Self-force: Numerical Implementations

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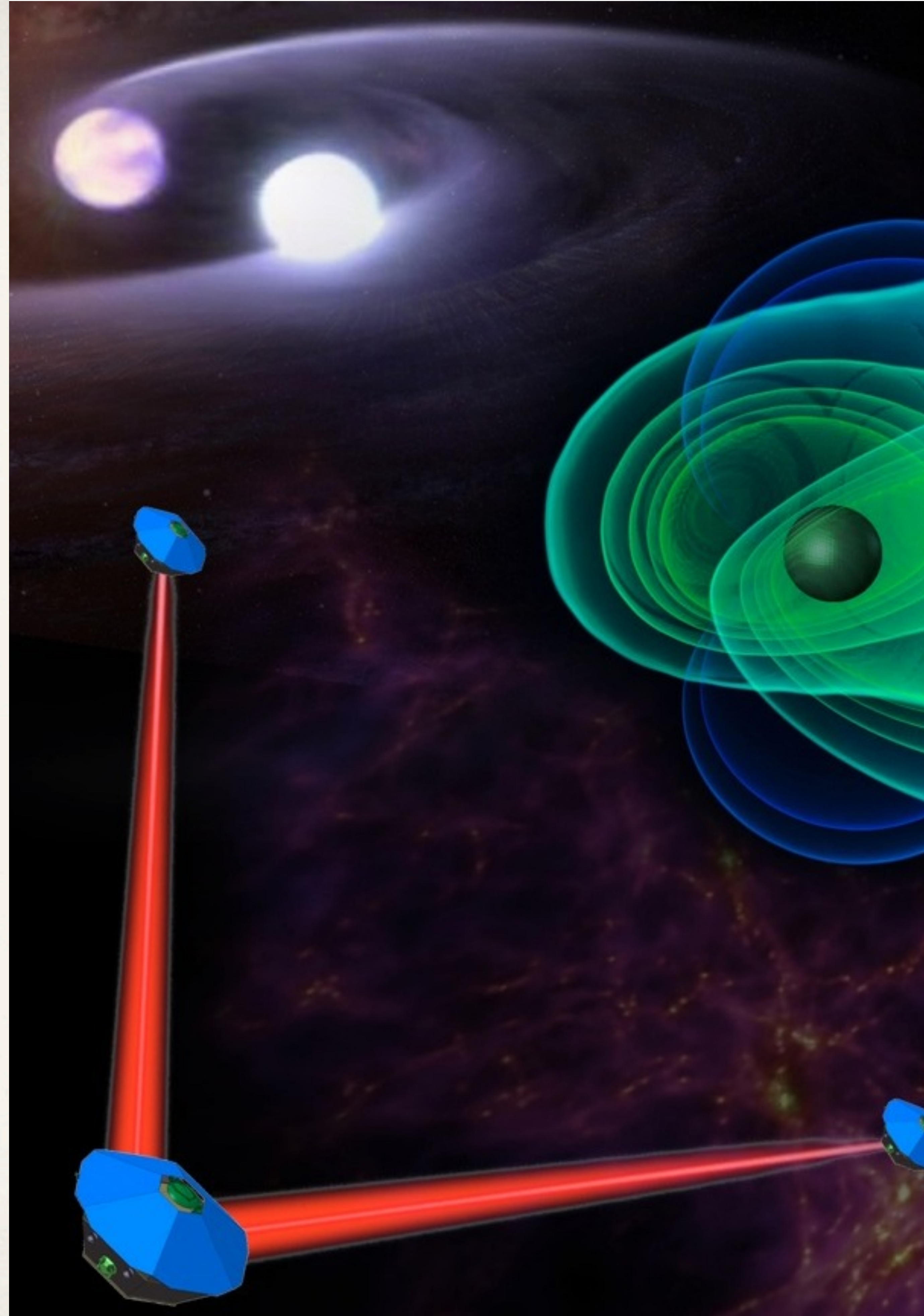
EMRIs

- A major goal of the Capra programme is to study EMRIs.
- Many orbits
- No strong reasons not to expect generic orbits.
- Larger black hole generally spinning.
- Ultimate goal: $\sim 10^4$ accurate gravitational self-force evolved generic orbits in Kerr



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Formal prescription at first order

- Foundations and formalism by now well understood at first order.
- Solve the coupled system of equations for the motion of a point particle and its retarded field.

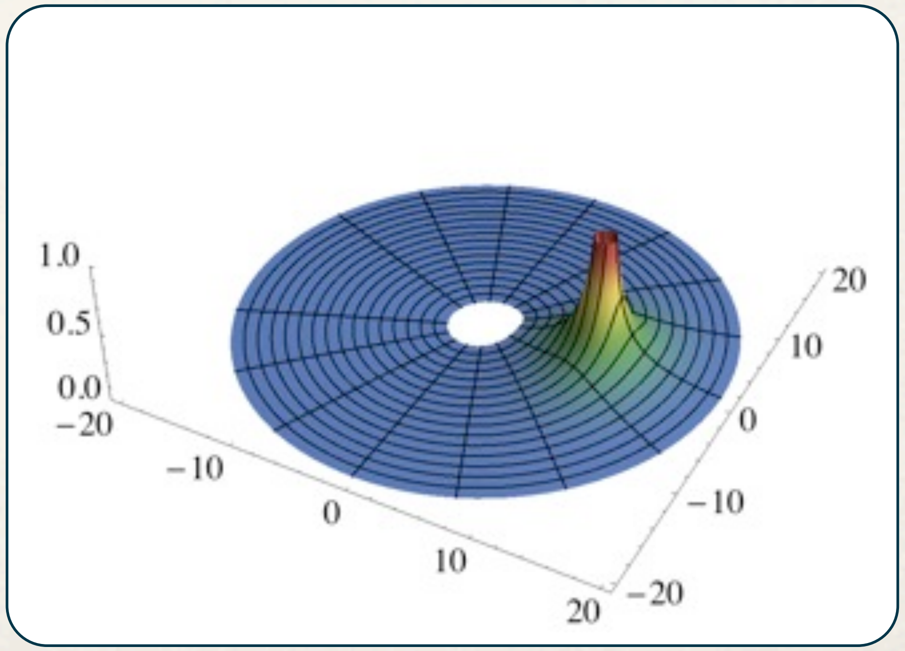
Scalar	Electromagnetic	Gravitational
$\Box\Phi^{\mathrm{ret}} = -4\pi q \int \frac{\delta^4(x - z(\tau))}{\sqrt{-g}} d\tau$	$\Box A^{\mathrm{ret}}_a - R_a{}^b A^{\mathrm{ret}}_b = -4\pi e \int g_{aa'} u^{a'} \sqrt{-g}\delta_4(x, z(\tau)) d\tau$	$\Box\bar{h}^{\mathrm{ret}}_{ab} + 2C_a{}^c{}_b{}^d \bar{h}^{\mathrm{ret}}_{cd} = -16\pi\mu \int g_{a'(a} u^{a'} g_{b)b'} u^{b'} \sqrt{-g}\delta_4(x, z(\tau)) d\tau$
$\Phi^{\mathrm{R}} = \Phi^{\mathrm{ret}} - \Phi^{\mathrm{S}}$	$A^{\mathrm{R}}_a = A^{\mathrm{ret}}_a - A^{\mathrm{S}}_a$	$\bar{h}^{\mathrm{R}}_{ab} = \bar{h}^{\mathrm{ret}}_{ab} - \bar{h}^{\mathrm{S}}_{ab}$
$f_a = \nabla_a \Phi^{\mathrm{R}}$	$f^a = g^{ab} u^c A^{\mathrm{R}}{}_{[c,b]}$	$f^a = k^{abcd} \bar{h}^{\mathrm{R}}_{bc;d}$

