Self-force in Schwarzschild Space-time via the Method of Matched Expansions

Marc Casals
University College Dublin

In collaboration with S.Dolan, A.Ottewill, B.Wardell

ArXiv: 1306.0884

Capra 16 - UCD, 16 July 2013

Outline

1. Method of Matched Expansions
2. QuasiLocal
3. Distant Past
4. Results
5. Conclusions

Self-force via Green Function

- **S-F for scalar charge ('MiSaTaQuWa eq.'):**

$$F_\mu(\tau) = q^2 \int_{-\infty}^{\tau^-} d\tau' \; \nabla_\mu G_{ret}(z(\tau), z(\tau')) + \text{local}$$

- **Retarded Green function defined by**

$$\Box G_{ret}(x, x') = \delta_4(x, x')$$ **with causality b.c.**

- **Similar for emag (spin=1) and gravitational (spin=2) fields**

- **Global structure of G_{ret} is crucial!**

The birth of a method: Capra 2 - Dublin'99

Radiation Reaction in a Normal Neighbourhood

WGA & Éanna É. Flanagan

August 999

(Work in progress)

Radiation Reaction

- Small particle mass μ) moves on a geodesic z^α τ of a curved background spacetime M $g_{\alpha\beta}$

$$\mu^2 R(g_{\alpha\beta} <<$$

- Particle's gravitation alters metric:

$$g_{\alpha\beta} \longrightarrow g_{\alpha\beta} + \gamma_{\alpha\beta}$$

- linearized gravity:

$$\mathcal{D}\gamma)_{\alpha\beta} \quad T_{\alpha\beta},$$

where

$$\mathcal{D}\gamma \quad _\beta \qquad \Box\gamma_{\alpha\beta} \qquad 2R^\mu{}_{\alpha\beta}\ \gamma_{\mu\nu}$$
$$T_{\alpha\beta} \qquad 16\pi\ \mu \int_{-\infty}^{\theta} \delta(x - z(\tau))u_\alpha\ \tau\ u_\beta(\dot{\tau})d\tau$$

and u^α $\partial_\tau z^\alpha$ τ

The Question

How does $\gamma_{\alpha\beta}$ change particles trajectory?

The Answer

- Find $\gamma_{\alpha\beta}$ using the Green's function $G_{\alpha\beta}{}^{\mu'\nu'}(x,x') = \mathcal{D}^{-1}$:

$$\gamma_{\alpha\beta}(x) = \int_{-\infty}^{0} G_{\alpha\beta}{}^{\mu'\nu'}(x,x')\; T_{\mu'\nu'}(x')\; d\tau.$$

- Express correction to particle path as an acceleration,

$$a^{\alpha} = A^{\alpha\beta\gamma\delta}(g,u)\; \gamma_{\beta\gamma;\delta}.$$

The Problem

- Green's functions for $\mathcal{D}$ difficult to calculate analytically $\rightarrow$ numerical mode sum.
- Green's function distributional on light cone $\rightarrow$ mode sum does not converge well near the cone.

Poisson &
Wiseman's Suggestion

- Do numerical mode sum (good estimate far from particle).
- Do normal neighbourhood analysis near particle.
- Match solutions.

Method of Matched Expansions

- **Non-local part of S-F:** $\int_{-\infty}^{\tau^-} d\tau' \; \nabla_\mu G_{ret}$

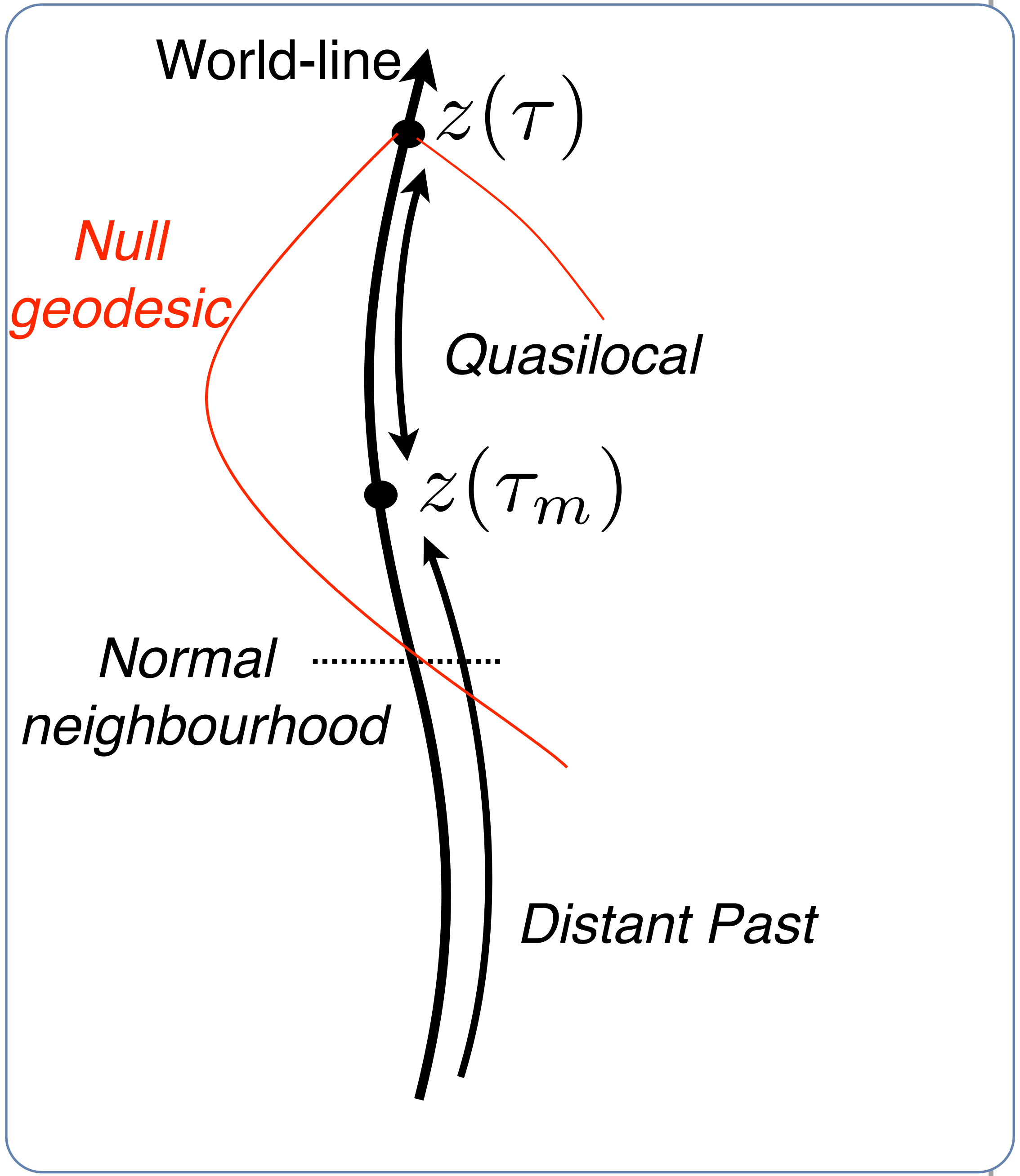
- **Matched expansions: choose** τ_m **:**

