Osculating Evolution to Model EMRI Resonances

Priscilla Canizares

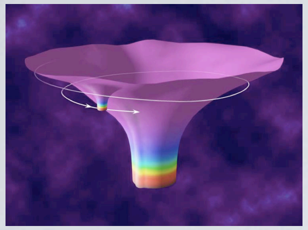
Institute of Astronomy. Cambridge, UK

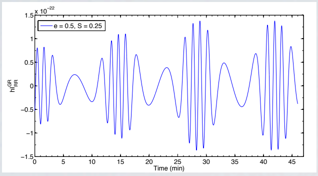
16th Capra meeting. Dublin

Outlook

Warburton, van de Meent, Tanaka, Isoyama and Dolan's talks

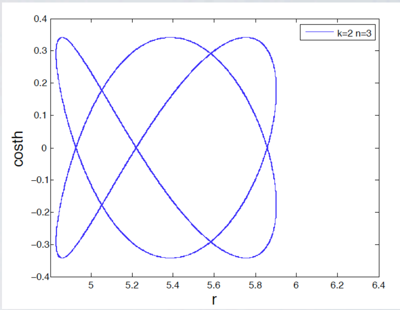
- Motivation: resonances in EMRIs systems
- Osculating element formalism
- Action angle variable formalism
- Osculating evolution + action angle variable for studying resonances



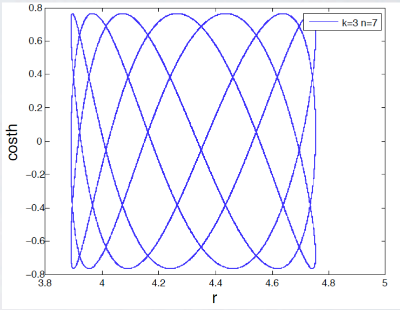


Motivation

- Extreme-mass-ratio inspirals (EMRIs) can cross several resonant points during its evolution. This resonant points affect the orbital dynamics, and hence the gravitational waves (GWs) emitted and their fluxes, and possibly could have an impact on GW detection and parameter estimation.



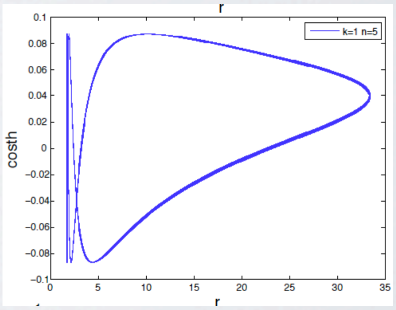
- Resonant behavior is not chaotic, but makes the system highly dependent on the parameters at the moment that it enters in a resonant region.



Modeling

- Resonances shift the orbital phase (by several tens to ~10^2 rd)

Detection



- The location of the resonances depend on the spacetime geometry of the system. Resonances might help to study strong-gravity aspects.

[Flanagan & Hinderer (2012), Flanagan et al (2012), Brink et al (2013)]

Motivation

- To determine whether resonances are important, it is necessary to model EMRI systems as they evolve through them, and evaluate its effects:

    - Need of a formalism to evolve the system through resonances

Osculating evolution + Action angle variable formalism

Osculating Evolution
Newtonian systems

Osculating Evolution. Newtonian Systems

Celestial mechanics: The method of variation of constants for treating highly nonlinear problems [$\vec{\boldsymbol{r}}=(x,\, y,\, z)$]

$$\ddot{\vec{\mathbf{r}}} + \frac{\mu}{r^2}\,\frac{\vec{\mathbf{r}}}{r} = 0 \longrightarrow \ddot{\vec{\mathbf{r}}} + \frac{\mu}{r^2}\,\frac{\vec{\mathbf{r}}}{r} = \Delta\vec{\mathbf{F}}$$

$$\vec{\boldsymbol{r}} \equiv \vec{\boldsymbol{r}}_{planet} - \vec{\boldsymbol{r}}_{sun} \qquad \mu \equiv G(m_{planet} + m_{sun})$$

- **Initial set of second order ordinary differential equations as a particular solution of an inhomogeneous or driven system.**