$$a^\alpha = (g^{\alpha\beta} + u^\alpha u^\beta) f_\beta \qquad \frac{dm}{d\tau} = u^\beta f_\beta$$

Numerical considerations

Several considerations arise when trying to turn this formal prescription into a practical numerical scheme:

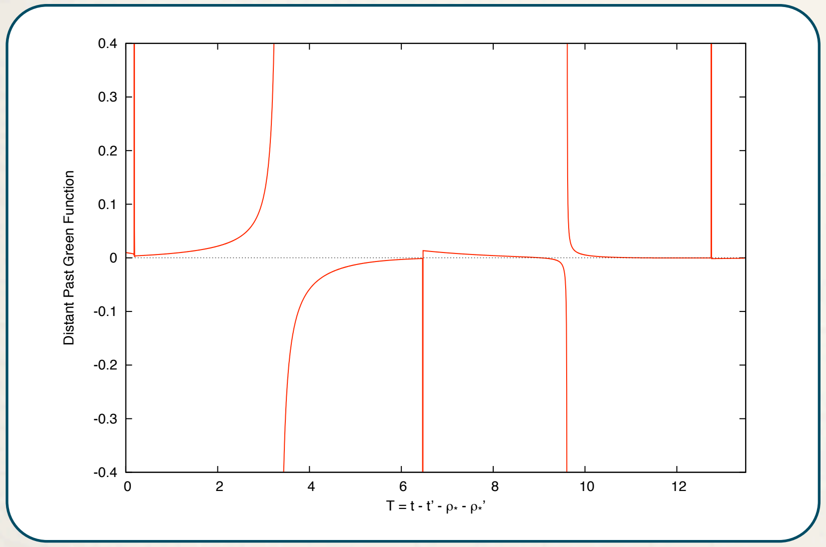
- System is coupled: $\mathbf{\Phi}^{\text{ret}}$ depends on the entire past world-line and the world-line depends on $\mathbf{\Phi}^{\text{ret}}$ => delay differential equation.
- δ-function sources are difficult to handle numerically.
- $\mathbf{\Phi}^{\text{ret}}$ diverges like 1/r near the world-line.



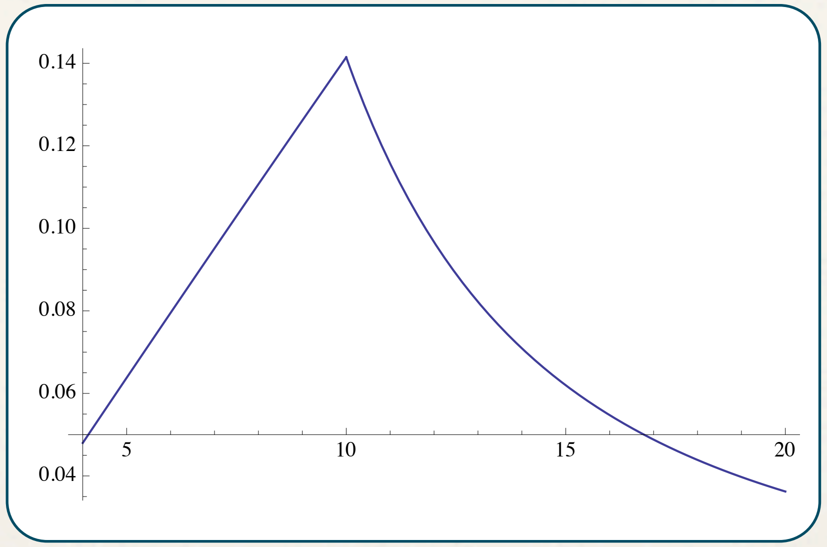
Approaches

- Several approaches have been developed for dealing with the numerical problems of point sources and singular fields.
- These broadly fall into three different categories