- before that point ('Quasilocal' region)

$$\int_{\tau_m}^{\tau^-} d\tau' \; \nabla_\mu G_{ret}$$

- after that point ('Distant Past')

$$\int_{-\infty}^{\tau_m} d\tau' \; \nabla_\mu G_{ret}$$



Method of Matched Expansions

- A priori no such τ_m need exist

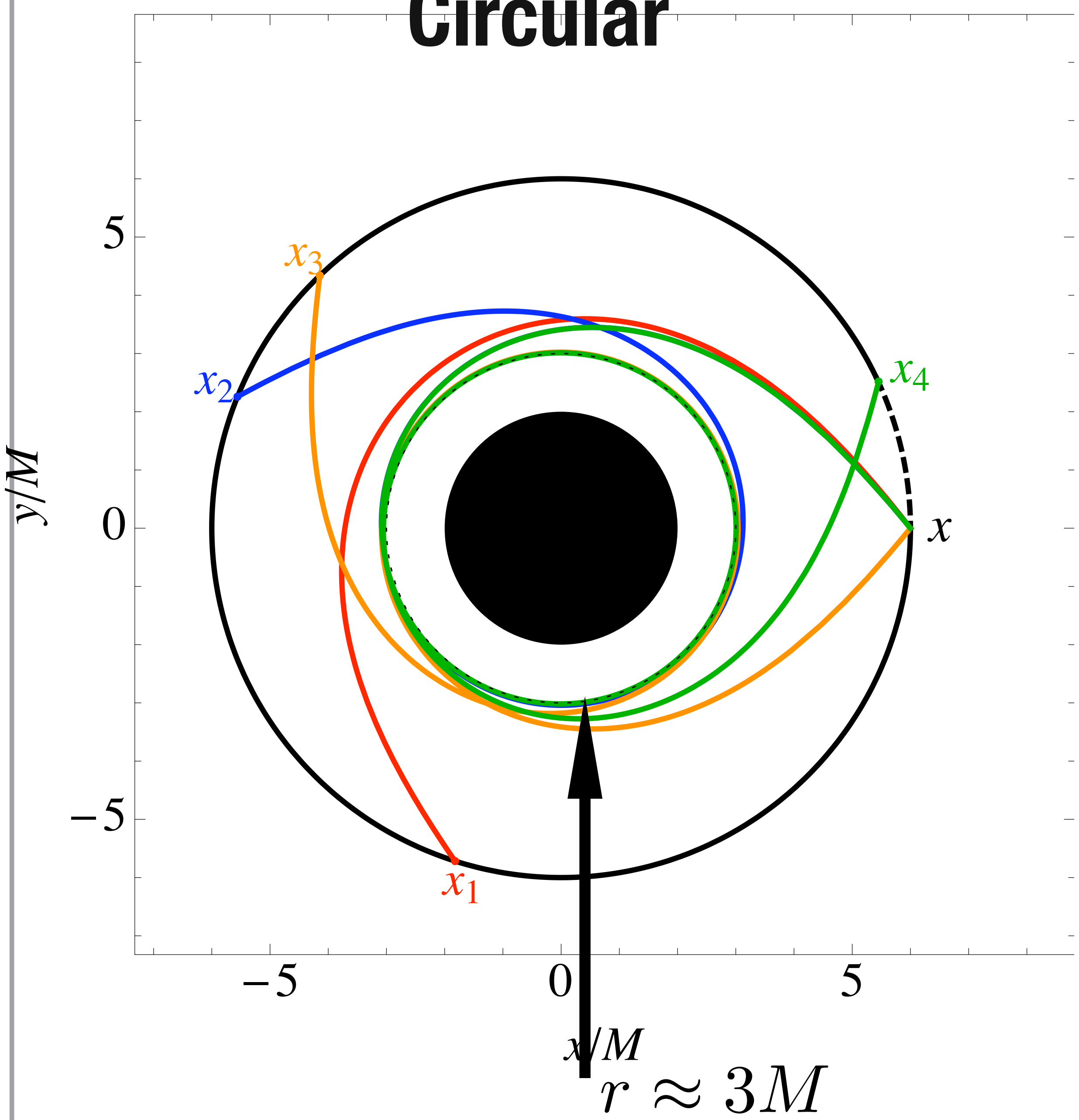
- Anderson&Wiseman'05: weak-field approx. in DP in Schwarzschild. "Poor" convergence.

- Casals,Dolan,Ottewill,Wardell'09: successful application of method of matched expansions in Nariai space-time $dS_2 \times \mathbb{S}^2$