Osculating Evolution. Newtonian Systems

- The constants of motion are allowed to evolve with time

$$\vec{\mathbf{r}} = \vec{\mathbf{f}}\,(C_1(t), ..., C_6(t),\, t)$$

Six constants, $C_i(t)$ for three, second order, differential equations

$$(\, x(t)\,,\ y(t)\,,\ z(t)\,,\ \dot{x}(t)\,,\ \dot{y}(t)\,,\ \dot{z}(t)\,)$$

and satisfy a specified constraint given by the forcing term

$$\frac{d\vec{\boldsymbol{r}}}{dt} = \frac{\partial \vec{\mathbf{f}}}{\partial t} + \sum_i \frac{\partial \vec{\mathbf{f}}}{\partial C_i}\frac{dC_i}{dt} \longrightarrow \ddot{\vec{\mathbf{r}}} + \frac{\mu}{r^2}\frac{\vec{\mathbf{r}}}{r} = \Delta\vec{\mathbf{F}}$$

- Three second order independent scalar differential for six variables

Ambiguity in parameterizing the orbit

The number of variables exceeds, by three, the number of equations $\{C_i, \dot{C}_i\}$ $(i = 1, ..., 6)$

Lagrange decided to make the instantaneous orbital elements $C_i(t)$ osculating, i.e., model trajectory as instantaneous ellipses tangential to the physical trajectory.

$$\boxed{\sum_i \frac{\partial \vec{\mathbf{f}}}{\partial C_i}\frac{dC_i}{dt} = 0}$$

To bear in mind

- Two time scales: the time scales associated with the solution of the homogeneous solutions, are embedded in the inhomogeneous problem with a driver that evolves with time in a different way

- The osculating conditions do not influence the shape of the physical trajectory and neither the rate of motion along that curve. We could choose some different supplementary condition

$$\boxed{\sum_i \frac{\partial \vec{\mathbf{f}}}{\partial C_i} \frac{d C_i}{dt} = \vec{\mathbf{\Phi}}(C_{1,\ldots,6}\,,\ t)}$$

the physical motion does not change (the expressions are more involved), but eventually could yield different solutions for the orbital elements.

- Accumulation of numerical errors could lead to a ”gauge shift” (non-zero RHS)

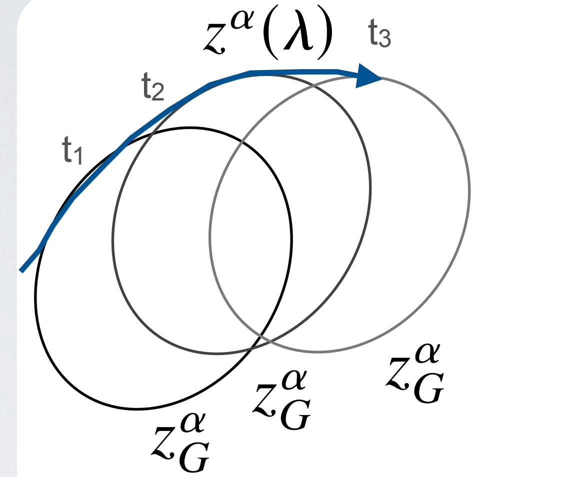
Efroimsky (2002)

Osculating Evolution
General Relativity

Osculating Evolution for EMRI Systems

Pound & Poisson (2008) developed a method to integrate the equations of motion that govern bound, accelerated orbits in Schwarzschild spacetime. Valid for arbitrary forces acting only within the orbital plane

$$\ddot{z}^{\alpha} + \Gamma^{\alpha}{}_{\beta\gamma}\dot{z}^{\beta}\dot{z}^{\gamma} = f^{\alpha},$$



$$I^{A} = (\vec{r}, \dot{\vec{r}})$$

$$I^{A} = \{p, e, w, T, \Phi\}$$

At each instant the true worldline is assumed to lie tangent to a reference geodesic: osculating orbit.

- The worldline evolves smoothly from one geodesic to other.
- The transition between osculating orbits corresponds to an evolution of the elements.

$$z^{\alpha}(\lambda) = z_G^{\alpha}(I^{A}(\lambda), \lambda),$$

$$\frac{dz^{\alpha}}{d\lambda}(\lambda) = \frac{\partial z_G^{\alpha}}{\partial \lambda}(I^{A}(\lambda), \lambda),$$

$$\xrightarrow{\lambda = \tau}$$

$$\frac{\partial z_G^{\alpha}}{\partial I^{A}}\dot{I}^{A} = 0,$$

$$\frac{\partial \dot{z}_G^{\alpha}}{\partial I^{A}}\dot{I}^{A} = f^{\alpha}.$$

reduces second-order differential equations to a set of coupled first-order equations.

