial^k h^R_{0i} n_j n_k + 8\partial_0 \partial_i h^R_{0k} n_j n^k - 8\partial_j \partial_k h^R_{00} n_i n^k - 8\partial_0 \partial_0 h^R_{jk} n_i n^k - 8\partial_i \partial_k h^R_{00} n_j n^k \\
& - 8\partial_0 \partial_0 h^R_{ik} n_j n^k - 72\delta_{ij} \partial_0 \partial^l h^R_{0k} n_k n_l + 48\partial_0 \partial^l h^R_{0k} n_i n_j n_k n_l + 36\delta_{ij} \partial_k \partial_l h^R_{00} n^k n^l + 36\delta_{ij} \partial_0 \partial_0 h^R_{kl} n^k n^l \\
& + 16\partial_k \partial_l h^R_{ij} n^k n^l + 8\partial_j \partial_l h^R_{ik} n^k n^l + 8\partial_i \partial_l h^R_{jk} n^k n^l - 8\partial_i \partial_j h^R_{kl} n^k n^l - 24\partial_k \partial_l h^R_{00} n_i n_j n^k n^l - 24\partial_0 \partial_0 h^R_{kl} n_i n_j n^k n^l \\
& - 4\delta_{ij} \partial_m \partial_p h^R_{kl} n^k n^l n^m n^p - 16\dot{A}^k \epsilon_{jkl} \mathcal{B}_i{}^l - 16\dot{A}^k \epsilon_{klm} n_j n^l \mathcal{B}_i{}^m + 16\dot{A}^k \epsilon_{jlm} n_k n^l \mathcal{B}_i{}^m + 16\epsilon_{jlm} h^R_{0k} n^k n^l \mathcal{B}_i{}^m \\
& - 16\dot{A}^k \epsilon_{ikl} \mathcal{B}_j{}^l - 16\dot{A}^k \epsilon_{klm} n_i n^l \mathcal{B}_j{}^m + 16\dot{A}^k \epsilon_{ilm} n_k n^l \mathcal{B}_j{}^m + 16\epsilon_{ilm} h^R_{0k} n^k n^l \mathcal{B}_j{}^m + 16\dot{A}^k \epsilon_{jkm} n_i n^l \mathcal{B}_l{}^m \\
& + 16\dot{A}^k \epsilon_{ikm} n_j n^l \mathcal{B}_l{}^m + 144\dot{A}^k \epsilon_{kmp} \delta_{ij} n^l n^m \mathcal{B}_l{}^p - \dot{A}^k \epsilon_{kmp} n_i n_j n^l n^m \mathcal{B}_l{}^p + 12A^k \epsilon_{jkl} \dot{\mathcal{B}}_i{}^l - 32\epsilon_{jkl} n^k \dot{\mathcal{B}}_i{}^l \\
& - 4A^k \epsilon_{klm} n_j n^l \dot{\mathcal{B}}_i{}^m + 20A^k \epsilon_{jlm} n_k n^l \dot{\mathcal{B}}_i{}^m + 12A^k \epsilon_{ikl} \dot{\mathcal{B}}_j{}^l - 32\epsilon_{ikl} n^k \dot{\mathcal{B}}_j{}^l - 4A^k \epsilon_{klm} n_i n^l \dot{\mathcal{B}}_j{}^m + 20A^k \epsilon_{ilm} n_k n^l \dot{\mathcal{B}}_j{}^m \\
& - 4A_j \epsilon_{ilm} n^k n^l \dot{\mathcal{B}}_k{}^m - 4A_i \epsilon_{jlm} n^k n^l \dot{\mathcal{B}}_k{}^m + 35\epsilon_{jlm} n_i n^k n^l \dot{\mathcal{B}}_k{}^m + 35\epsilon_{ilm} n_j n^k n^l \dot{\mathcal{B}}_k{}^m + 4A^k \epsilon_{jkm} n_i n^l \dot{\mathcal{B}}_l{}^m \\
& + 4A^k \epsilon_{ikm} n_j n^l \dot{\mathcal{B}}_l{}^m + 48A^k \epsilon_{kmp} \delta_{ij} n^l n^m \dot{\mathcal{B}}_l{}^p - 32A^k \epsilon_{kmp} n_i n_j n^l n^m \dot{\mathcal{B}}_l{}^p + 8A^k \epsilon_{jmp} n_i n_k n^l n^m \dot{\mathcal{B}}_l{}^p \\
& + 8A^k \epsilon_{imp} n_j n_k n^l n^m \dot{\mathcal{B}}_l{}^p - 16h^R_{00} \mathcal{E}_{ij} - 64h^R_{kl} n^k n^l \mathcal{E}_{ij} - 96B_{[jk]} \mathcal{E}_i{}^k - 48h^R_{jk} \mathcal{E}_i{}^k + 16h^R_{00} n_j n^k \mathcal{E}_{ik} + 16h^R_{jk} n^k n^l \mathcal{E}_{il} \\
& - 96B_{[ik]} \mathcal{E}_j{}^k - 48h^R_{ik} \mathcal{E}_j{}^k + 16h^R_{00} n_i n^k \mathcal{E}_{jk} + 16h^R_{ik} n^k n^l \mathcal{E}_{jl} - 72\delta_{ij} h^R_{00} n^k n^l \mathcal{E}_{kl} - 16h^R_{ij} n^k n^l \mathcal{E}_{kl} + 48h^R_{00} n_i n_j n^k n^l \mathcal{E}_{kl} \\
& + 96\delta_{ij} B_{[km]} n^k n^l \mathcal{E}^m_l + 48\delta_{ij} h^R_{km} n^k n^l \mathcal{E}_l{}^m - 96B_{[km]} n_i n_j n^k n^l \mathcal{E}_l{}^m - 48h^R_{km} n_i n_j n^k n^l \mathcal{E}_l{}^m \\
& + 16\delta_{ij} h^R_{kl} n^k n^l n^m n^p \mathcal{E}_{mp} - 96A_0 \dot{\mathcal{E}}_{ij} + 48A_0 \delta_{ij} n^k n^l \dot{\mathcal{E}}_{kl} - 48A_0 n_i n_j n^k n^l \dot{\mathcal{E}}_{kl} + 48A^k \mathcal{E}_{ijk} - 96n^k \mathcal{E}_{ijk} + 48A^k n_k n^l \mathcal{E}_{ijl} \\
& - 48A^k \delta_{ij} n^l n^m \mathcal{E}_{klm} + 48A^k n_i n_j n^l n^m \mathcal{E}_{klm} + 6\delta_{ij} n^k n^l n^m \mathcal{E}_{klm} - 16A^k \delta_{ij} n_k n^l n^m n^p \mathcal{E}_{lmp}\Big] + O(r^2)
\end{aligned}
\tag{B6}
$$