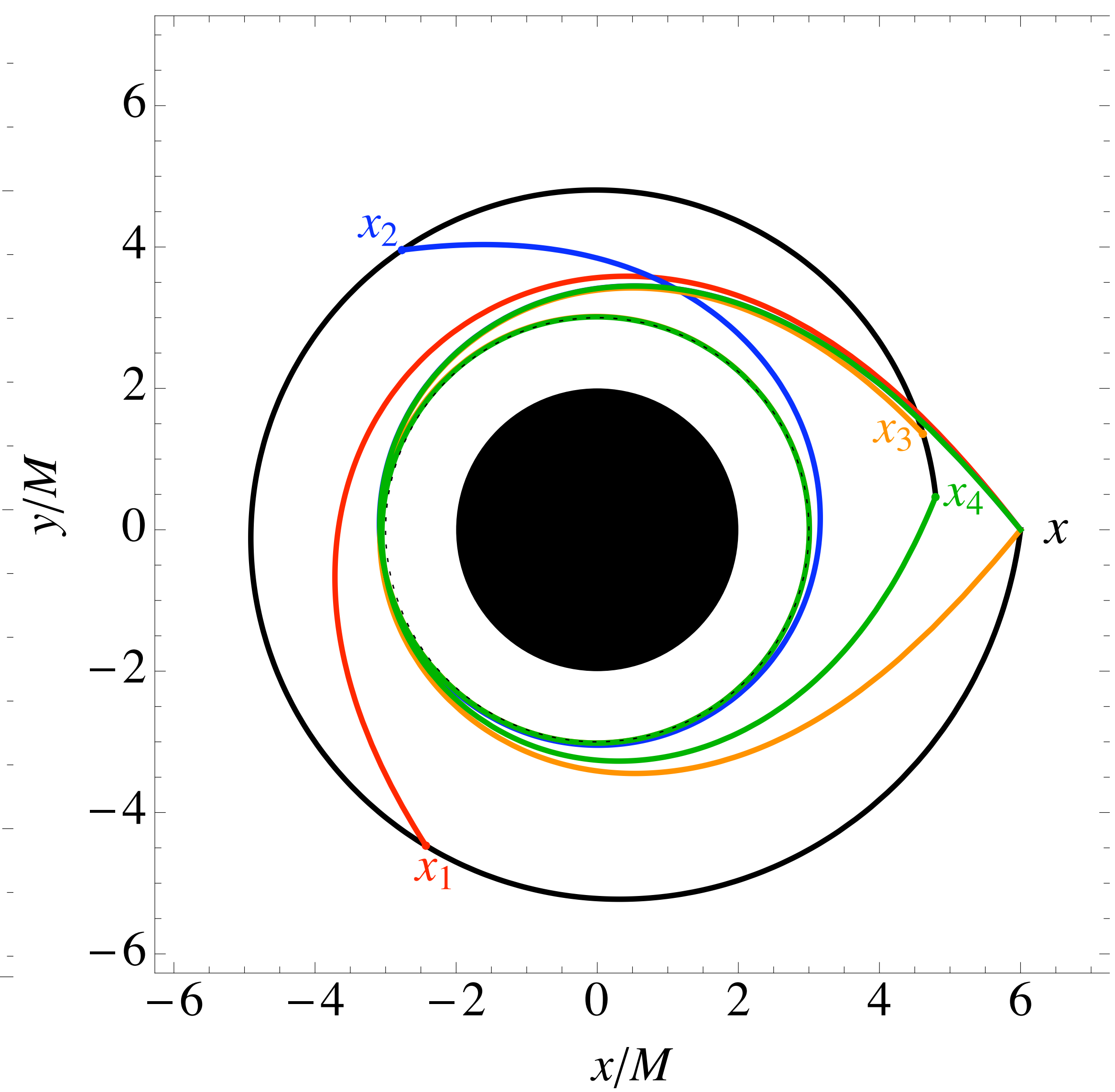
Method of Matched Expansions

- **Here we apply it to Schwarzschild. Scalar charge at r=6M in 2 geodesics:**

Circular



Eccentric (p=7.2M, e=0.5)

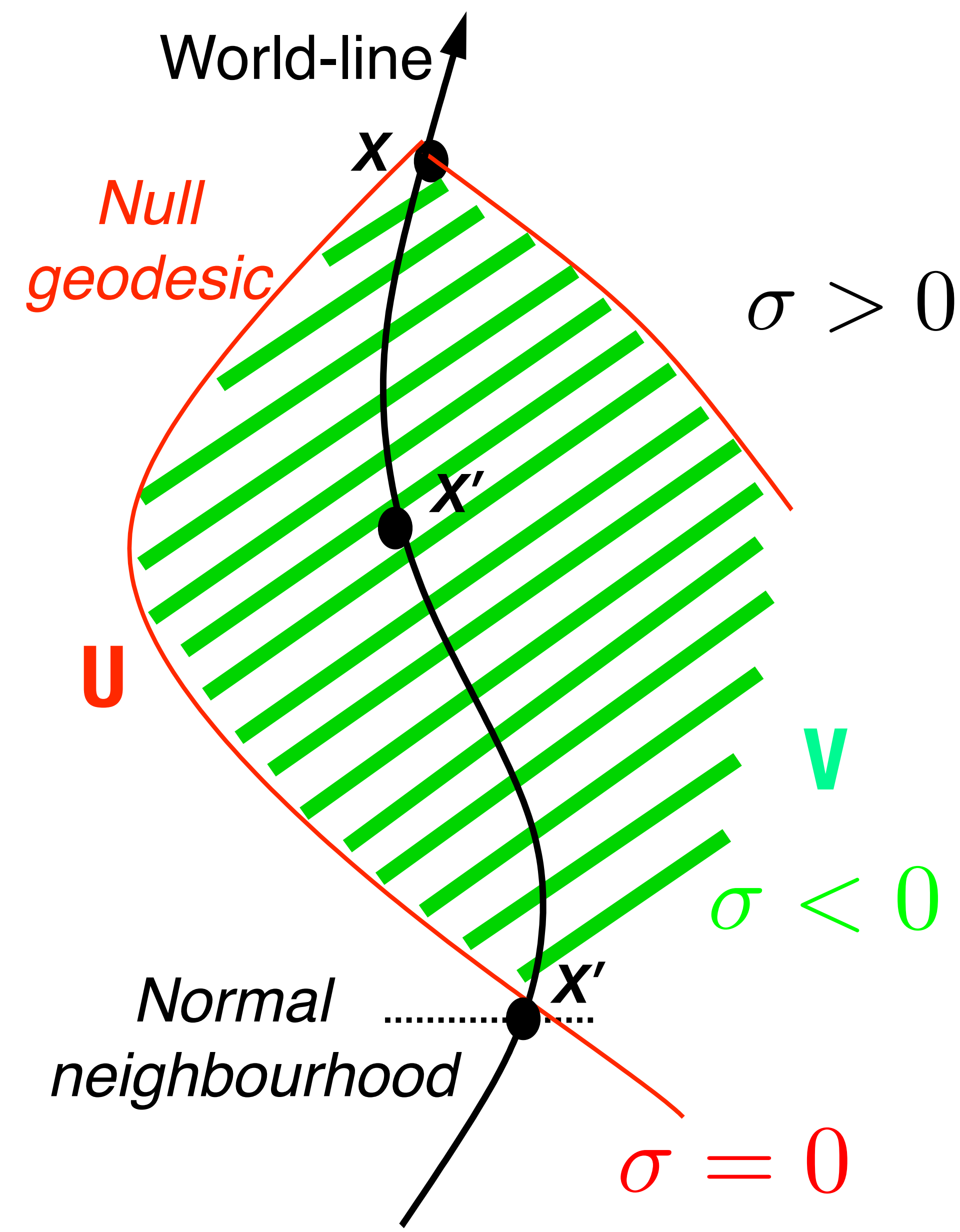


Quasilocal - Hadamard form

$$G_{ret}(x,x') = \underbrace{\theta(\Delta t)}_{\neq 0 \text{ in the past}} \left\{ \underbrace{U(x,x')\,\delta(\sigma)}_{\neq 0 \text{ on light cone}} + \underbrace{V(x,x')\,\theta(-\sigma)}_{\neq 0 \text{ inside light cone}} \right\}$$

- σ : square of **geodesic distance**
- **U & V regular bitensors**
- **Only valid in normal neighbourhood**
- **It renders regularization trivial**

$$\int_{\tau_m}^{\tau^-} d\tau' \, \nabla_\mu G_{ret} = \int_{\tau_m}^{\tau} d\tau' \, \nabla_\mu V$$



Quasilocal - Hadamard form

- **Calculate V with, e.g., coordinate expansion using WKB**

$$V(x,x') = \sum_{i,j,k=0}^{\infty} v_{ijk}(r)\ (t-t')^{2i}(1-\cos\gamma)^{j}(r-r')^{k}$$