Osculating Evolution for EMRI Systems

Although the method is valid for any perturbing force, it is most useful when the force is small, since we can use it to construct an averaged evolution.

Gair et al (2011) Generalization to Kerr spacetime.

Identify the orbit at any time with a geodesic $I^A = \{E, L_z, \mathcal{Q}, \psi_0, \phi_0, \chi_0\}$ of the unperturbed system ($\delta f^\mu = 0$) that passes through the same position with the same velocity.

$$\frac{Du^\mu}{D\tau} = \frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} - f^\mu_{\mathrm{geo}} = \delta f^\mu$$

The osculating elements evolve as $\dot{I} = \nabla_r I \cdot \dot{\mathbf{r}} + \nabla_v I \cdot \ddot{\mathbf{r}} = \nabla_v I \cdot \delta f$

Final equation follows from orthogonality of acceleration to velocity, i.e., $u_\mu \delta f^\mu = 0$

Osculating Evolution for EMRI Systems

The two relevant phase angles are defined by writing

$$r = \frac{p}{1 + e\cos(\psi - \psi_0)} \qquad \cos^2\theta = \cos^2\theta_{\max}\cos^2(\chi - \chi_0)$$

The Boyer-Lindquist coordinates t and ϕ are not oscillatory. The osculating element equations for $t - t_0$ and $\phi - \phi_0$ are equivalent to integrating the geodesic equations for these coordinates with the evolving orbit on the right hand side.

Kerr geodesics parameterized with "Mino" time: $\mathrm{d}\lambda = \frac{1}{\Sigma}\mathrm{d}\tau$ The radial and polar motion decouple

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2 = [E(r^2+a^2) - aL_z]^2 - \Delta[r^2 + (L_z - aE)^2 + Q] \equiv V_r(r),$$

$$\left(\frac{\mathrm{d}\theta}{\mathrm{d}\lambda}\right)^2 = Q - \cot^2\theta L_z^2 - a^2\cos^2\theta(1 - E^2) \equiv V_\theta(\theta),$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}\lambda} = \mathrm{cosec}^2\theta L_z + aE\left(\frac{r^2+a^2}{\Delta} - 1\right) - \frac{a^2L_z}{\Delta} \equiv V_\phi(r,\theta),$$

$$\frac{\mathrm{d}t}{\mathrm{d}\lambda} = E\left[\frac{(r^2+a^2)^2}{\Delta} - a^2\sin^2\theta\right] + aL_z\left(1 - \frac{r^2+a^2}{\Delta}\right) \equiv V_t(r,\theta).$$

Osculating Evolution for EMRI Systems

Equations for (E, L_z, Q) are derived from covariant equations of motion

$$\begin{aligned}
\dot{E} &= -\delta f_t \\
\dot{L}_z &= \delta f_\phi \\
\dot{K} &= \dot{E}\frac{2}{\Delta}(\varpi^4 E - a\varpi^2 L_z) + \dot{L}_z\frac{2}{\Delta}(a^2 L_z - a\varpi^2 E) - 2\Delta u_r \delta f_r
\end{aligned}$$

where $K = Q + (L_z - aE)^2, \quad \varpi^2 = r^2 + a^2$

This formulation lead to apparent singularities at the turning points that are inconvenient for numerical implementation:

The equations for phase constants appear singular at turning points

$$\dot{\chi}_0 = -\frac{1}{\partial\theta/\partial\chi_0}\left(\frac{\partial\theta}{\partial E}\dot{E} + \frac{\partial\theta}{\partial L_z}\dot{L}_z + \frac{\partial\theta}{\partial Q}\dot{Q}\right)$$

$$\dot{\psi}_0 = -\frac{1}{\partial r/\partial\psi_0}\left(\frac{\partial r}{\partial E}\dot{E} + \frac{\partial r}{\partial L_z}\dot{L}_z + \frac{\partial r}{\partial Q}\dot{Q}\right)$$

