Self-force on a scalar particle on a generic orbit in Kerr spacetime

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in collaboration with

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Outline

Goals, overall plan of the computation

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Brief review of effective-source (puncture-function) method

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Some details of the computation

- time-domain evolution with AMR
- *m*-mode decomposition and 2+1D evolution
- worldtube scheme
  - moving the worldtube
- computing the puncture function and effective source
  - efficient numerical computation of the ϕ integrals
  - elliptic integrals $\Rightarrow$ **much** more efficient computation of the ϕ integrals
  - efficiently evaluating the series expansion
- finite differencing for C^2 functions near the particle

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Preliminary results

Conclusions

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  phase error $\lesssim 0.01$ radians over $\sim 10^5$ orbits of inspiral)
- as efficient as possible (orbital evolution!)

This is work in progress: some goals accomplished, some not yet!

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Worldtube scheme to treat far-from-the-particle region

- wordtube moves in (r, θ) to follow the particle around its orbit

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We construct a "puncture function" φ_p which approximates the Detweiler-Whiting singular field $\varphi_{\text{singular}}$, such that for some $n > 0$

- $\varphi_p - \varphi_{\text{singular}} = \mathcal{O}\left(\left|x - x_{\text{particle}}(t)\right|^n\right)$ close to the particle
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In practice we choose $n = 4$,
$\Rightarrow S_{\text{effective}}$ is C^0 near the particle; φ_r is C^2 near the particle

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m-mode decomposition

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in 3+1D, we Fourier-decompose and solve for each Fourier mode in 2+1D:

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  [numerically solve this for each m in 2+1D]

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- **move the wordtube in $(\mathbf{r}, \theta)$ to follow particle around eccentric orbit**

- **more efficient than numerical evolution in 3+1D**
  - **can use different numerical parameters for each m**
  - multiple 2+1D evolutions vs. single 3+1D evolution

Series expansions for the puncture fn and effective src

For a given t, Barry Wardell's puncture function φ_p is a series expansion

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where $\delta x^i := x^i - x^i_{\text{particle}}$

and where the N_{ijk} and D_{ijk} coefficients (there are 30–50 of them) depend on the central black hole's mass and spin and on the particle position and 4-velocity, but do **not** depend on δx^i.

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N_{ijk} and D_{ijk} coefficients computed by Barry Wardell via lengthly symbolic-algebra manipulations of the series expansions. (Details omitted here.)

Computing the m-mode effective src (and puncture fn)

We need to compute the Fourier integral

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- **this is $\sim 300\times$ faster than even an optimized numerical ϕ integration**
- the elliptic-integral form should also be more accurate

Initial data, boundary conditions

Initial data:

- start evolution with arbitrary initial data ($\varphi_{\mathsf{num},m} = 0$)
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  generic orbit: evolve until $\varphi_{\mathrm{num},m}$ is the same for different initial data choices (integrated in parallel)

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- in practice: for $\varphi_{\mathrm{num},m} = 0$ initial data, boundary reflections are only significant when outgoing junk radiation reaches the boundaries
  $\Rightarrow$ domain only needs to be about $\frac{1}{2}$ the causally-disconnected size to reduce boundary reflections to a negligible level

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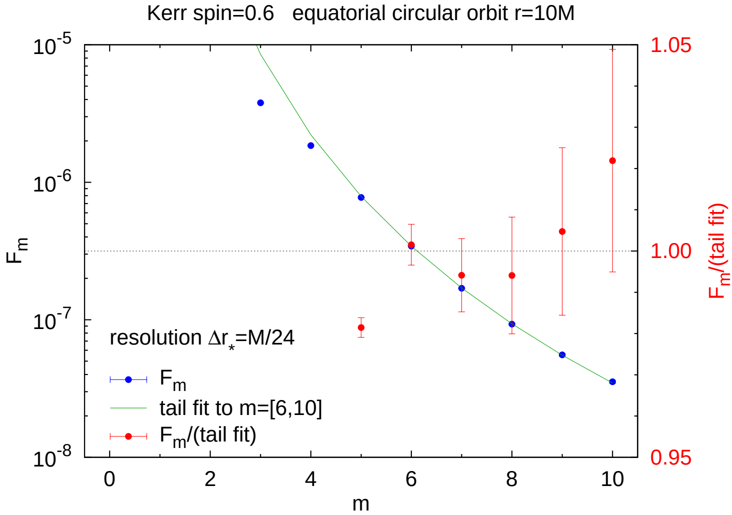
- code currently uses a uniform grid in (r_*, θ)
- typical worldtube size $\sim 4M$ in r_*, $\pi/8$ (22.5°) in θ
- 4th order spatial finite differencing,
  4th order Runge-Kutta time integration