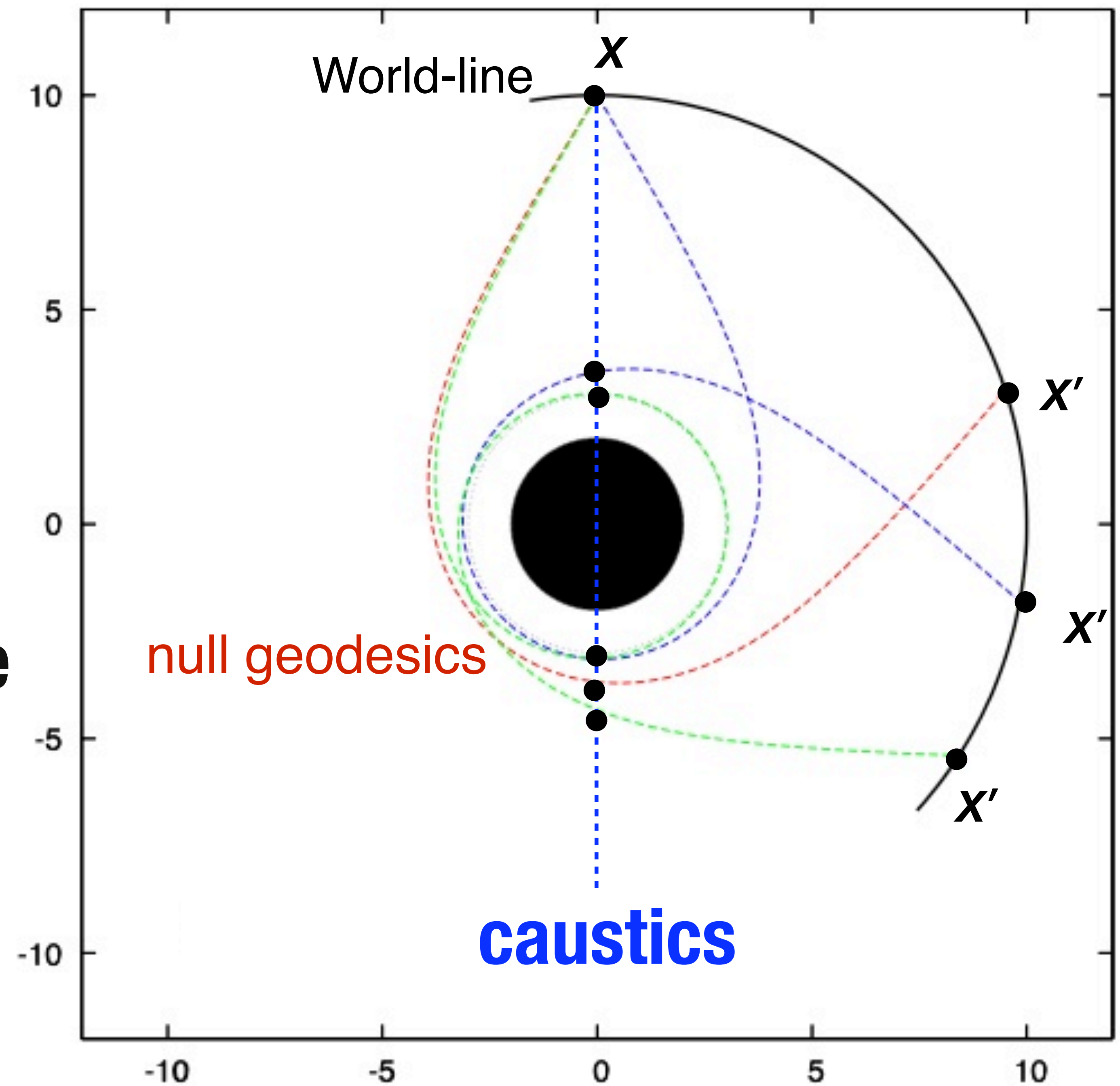
- **Improve accuracy&domain of validity via knowledge of singularity structure at 1st light-crossing and use of Padé approximants**

Distant past - Singularities of Green function

- "Propagation of singularities theorems": outside normal nbd, $G_{ret}(x,x')$ is singular along null geodesics (ie, $\sigma = 0$)

- Form of singularity outside the normal nbd?

- Caustics: focus points/where light cone intersects itself



Timelike circular geodesic in Schwarzschild (r=10M)

Distant Past: Black Hole Spectroscopy

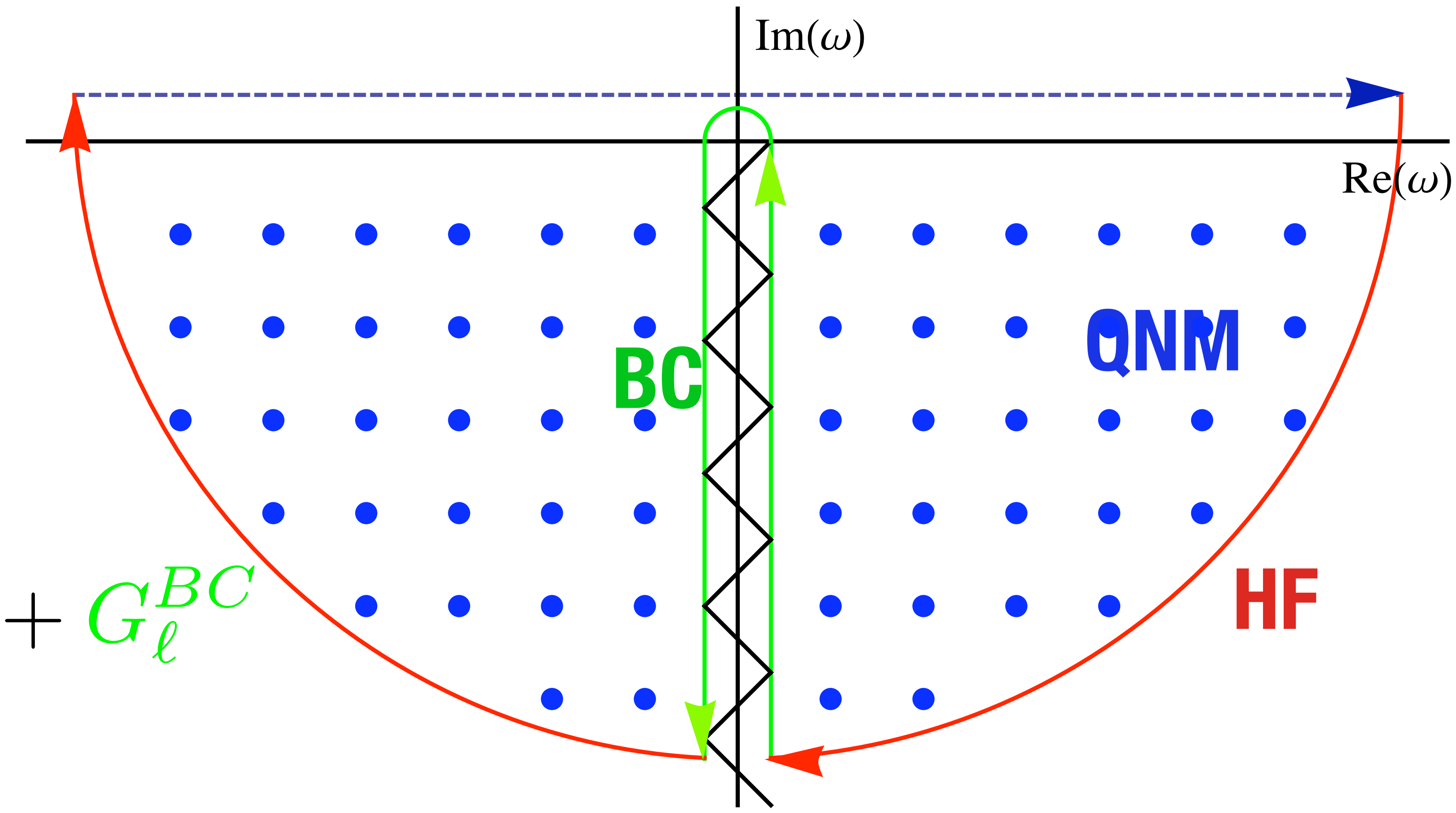
- **Multipolar decomposition:**

$$G_{ret}(x,x') = \frac{1}{rr'}\sum_{\ell=0}^{\infty}(2\ell+1)P_\ell(\cos\gamma)G_\ell^{ret}(r,r';t)$$