Osculating Evolution for EMRI Systems

If we take the radial geodesic equation $\Sigma^2\dot{r}^2 = V_r(r, L_z, E, Q)$ and differentiate with respect to (E, L_z, Q) and then combine the expressions, we find an alternative expression for $\dot{\psi}_0$

$$
\begin{aligned}
\dot{\psi}_0 \;\; &= \;\; 2\frac{\dot{\psi}_{\rm geo}}{\partial V_r/\partial r}\left[\Sigma^2\left(\dot{E}\frac{\partial \dot{r}}{\partial E} + \dot{L}_z\frac{\partial \dot{r}}{\partial L_z} + \dot{Q}\frac{\partial \dot{r}}{\partial Q}\right) + 2\Sigma r\dot{r}\left(\dot{E}\frac{\partial r}{\partial E} + \dot{L}_z\frac{\partial r}{\partial L_z} + \dot{Q}\frac{\partial r}{\partial Q}\right)\right. \\
&\qquad \left. -2\Sigma a^2\cos\theta\sin\theta\dot{r}\left(\dot{E}\frac{\partial \theta}{\partial E} + \dot{L}_z\frac{\partial \theta}{\partial L_z} + \dot{Q}\frac{\partial \theta}{\partial Q}\right) - \Sigma^2\delta f^r\right]
\end{aligned}
$$

where $\Sigma^2 = r^2 + a^2\cos^2\theta$. We can derive a similar expression for $\dot{\chi}_0$ using the polar geodesic equation. $\partial V_r/\partial r$

Osculating Evolution for EMRI Systems

We can also find a manifestly non-singular form of the equations by decomposing the force on the Kinnersley tetrad

$$\vec{\delta} f = -\delta f_n \vec{l} - \delta f_l \vec{n} + \delta f_m^* \vec{m} + \delta f_m \vec{m}^*$$

Osculating conditions given in terms of the acceleration components

$$\begin{aligned} \mathcal{A}_I &= r R_f + a I_f \cos\theta \\ \mathcal{A}_{II} &= r I_f - a R_f \cos\theta \\ \mathcal{A}_{III} &= R_u R_f + I_u I_f \end{aligned}$$

$$\begin{aligned} R_f &= \frac{1}{\sqrt{2}}(\delta f_m + \delta f_m^*) \\ I_f &= \frac{i}{\sqrt{2}}(\delta f_m - \delta f_m^*) \end{aligned}$$

then, the evolution of the orbital constants is given by

$$\frac{dE}{d\lambda} = \frac{u_r a_n \Delta}{u_n} - \frac{\Delta \mathcal{A}_{III}}{2u_n} - a \sin\theta \mathcal{A}_{II}$$

$$\frac{dL_z}{d\lambda} = \frac{a \sin^2\theta u_r a_n \Delta}{u_n} - \frac{a \sin^2\theta \Delta \mathcal{A}_{III}}{2u_n} - \varpi^2 \sin\theta \mathcal{A}_{II}$$

$$\frac{dK}{d\lambda} = 2\Sigma^2 \mathcal{A}_{III}$$

Osculating Evolution for EMRI Systems

The evolution of the phase constants can be found to be

$$
\begin{aligned}
\frac{d\chi_\theta}{d\lambda} &= \sqrt{\beta(z_+ - z)}\left[1 + \frac{(1-z_-)\Sigma\mathcal{A}_I\cos\chi_\theta}{\beta\sqrt{z_-}(z_+ - z_-)\sin\theta}\right] \\
&\quad + \frac{\cos\chi_\theta\sin\chi_\theta\mathcal{H}a\Delta(\mathcal{A}_{III} - 2u_r a_n)}{2(z_+ - z_-)\beta u_n} + \frac{\cos\chi_\theta\sin\chi_\theta\mathcal{G}\mathcal{A}_{II}}{\beta(z_+ - z_-)}
\end{aligned}
$$

$$
\begin{aligned}
\frac{d\psi_r}{d\lambda} &= \mathcal{P} + \frac{\mathcal{C}\mathcal{A}_{III}\sin\psi_r}{2(1+e\cos\psi_r)u_n} + \frac{\mathcal{D}\Sigma\mathcal{A}_{III}\mathcal{P}}{2(1+e\cos\psi_r)^2 u_n} \\
&- \quad \frac{a\mathcal{E}\sin\theta\sin\psi_r\mathcal{A}_{II}}{1+e\cos\psi_r} + \frac{\mathcal{P}a_n}{u_n(1+e\cos\psi_r)^2} \\
&\times \quad \left[(1-e)^2(1-\cos\psi_r)\frac{\Sigma_1 F_1}{\kappa_1} + .(1+e)^2(1+\cos\psi_r)\frac{\Sigma_2 F_2}{\kappa_2}\right]
\end{aligned}
$$