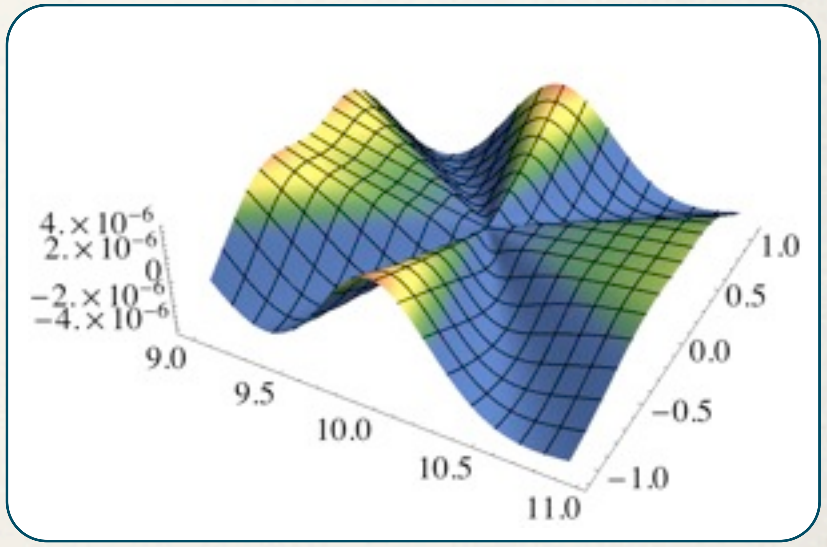
Green function



Mode-sum



Effective Source



Green function regularization

- MiSaTaQuWa equation gives the regularized self-force in terms of local components and a *tail* term.

$$f^a = (\text{local terms}) + \lim_{\epsilon \to 0} q^2 \int_{-\infty}^{\tau - \epsilon} \nabla^a G_{\text{ret}}(x, x') d\tau'$$

- Local terms are easily calculated.
- Tail contains contribution to the self-force from the past.
- Integral of the retarded Green function over the entire past world-line.

Green function regularization

$$f^a = (\text{local terms}) + \lim_{\epsilon \to 0} q^2 \int_{-\infty}^{\tau - \epsilon} \nabla^a G_{\text{ret}}(x, x') d\tau'$$

- MiSaTaQuWa equation suggests an natural approach for computing a regularized self-force.
- If we can compute the Green function along the world-line, then we're done: just integrate this to get the regularized self-force for any orbit.
- The difficulty is in developing a strategy for computing the Green function over a sufficiently large portion of the world-line.

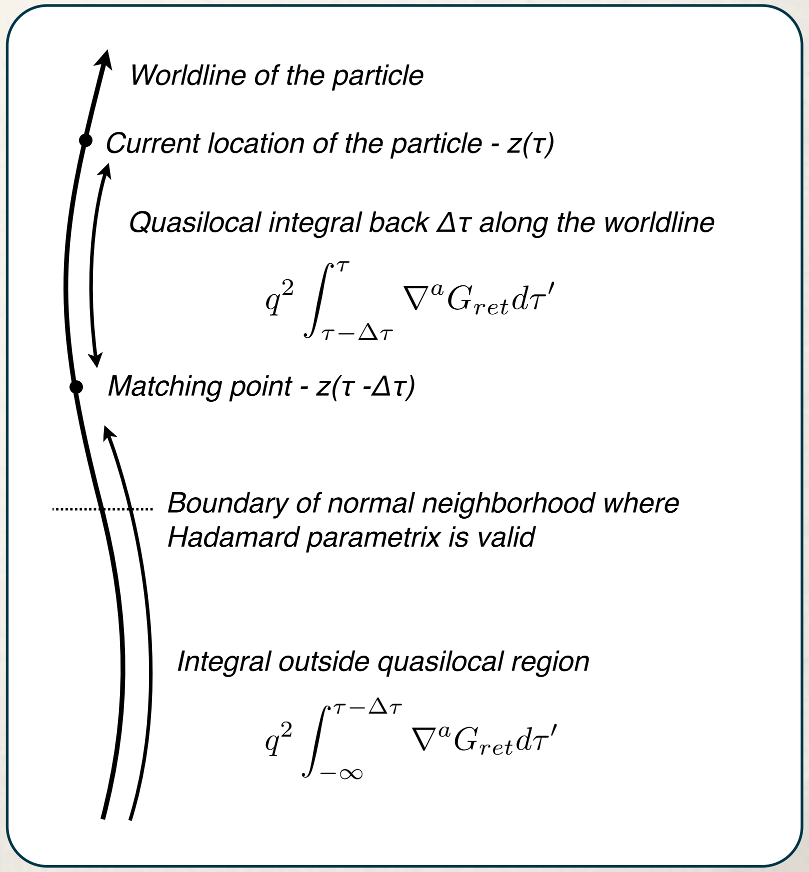
Matched expansions

- One approach is to compute the Green function using matched asymptotic expansions [W. G. Anderson and A. G. Wiseman, Class. Quantum Grav. 22, S783 (2005); M. Casals, S. R. Dolan, A. C. Ottewill, and B. Wardell, Phys. Rev. D 79, 124043 (2009)]