Falloff of F^m at large m for equatorial circular orbit

Should have $F_m \sim \frac{k_4}{m^4} + \frac{k_5}{m^5} + \frac{k_6}{m^6} + \cdots$ at large m



Self-force for equatorial circular orbit

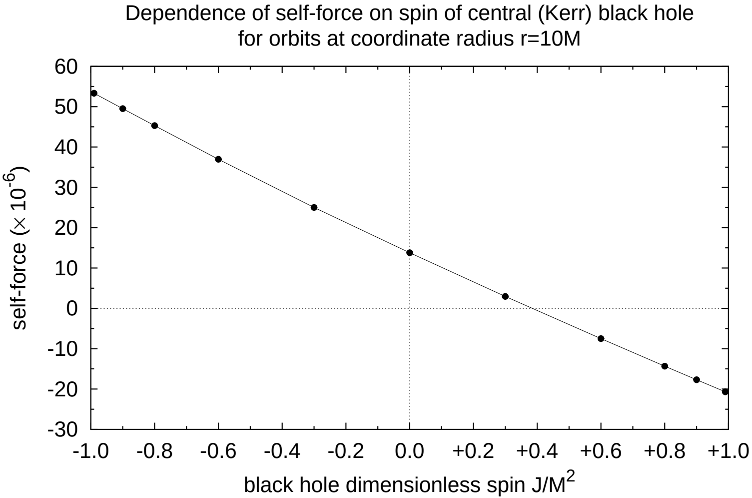
Kerr spin 0.6, particle in equatorial circular orbit at $r = 10M$

Self-force in units of $10^{-6} q^2/M^2$:

this work:	-7.4999 ± 0.0060	(resolution $\Delta r_* = M/24$)
	of which $1/m^4$ tail sum ($m \geq 11$) is -0.8%	
	$1/m^5$ tail sum ($m \geq 11$) is -0.4%	
	total tail sum ($m \geq 11$) is -1.2%	
Niels Warburton:	-7.491205	(very accurate frequency-domain calculation)
difference (error):	-0.0087 ± 0.0060	($\sim 0.1\%$)

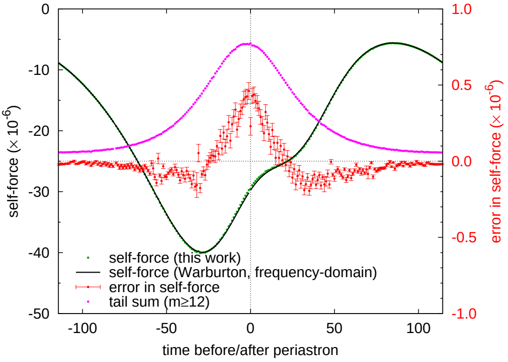
Dependence of self-force on central black hole spin

particle in equatorial circular orbit at $r = 10M$
resolution: $\Delta r_* = M/16$



Self-force for low-eccentricity orbit

Kerr spin 0.6, particle in equatorial eccentric orbit: $p = 8M$, $e = 0.2$
resolution: $\Delta r_* = M/12$