- **Fourier transform:**

$$G_\ell^{ret}(r,r';t) \equiv \int_{-\infty+ic}^{\infty+ic} d\omega\ G_\ell(r,r';\omega)e^{-i\omega t}$$

Complex-Frequency Plane



- **Residue theorem:**

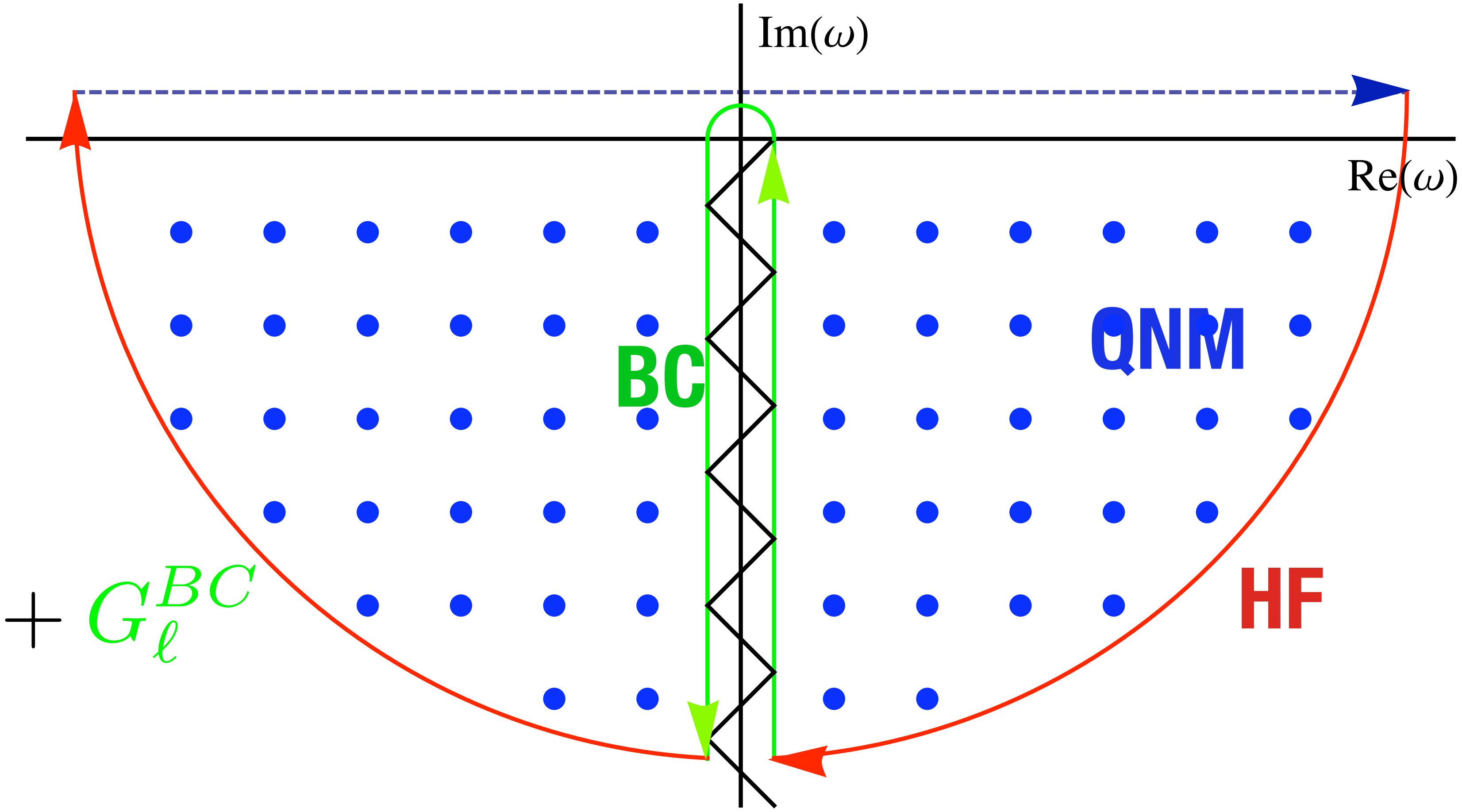
$$G_\ell^{ret} = G_\ell^{HF} + G_\ell^{QNM} + G_\ell^{BC}$$

G_ℓ^{HF} **Integral along high-frequency arc. Zero in Distant Past.**

G_ℓ^{QNM} **Sum over residues of poles (quasinormal modes)**

G_ℓ^{BC} **Integral around branch cut**

Complex-Frequency Plane



- **Residue theorem:**

$$G_\ell^{ret} = \cancel{G_\ell^{HF}} + G_\ell^{QNM} + G_\ell^{BC}$$

G_ℓ^{HF} **Integral along high-frequency arc. Zero in Distant Past.**

G_ℓ^{QNM} **Sum over residues of poles (quasinormal modes)**

G_ℓ^{BC} **Integral around branch cut**

Radial Equation

- **Green function modes:** $G_\ell(r, r'; \omega) = \dfrac{R_\ell^{in}(r_<, \omega)\, R_\ell^{up}(r_>, \omega)}{W(\omega)}$

- $R_\ell^{in/up}$ **are slns. of radial ODE ('Regge-Wheeler eq.') for the perturbation:**

$$\left[\frac{d^2}{dr_*^2} + \omega^2 - V(r)\right] R_\ell(r, \omega) = 0 \qquad V(r) = \left(1 - \frac{1}{r}\right)\left[\frac{\ell(\ell+1)}{r^2} + \frac{(1-s^2)}{r^3}\right]$$