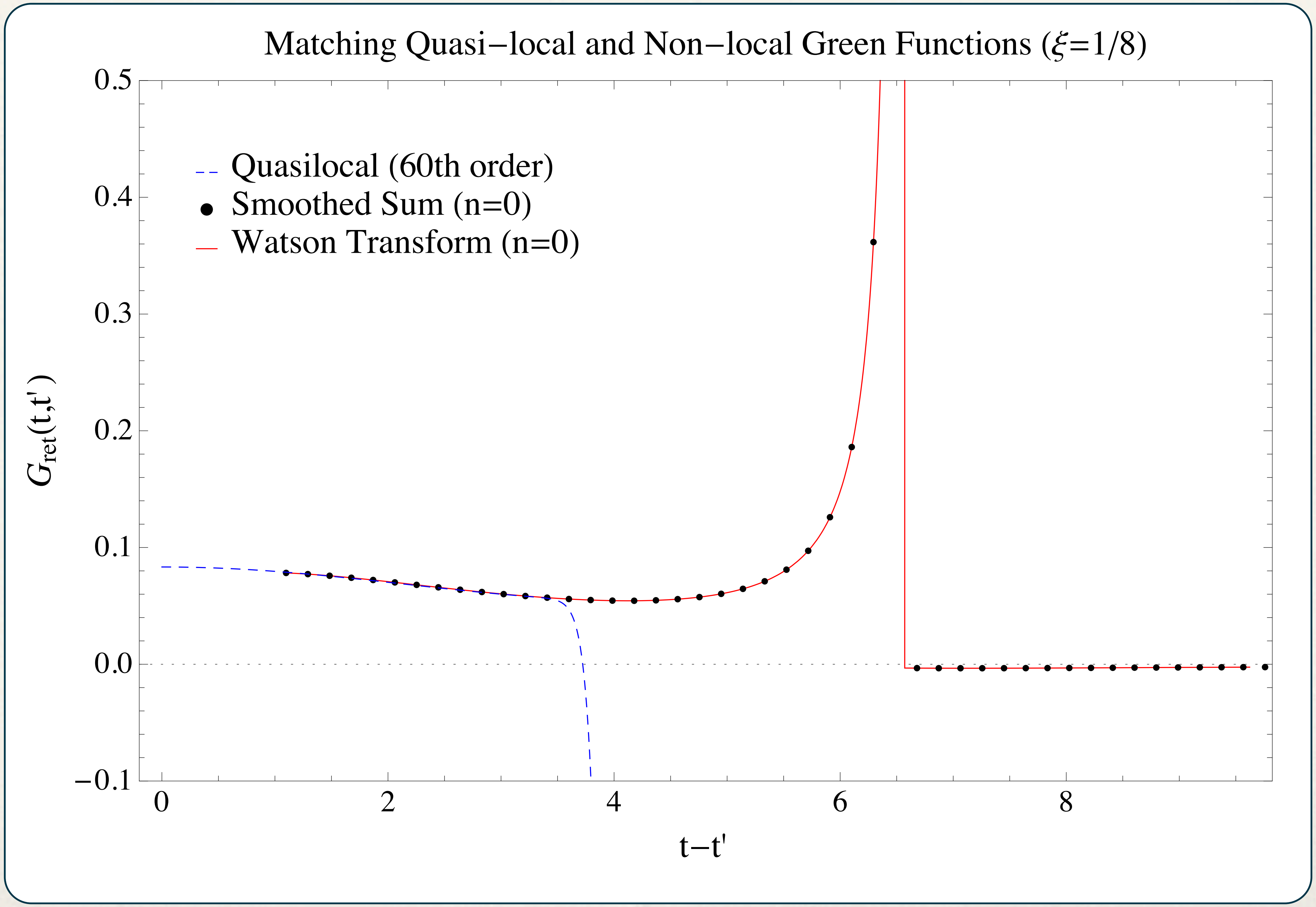
- Separately compute expansions of the Green function in the recent past and in the distant past.

- Recent past obtained through a series expansion of the Hadamard form for the Green function, distant past through a quasi-normal mode sum and branch-cut integral.

- Stitch together expansions in an overlapping matching region to give the full Green function.

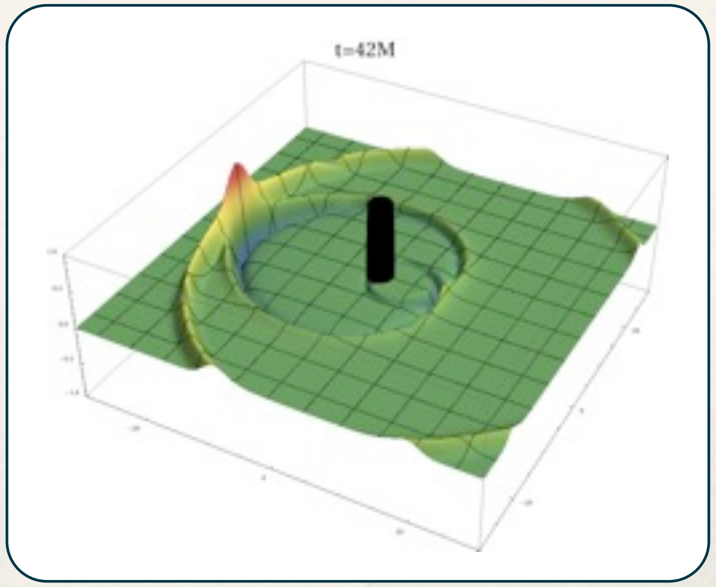


Matched expansions



Numerical time-domain evolution

- Another option is to numerically compute the Green function using time-domain evolution [A. Zenginoğlu, C. R. Galley, arXiv:1206.1109].
- Numerically evolve a wave equation for the Green function, rather than the field.
- Still need to make use of quasi-local expansion for the recent past, but distant past calculation is much easier than with quasi-normal modes + branch cut.



Green function regularization

- Advantages:
  - Just need to compute the Green function once and we have the self-force for all orbits.
  - Avoids numerical cancellation by directly computing the regularized field.
  - May yield geometric insight
  - Green function can be applied to other problems
- Disadvantages:
  - Computing the Green function can be hard.
  - Have to compute the Green function for all pairs of points x and x'.
  - Not naturally suited to self-consistent evolution.
  - Second order not so well understood.

Mode-sum regularization

- The retarded field diverges close to the world-line

$$\Phi^{\rm ret} \sim \frac{1}{r}$$

- If we decompose this into spherical harmonics,

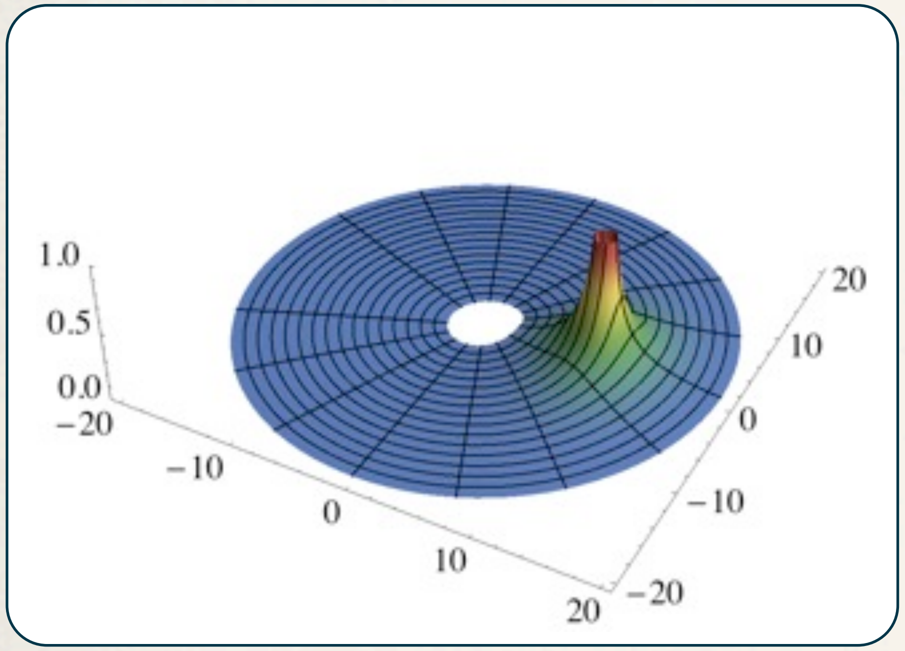
$$\Phi^{\rm ret}_{lm}(t,r) = \int \Phi^{\rm ret} Y^{lm*}(\theta,\phi) d\Omega$$