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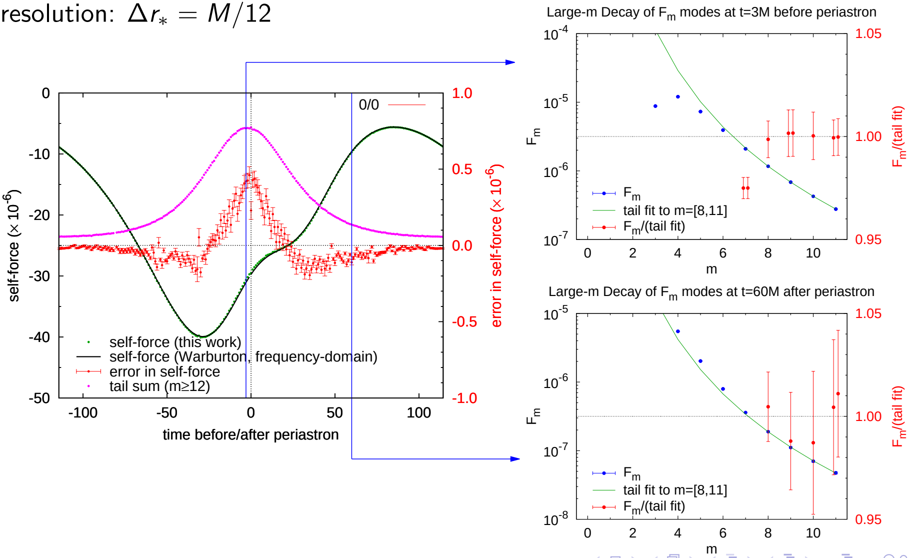
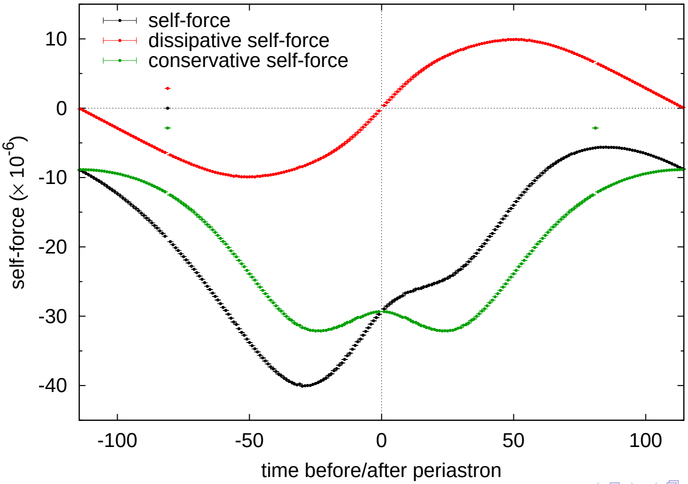


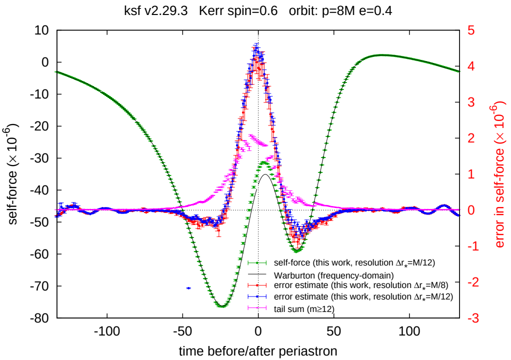
Conservative and dissipative parts of the self-force

Kerr spin 0.6, particle in equatorial eccentric orbit: $p = 8M$, $e = 0.2$
resolution: $\Delta r_* = M/12$



Self-force for moderately eccentric orbit

Kerr spin 0.6, particle in equatorial eccentric orbit: $p = 8M$, $e = 0.4$
resolution: $\Delta r_* = M/12$



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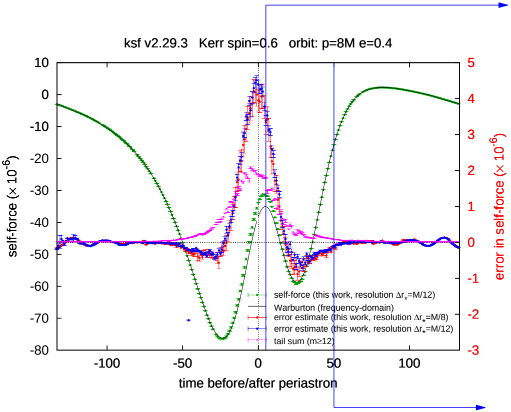