$$r_* = r_*(r) \in (-\infty, \infty) \qquad s = 0, 1, 2$$

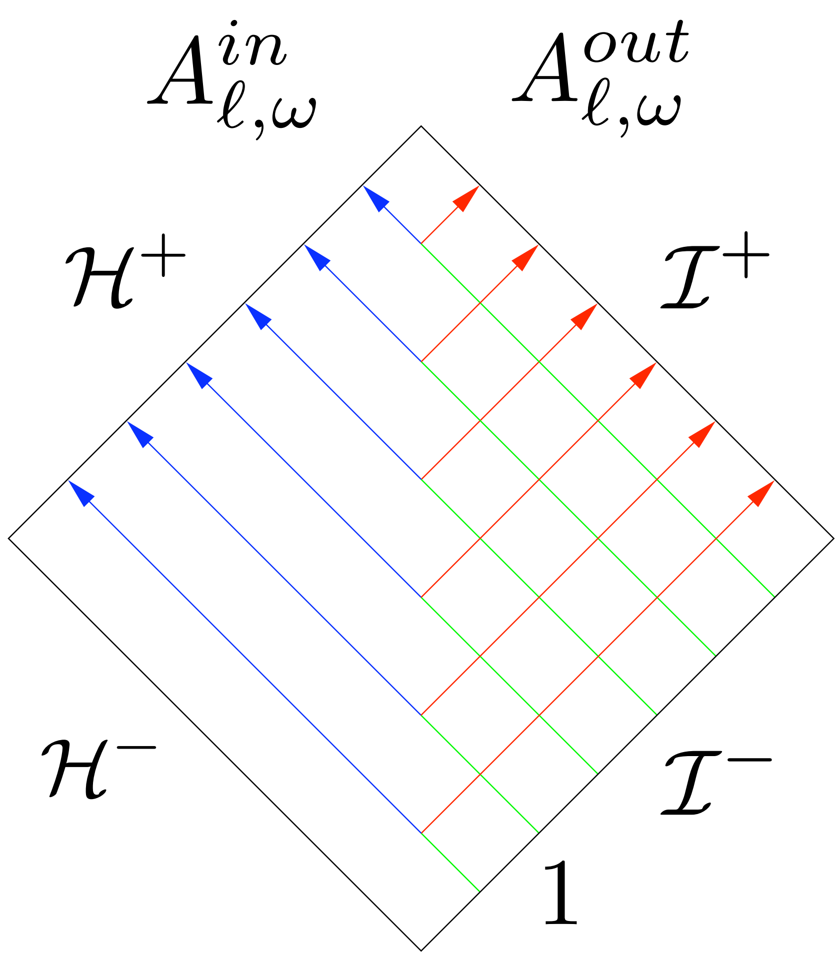
$$2M = 1$$

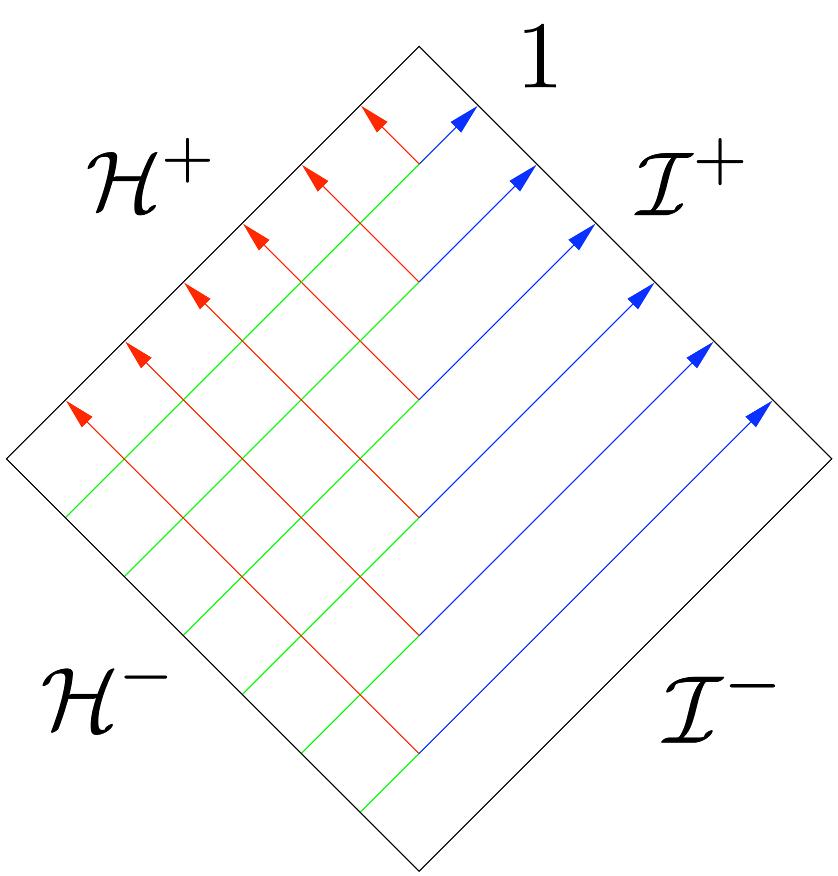
Radial solutions

- **Two lin. indep. slns.:**

$$R^{in}_{\ell} \sim e^{-i\omega r_*}$$
$$r_* \to -\infty$$

$$R^{up}_{\ell} \sim e^{+i\omega r_*}$$
$$r_* \to \infty$$





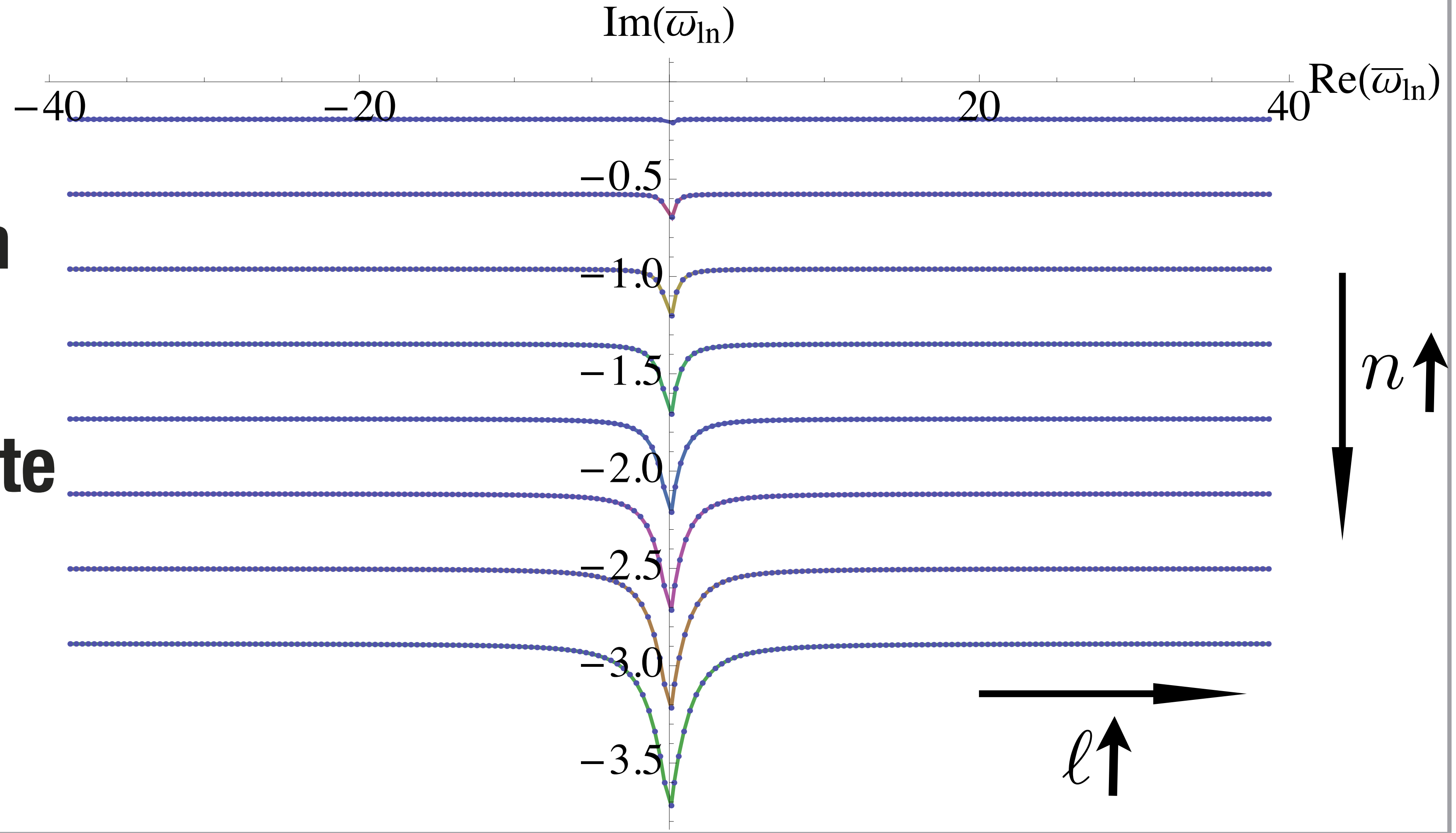
Quasinormal Modes

- **QNM frequencies**: simple poles of $G_\ell = \dfrac{R_\ell^{in}(r_<,\omega)\,R_\ell^{up}(r_>,\omega)}{W(\omega)}$ in the complex-ω plane: $W(\omega_{ln}) = 0$

- **Boundary conditions:** $\underset{r_*\to-\infty}{e^{-i\omega_{\ell n}r_*} \sim R_\ell^{in}} \propto \underset{r_*\to\infty}{R_\ell^{up} \sim e^{+i\omega_{\ell n}r_*}}$