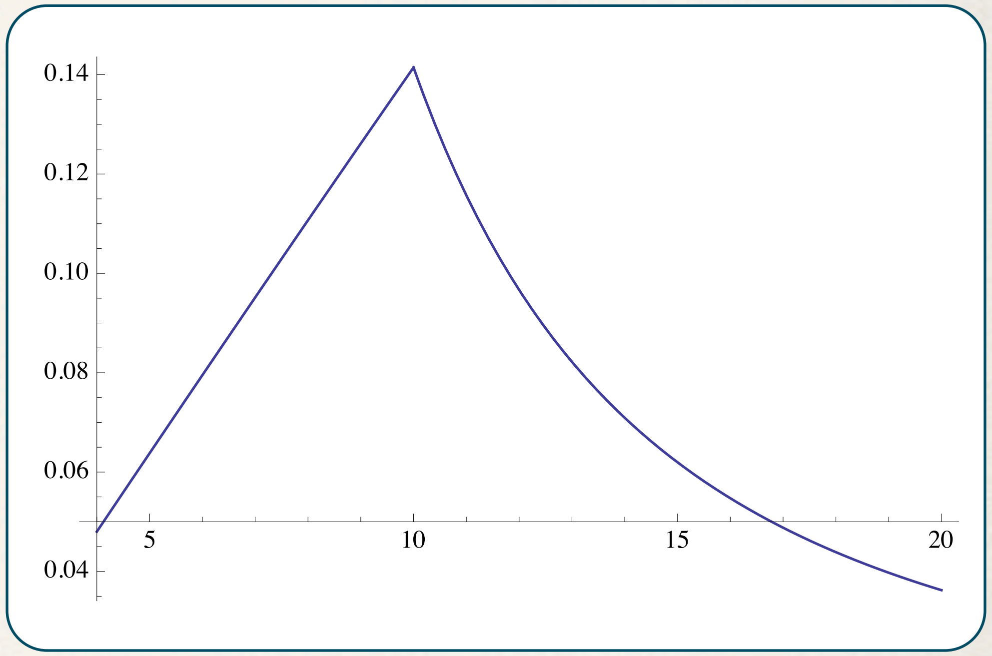
  the singularity is "smeared out" over a 2-sphere. Each *l,m* mode is then finite on the world-line

$$\Phi^{\rm ret}_{lm} \sim |r|$$

- L. Barack and A. Ori, Phys. Rev. D 61, 061502

Mode-sum regularization





Mode-sum regularization

- Solve a 2D wave equation for each *l,m* mode

$$\left[\frac{\partial^2}{\partial r_*^2} - \frac{\partial^2}{\partial t^2} - V_l\right]\Phi_{lm}^{\rm ret} = S_{lm}\delta(r - r_0(t))$$

- Similar equations for electromagnetic and gravitational cases.

- Solution can be found in time domain as either 1+1D or characteristic evolution. δ-function needs careful treatment through particular finite differencing schemes/multi-domain methods.

- In the frequency domain this becomes an ordinary differential equation for each l,m,ω. This is particularly convenient for orbits where the number of frequencies is small (e.g. circular orbits). δ-function appears as matching condition between two homogeneous solutions.

Mode-sum regularization

- Solve a 2D wave equation for each *l*,*m* mode

$$\left[\frac{\partial^2}{\partial r_*^2} - \frac{\partial^2}{\partial t^2} - V_l\right]\Phi^{\text{ret}}_{lm} = S_{lm}\delta(r - r_0(t))$$

- Differentiate and sum over *m* to get *l*-modes of the unregularized self-force

$$f^a_l = \sum_{m=-l}^{l} \nabla^a \Phi^{\text{ret}}_{lm}$$

- Each *l*-mode is finite, but their sum diverges like *l*

$$f^a_l \sim l \quad \text{as} \quad l \to \infty$$

Mode-sum regularization

- In order to regularize, decompose Φ^S into spherical harmonic modes

$$\Phi^{\rm S}_{lm}(t,r) = \int \Phi^{\rm S} Y^{lm*}(\theta,\phi) d\Omega$$

  and subtract mode by mode.