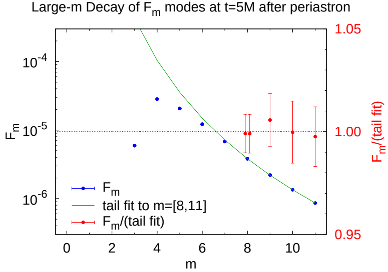

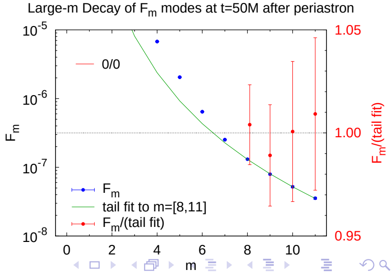

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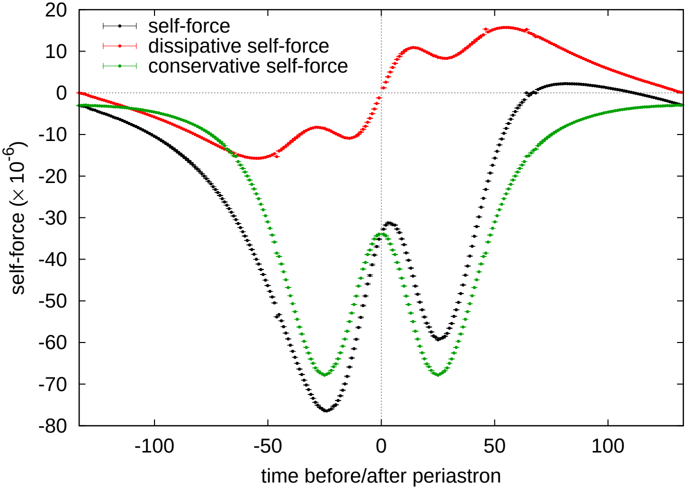
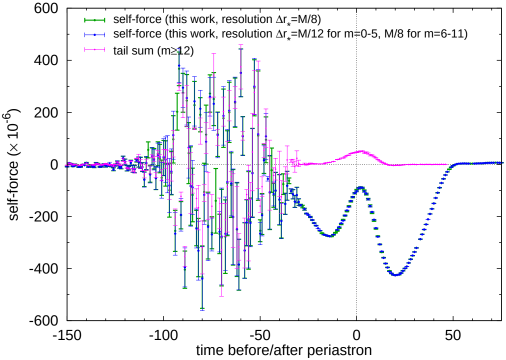


“Self-force” for highly eccentric orbit

Kerr spin 0.6, particle in equatorial eccentric orbit: $p = 8M$, $e = 0.9$
resolution: mixture of $\Delta r_* = M/12$ and $M/8$



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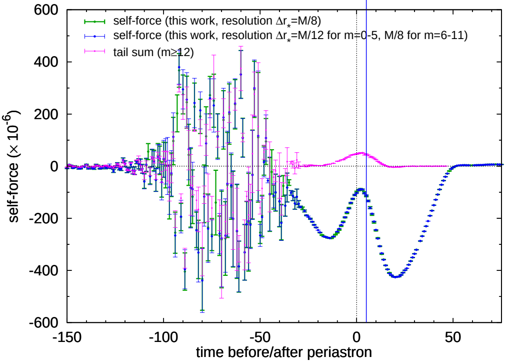

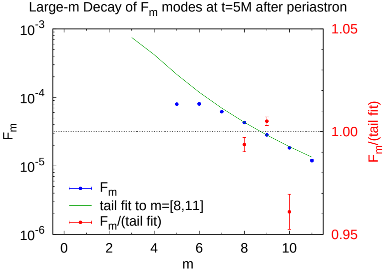

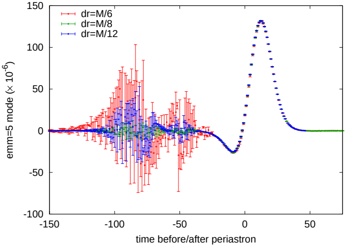

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- eccentric-orbit elliptic integrals and AMR shoulod greatly improve this; better finite differencing at the particle should also help