$\mathrm{Re}(\omega_{\ell n})$: freq. of oscillation

$\mathrm{Im}(\omega_{\ell n})$: decay rate



Quasinormal Modes

- **QNM sum:**

$$G_\ell^{QNM}(r, r'; \Delta t) = \sum_{n=0}^{\infty} \mathrm{Re} \left(\frac{R_\ell^{in}(r,\omega) R_\ell^{in}(r',\omega)}{\omega A_{\ell,\omega}^{out} \frac{\partial A_{\ell,\omega}^{in}}{\partial \omega}} e^{-i\omega \Delta t} \right) \Bigg|_{\omega=\omega_{\ell,n}}$$

- **n-sum convergent for** $\Delta t \gtrapprox |r_*| + |r'_*|$

- ℓ **-sum leads to divergences at light-crossing times**

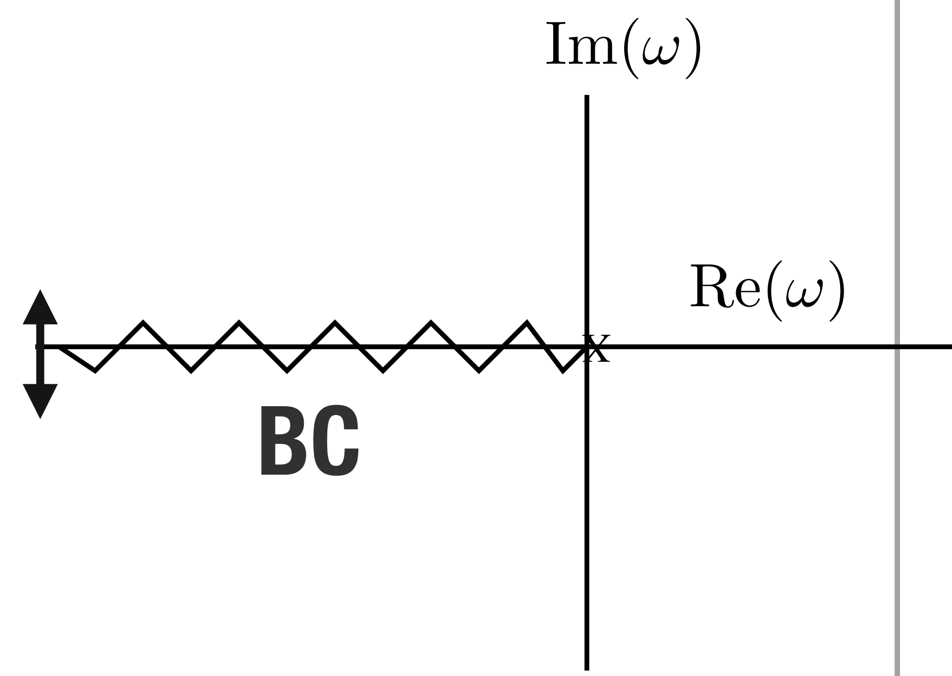
Branch Cut

- **Ahem...what is a BC??**

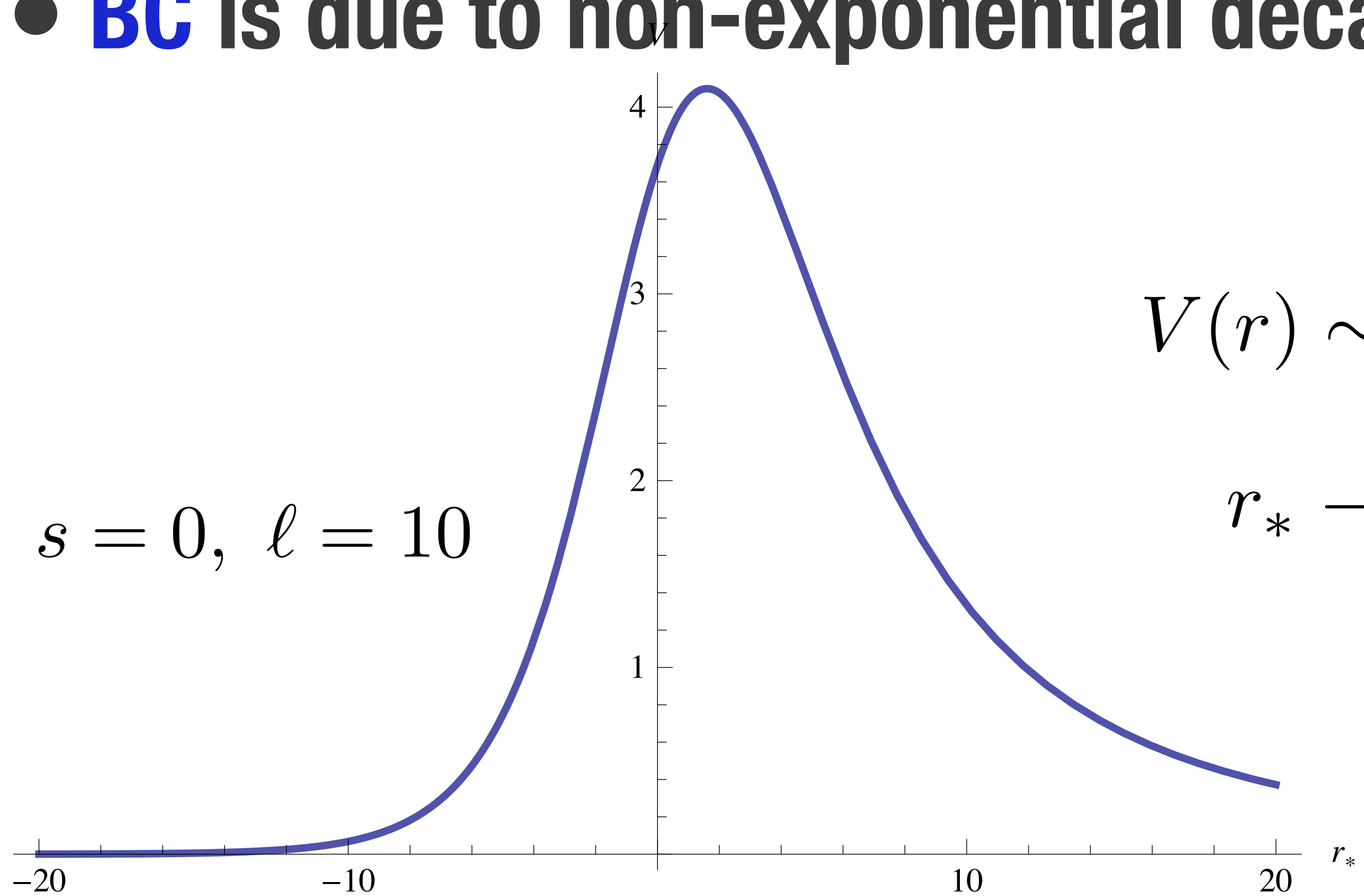
Ex: $\ln\omega = \ln|\omega| + i\arg(\omega)$