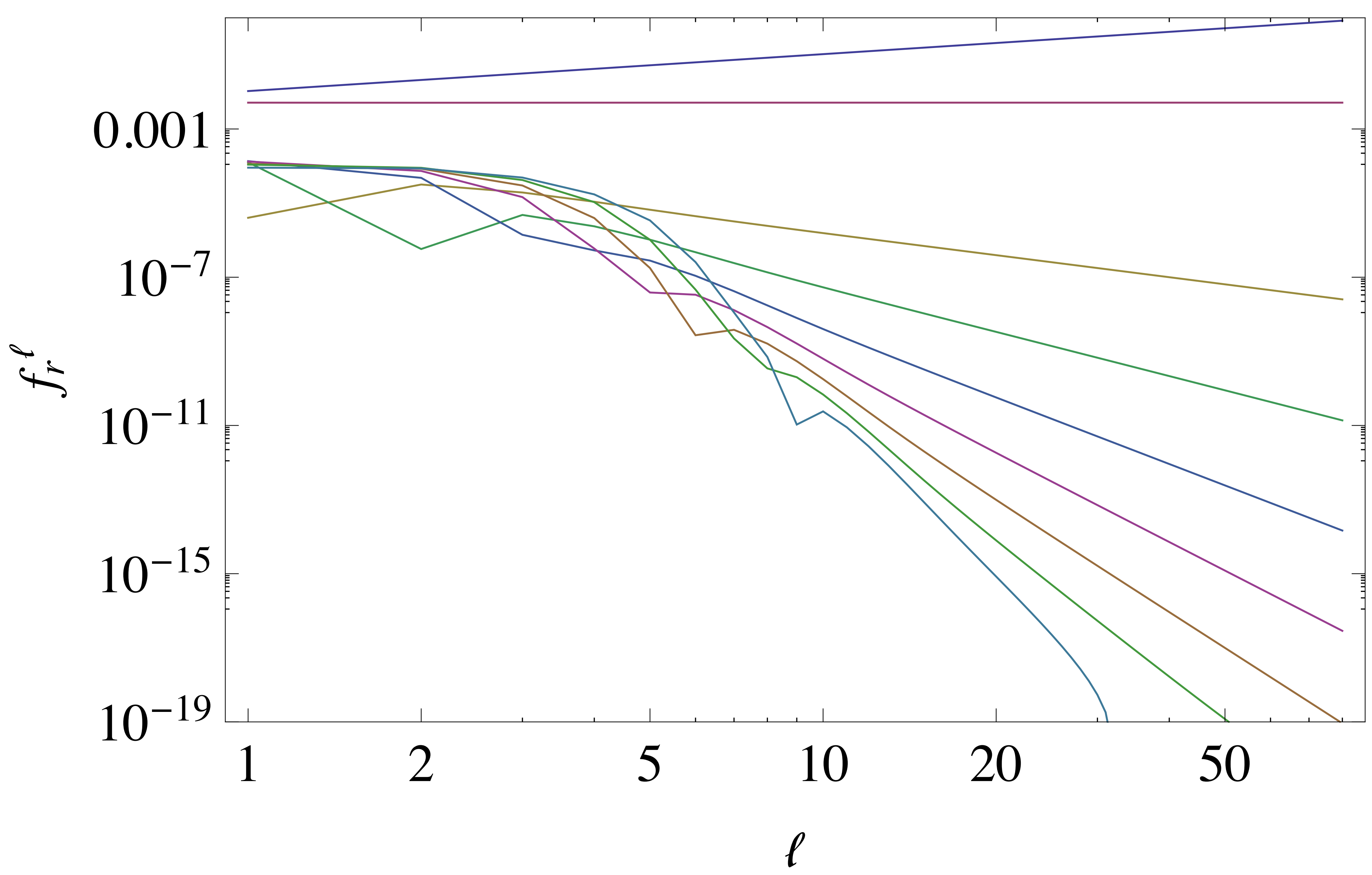
- Typically only know Φ^S approximately as an expansion for large *l*.

- Coefficients of this expansion are known as *regularization parameters.*

- Compute a regularized self-force by subtracting regularization parameters from unregularized self-force

$$[f^a_l]^{\rm R} = \sum_{l=0}^{\infty} f^a_l - A_l(l+\tfrac{1}{2}) - B_l - \cdots$$

Mode-sum regularization



Mode-sum regularization

- Advantages:
  - Suitable for fast, high-accuracy frequency domain calculations.
  - In time domain leads to fast, accurate 1+1D evolutions.
  - Relatively easy to implement
- Disadvantages:
  - Requires numerical cancellation of large quantities.
  - Not particularly suited to Kerr due to use of spherical harmonic decomposition.
  - No clear extension to second order yet.
  - Not naturally suited to self-consistent evolution.

Mode-sum regularization

- The success of the mode-sum approach is clear from the number of papers based on it.

R. Haas and E. Poisson, Phys. Rev. D74, 044009 (2006)
L. Barack and L. M. Burko, Phys. Rev. D62, 084040 (2000)
L. M. Burko, Phys. Rev. Lett. 84, 4529 (2000)
S. Detweiler, E. Messaritaki, and B. F. Whiting, Phys. Rev. D67, 104016 (2003)
L. M. Diaz-Rivera, E. Messaritaki, B. F. Whiting, and S. L. Detweiler, Phys. Rev. D70, 124018 (2004)
R. Haas, Phys. Rev. D75, 124011 (2007)
P. Canizares and C. F. Sopuerta, Phys. Rev. D79, 084020 (2009)
P. Canizares, C. F. Sopuerta, and J. L. Jaramillo, Phys. Rev. D82, 044023 (2010)
L. Barack and N. Sago, Phys. Rev. D75, 064021 (2007)
L. Barack and C. O. Lousto, Phys. Rev. D66, 061502 (2002)
N. Sago, L. Barack, and S. Detweiler, Phys. Rev. D 78, 124024 (2008)
S. Detweiler, Phys. Rev. D 77, 124026 (2008)
N. Sago, Classical and Quantum Gravity 26, 094025 (2009)
L. Barack and N. Sago, Phys. Rev. D81, 084021 (2010)
T. S. Keidl, A. G. Shah, J. L. Friedman, D.-H. Kim, and L. R. Price (2010)
A. Shah, T. Keidl, J. Friedman, D.-H. Kim, and L. Price (2010)
N. Warburton and L. Barack, Phys. Rev. D81, 084039 (2010)
N. Warburton and L. Barack, in preparation.
J. Thornburg (2010)
R. Haas (2011)
N. Warburton, S. Akcay, L. Barack, J. R. Gair, and N. Sago, Phys.Rev. D85, 061501 (2012)
S. Hopper and C. R. Evans, Phys.Rev. D82, 084010 (2010)
T. S. Keidl, A. G. Shah, J. L. Friedman, D.-H. Kim, and L. R. Price, Phys.Rev. D82, 124012 (2010)
A. G. Shah, T. S. Keidl, J. L. Friedman, D.-H. Kim, and L. R. Price, Phys.Rev. D83, 064018 (2011)
L. Barack and A. Ori, Phys. Rev. D 66, 084022 (2002)
N. Mino and Sasaki, Prog. Theor. Phys. 108, 1039 (2002)
L. Barack and N. Sago, Phys.Rev. D83, 084023 (2011)

Effective source regularization

- Derive an evolution equation for Φ^{R}
  [Barack and Golbourn (2007), Detweiler and Vega (2008)]

- Always work with Φ^{R} instead of Φ^{ret}.