Where $\chi_\theta = \chi - \chi_0$ **and** $\psi_r = \psi - \psi_0$ **.**

Action Angle Variables

Action Angle Variables

Geodesic evolution equations in generalized action angle variables

$$\Omega_\alpha(\mathbf{J}) \equiv \dot{q}_\alpha = \frac{\partial H(\mathbf{J})}{\partial J_\alpha}$$

$$\dot{J}_\alpha = -\frac{\partial H(\mathbf{J})}{\partial q_\alpha} = 0.$$

$$\longrightarrow$$

$$q_\alpha(t) = \Omega_\alpha(\mathbf{J}_0)t + q_{\alpha 0},$$

$$J_\alpha(t) = J_{\alpha 0},$$

Forced motion

$$\frac{dq_\alpha}{d\tau} = \omega_\alpha(\mathbf{J}) + \varepsilon g_\alpha^{(1)}(q_\theta, q_r, \mathbf{J}) + O(\varepsilon^2),$$

$$\epsilon = \mu/M$$

$$\frac{dJ_\nu}{d\tau} = \varepsilon G_\nu^{(1)}(q_\theta, q_r, \mathbf{J}) + \varepsilon^2 G_\nu^{(2)}(q_\theta, q_r, \mathbf{J}) + O(\varepsilon^3).$$

$J_i = (E, L_z, Q)$ **Conserved quantities of the geodesic motion**

$q_\alpha = (q_t, q_r, q_\theta, q_\phi)$ **Generalized angle variables associated with** t, r, θ, ϕ

$g_\alpha^{(1)}, G_\nu^{(1)}$ $G_\nu^{(2)}$ **Given by the first and second orders self-forces (2π periodic in q_θ, q_r)**

Hinderer & Flanagan (2012)

Action Angle Variables

- The forcing terms can be expanded as a double Fourier series

$$G_\nu^{(1)}(q_\theta, q_r, \mathbf{J}) = \sum_{k,n} G_{\nu kn}^{(1)}(\mathbf{J}) e^{i(kq_\theta + nq_r)}$$

- In terms of λ , the motions in r, θ become periodic

$$r(\lambda + \Lambda_r) = r(\lambda), \quad \theta(\lambda + \Lambda_\theta) = \theta(\lambda)$$

and we can define angle variables $w^{r,\theta} = \Upsilon_{r,\theta}\lambda$ were $\Upsilon_{r,\theta} = 2\pi/\Lambda_{r,\theta}$

$$\Lambda_r = 2\int_{r_{\mathrm{peri}}}^{r_{\mathrm{ap}}} \frac{dr}{R(r)^{1/2}}, \qquad \Lambda_\theta = 4\int_{\theta_{\mathrm{min}}}^{\pi/2} \frac{d\theta}{\Theta(\theta)^{1/2}}.$$

Hinderer & Flanagan (2012) Drasco (2007)

Action Angle Variables

- Bound orbits can be uniquely expressed as:

$$r(\lambda) = \sum_{n=0}^{\infty} \hat{r}_n \cos[n\Upsilon_r(\lambda - \lambda_r)],$$

$$\theta(\lambda) = \sum_{k=0}^{\infty} \hat{\theta}_k \cos[k\Upsilon_\theta(\lambda - \lambda_\theta)],$$

$$t(\lambda) = \Gamma(\lambda - \lambda_t) + \sum_{k=1}^{\infty} \hat{t}_k^\theta \sin[k\Upsilon_\theta(\lambda - \lambda_\theta)] + \sum_{n=1}^{\infty} \hat{t}_n^r \sin[n\Upsilon_r(\lambda - \lambda_r)],$$

$$\phi(\lambda) = \Upsilon_\phi(\lambda - \lambda_\phi) + \sum_{k=1}^{\infty} \hat{\phi}_k^\theta \sin[k\Upsilon_\theta(\lambda - \lambda_\theta)] + \sum_{n=1}^{\infty} \hat{\phi}_n^r \sin[n\Upsilon_r(\lambda - \lambda_r)],$$

Drasco (2007)