- No distributional sources and no singular fields.

- If Φ^{S} is chosen appropriately, then we can directly use Φ^{R} in the equations of motion.

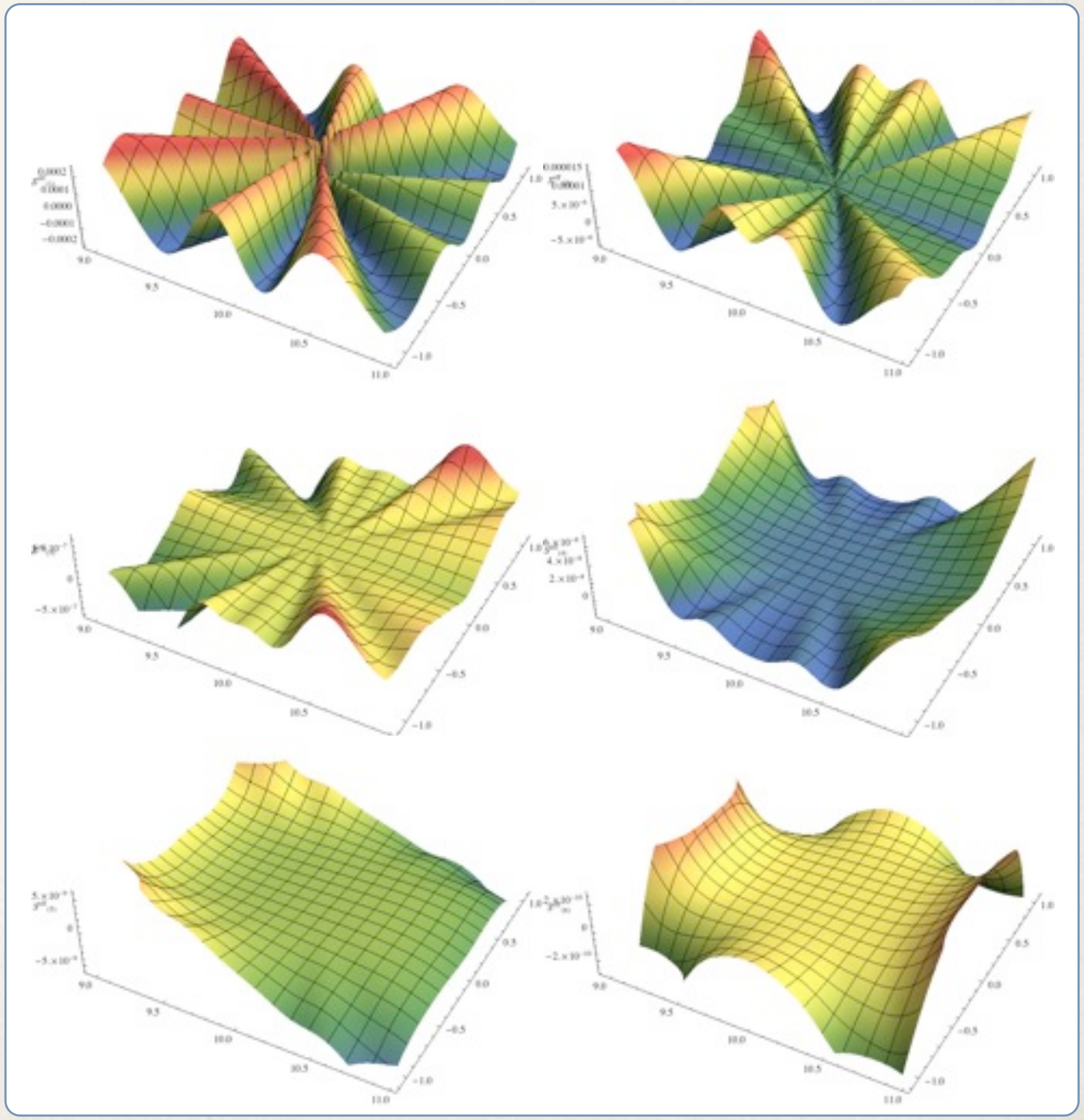
$$
\begin{aligned}
\Box\Phi^{\mathrm{R}} &= \Box\Phi^{\mathrm{ret}} - \Box\Phi^{\mathrm{S}} \\
&= -4\pi q \int \frac{\delta^4(x - z(\tau))}{\sqrt{-g}} d\tau - \Box\Phi^{\mathrm{S}}
\end{aligned}
$$

$$
\frac{Du^\alpha}{d\tau} = a^\alpha = \frac{\bar{q}}{m(\tau)}(g^{\alpha\beta} + u^\alpha u^\beta)\nabla_\beta\Phi^{\mathrm{R}}
$$

$$
\frac{dm}{d\tau} = -\bar{q}u^\beta\nabla_\beta\Phi^{\mathrm{R}}
$$

Effective source regularization

- If Φ^{S} is exactly the Detweiler-Whiting singular field, Φ^{R} is a solution of the homogeneous wave equation.
- If Φ^{S} is only approximately the Detweiler-Whiting singular field, then the equation for Φ^{R} has an effective source, *S*.
- *S* is typically finite, but of limited differentiability on the world line.



Window function

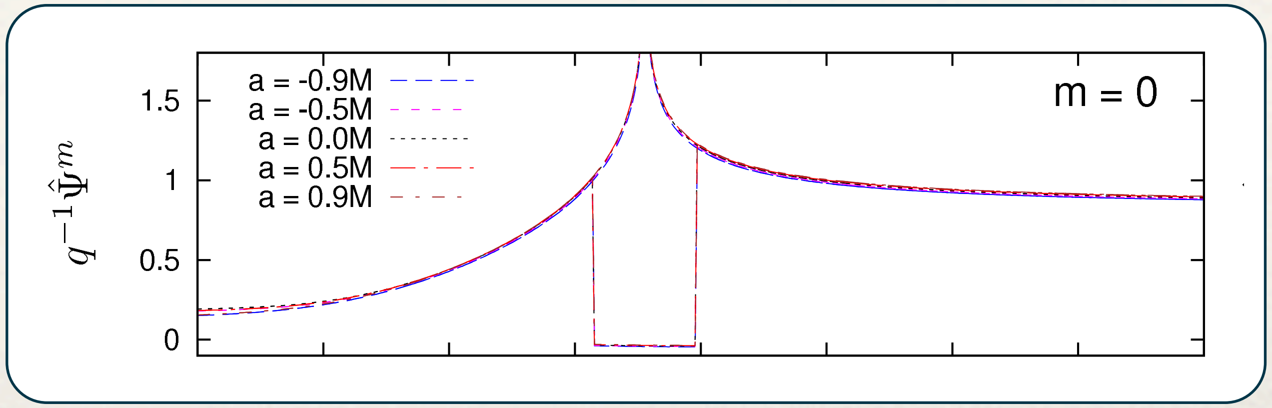
- Detweiler-Whiting singular field defined through a Hadamard form Green function which is not defined globally.
- Need to introduce a method for restricting the singular field to a region near the particle.
- Two common approaches: window function and world-tube.
- In window function approach, we multiply the singular field by a function which is 1 at the particle and goes to 0 far away:

$$\Box\Phi^{\mathrm{R}} = -\Box(W\Phi^{\mathrm{S}})$$

World tube

- Alternatively, introduce a tube around the particle.
- Inside the tube, solve for the regularized field, outside solve homogeneous equation for the full retarded field.
- On the world-tube boundary, apply the boundary condition

$$\Phi^{\mathrm{ret}} = \Phi^{\mathrm{S}} + \Phi^{\mathrm{R}}$$



2+1D m-mode scheme

- Exploit the azimuthal symmetry of the Kerr spacetime by decomposing into Fourier modes

$$\Phi_m(r,\theta,t) = \int_{-\pi}^{\pi} \Phi e^{-im\phi} d\phi$$

- Evolve 2+1D wave equation in (r,θ,t) for each *m*-mode.

$$\Box_m \Phi_m(r,\theta,t) = S_m$$

- Potentially improved efficiency relative to 3+1D.

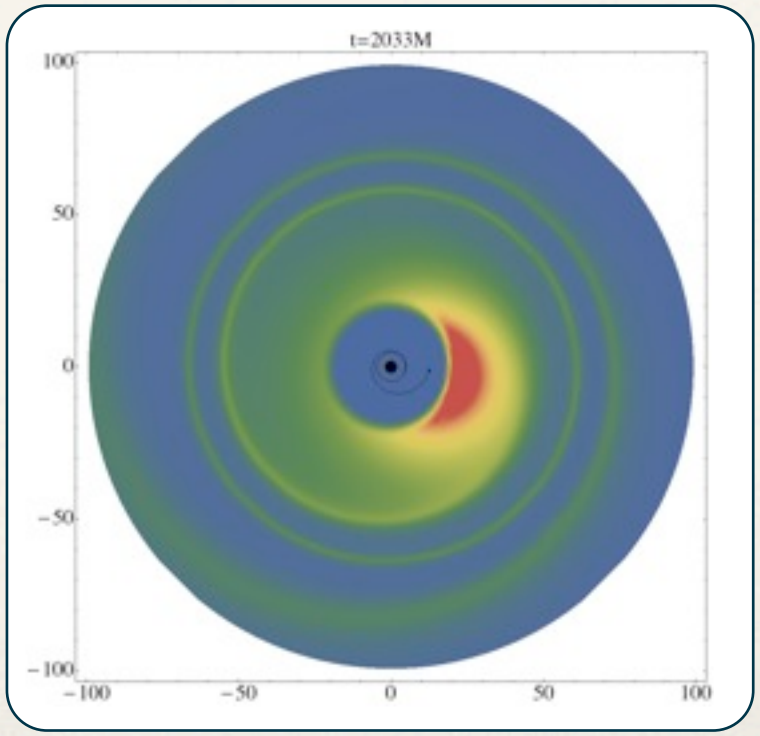
- Needs a 2D effective source. This may be computed by either numerically or analytically integrating the 3D effective source.

3+1D scheme

- Evolve the full 3+1 dimensional wave equation.

$$\Box\Phi = S$$

- Reuse much of the code and tools developed for Numerical Relativity.



Self-consistent evolution

- Solve the coupled system of equations for the motion of the particle and its regularized field.
- $\Phi^{\mathrm{R}} = \Phi^{\mathrm{ret}}$ in the wave zone
- Φ^{R} finite and (typically) twice differentiable on the world-line

$$\Box \Phi^{\mathrm{R}} = S(x|z(\tau), u(\tau))$$

$$\frac{Du^{\alpha}}{d\tau} = a^{\alpha} = \frac{\bar{q}}{m(\tau)}(g^{\alpha\beta} + u^{\alpha}u^{\beta})\nabla_{\beta}\Phi^{\mathrm{R}}$$

$$\frac{dm}{d\tau} = -\bar{q}u^{\beta}\nabla_{\beta}\Phi^{\mathrm{R}}$$

Effective source regularization

- Advantages:
  - Everything is finite. No distributional sources or singular fields.
  - Does not rely on any underlying symmetry. Can be applied to generic orbits in generic spacetimes.
  - Naturally suited to self-consistent evolution.
  - Possible extension to second order
- Disadvantages:
  - Relatively costly computationally when evolved in 2+1D or 3+1D.
  - Effective source is often a very complicated expression.
  - Problems with evolving Lorenz gauge metric perturbations in time domain.

Conclusions and prospects

- Many different approaches to numerical self-force calculations.
- Each has advantages and disadvantages; no clear winner.
- All existing calculations purely at first perturbative order.
- Key component of most calculations is knowledge of the Detweiler-Whiting singular field. The more accurately this is known, the more accurately we can compute the self-force
- Recent formal work to extend things to second order [Detweiler (2012), Pound (2012), Gralla (2012)]. Second order singular field and effective source.