Osculating Evolution + Action Angle Variable

Osculating Evolution + Action Angle Variable

Geodesic parameterized by: $I^A = \{E, Lz, \mathcal{Q}, \psi_0, \chi_0, \phi_0\}$

Use the fact that we can map the BL coordinates r, θ **onto** $w^{r,\theta} = \Upsilon_{r,\theta}\lambda$ **:**

$$r = r(\ w_r)$$

$$\Upsilon_r = \frac{\pi\sqrt{(1-\mathcal{E}^2)(r_1-r_3)(r_2-r_4)}}{2K(k_r)}$$

$$k_r = \sqrt{\frac{r_1-r_2}{r_1-r_3}\frac{r_3-r_4}{r_2-r_4}},$$

$$\theta = \theta(\ w_\theta)$$

$$\Upsilon_\theta = \frac{\pi\mathcal{L}_z\sqrt{\epsilon_0 z_+}}{2K(k_\theta)}$$

$$k_\theta = \sqrt{\frac{z_-}{z_+}}$$

$$z_- = \cos^2\theta_{min}$$

$$z_+ = \frac{\mathcal{Q}}{L_z^2\epsilon_0 z_-}$$

$$\epsilon_0 = \frac{a^2(1-E^2)}{L_z}$$

Write $w^r(\psi) = \Upsilon_r\lambda(\psi)$ $w^\theta(\chi) = \Upsilon_\theta\lambda(\chi)$ **by inverting:**

$$\frac{d\chi}{d\lambda} = \sqrt{\beta(z_+ - z)}$$

$$\frac{d\psi}{d\lambda} = \frac{M\sqrt{1-E^2}[(p-p_3)-\varepsilon(p+p_3\cos\psi)]^{1/2}[(p-p_4)+\varepsilon(p-p_4\cos\psi)]^{1/2}}{1-\varepsilon^2}$$

Osculating Evolution + Action Angle Variable

Applying the osculating conditions to $w^{r,\theta} = \Upsilon_{r,\theta}\lambda$ we obtain the evolution of the positional elements in terms of the fundamental frequencies

$$\boxed{\begin{aligned} \frac{d\psi_0}{d\tau} &= \left(\frac{d\Upsilon_r}{dE}\frac{dE}{d\tau} + \frac{d\Upsilon_r}{dQ}\frac{dQ}{d\tau}\right)\frac{\Sigma}{\Upsilon_r}\frac{d\psi}{d\tau} \\ \frac{d\chi_0}{d\tau} &= \left(\frac{d\Upsilon_\theta}{dE}\frac{dE}{d\tau} + \frac{d\Upsilon_\theta}{dL_z}\frac{dL_z}{d\tau} + \frac{d\Upsilon_\theta}{dQ}\frac{dQ}{d\tau}\right)\frac{\Sigma}{\Upsilon_\theta}\frac{d\chi}{d\tau} \end{aligned}}$$

we can keep track of the initial phases when we cross a resonant point.

Using $r = r(\,w_r)$, $\theta = \theta(\,w_\theta)$ we can expand the self-force into a Fourier series

$$G^{(1)}_\nu(\omega_r, \omega_\theta, \mathbf{J}) = \sum_{\mathbf{k},\mathbf{n}} \mathbf{G}^{(1)}_\nu(\mathbf{J})\mathbf{e}^{\mathbf{i}(\mathbf{r}(\omega_\mathbf{r})+\theta(\omega_\theta))}$$

Osculating Evolution + Action Angle Variable

- ✓ From the initial orbital parameters $(e, p, \iota) \longrightarrow (E, L, Q) \longrightarrow (\Upsilon_t, \Upsilon_r, \Upsilon_\theta, \Upsilon_\phi)$
- ✓ Compute $\dot{E}, \dot{L}_z, \dot{Q}$ at leading post-Newtonian order
- ✓ Compute the osculating conditions for the initial positional elements ψ_0, χ_0
- Studying the evolution of the trajectory

Summary & Prospects

Osculating evolution provides a natural scheme to evolve EMRI systems through resonances. Using the action angle variable formalism the initial phases can be evolved through resonances and incorporate their effects on the EMRI evolution.

Knowing the evolution of the positional elements we could build a Kludge waveform model including resonant effects.

- Basis for modeling resonant transitions in a easy way
- Is an adiabatic evolution enough for studying resonances?
- Which resonances are important?
- Effect of resonances in parameter estimation.

Thank you for your attention!