$\arg(\omega) \in (-\pi, \pi]$

$\Delta(\ln\omega) = 2\pi$



- **BC is due to non-exponential decay of potential at radial infinity:**



$s = 0,\ \ell = 10$

$$V(r) \sim \frac{\ell(\ell+1)}{r_*^2} + \frac{2\ell(\ell+1)r_h\ln(r_*/r_h)}{r_*^3}$$

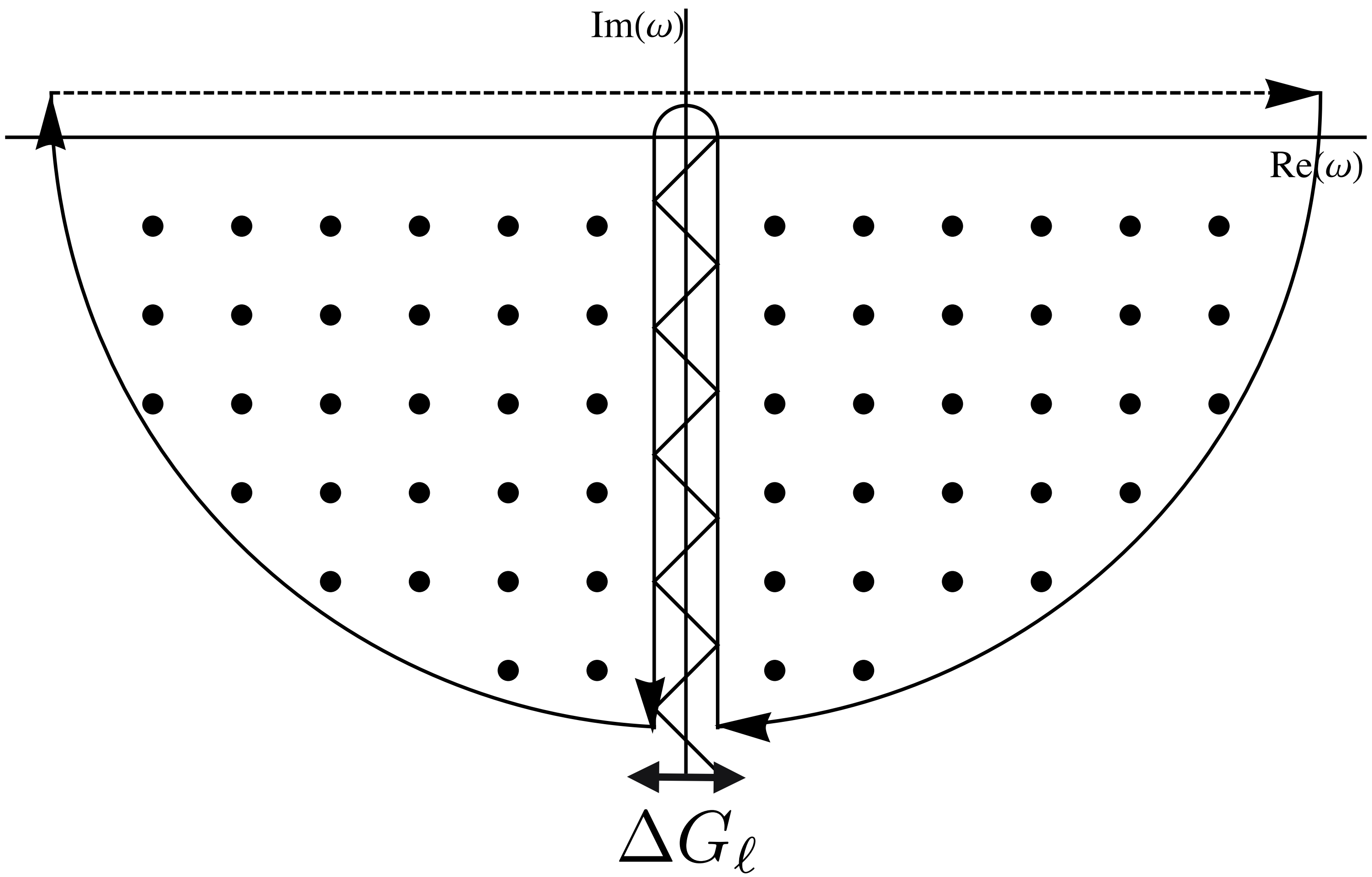
$r_* \to \infty$

Branch Cut

- **BC integral** $G_\ell^{BC}(r,r';t) = \int_0^\infty d\nu\ \Delta G_\ell(r,r';-i\nu)e^{-\nu t}$

$\omega = -i\nu$

$$\Delta R_\ell^{up}(r,-i\nu) \equiv \lim_{\epsilon\to 0}\left[R_\ell^{up}(r,\epsilon - i\nu) - R_\ell^{up}(r,-\epsilon - i\nu)\right]$$



- **BC modes:**

$$\Delta G_\ell(r,r';-i\nu) = 2i\nu \frac{\Delta R_\ell^{up}(r,-i\nu)}{R_\ell^{up}(r,+i\nu)} \frac{R_\ell^{in}(r,-i\nu)R_\ell^{in}(r',-i\nu)}{|W(-i\nu)|^2}$$

- ν **-integral convergent for** $\Delta t \gtrapprox |r_*| + |r'_*|$

Methods for QNMs and BC

- **QNM** and **small-** $|\omega|$ **BC** asymptotics by method of **Mano, Suzuki, Takasugi**:

match series of hypergeometric functions (convergent $\forall r \neq \infty$ with series of Coulomb functions (convergent $\forall r \neq r_h$) - "Functional Methods" discussion)

Methods for QNMs and BC

- **Mid-$|\omega|$ BC** by using series of **confluent hypergeometric functions** (Leaver'86)

$$R_\ell^{up} \propto \sum_{n=0}^{\infty} a_n (1-2\nu)_n \; U(s+1-2\nu+n, 2s+1, -2\nu r)$$

New series on BC:

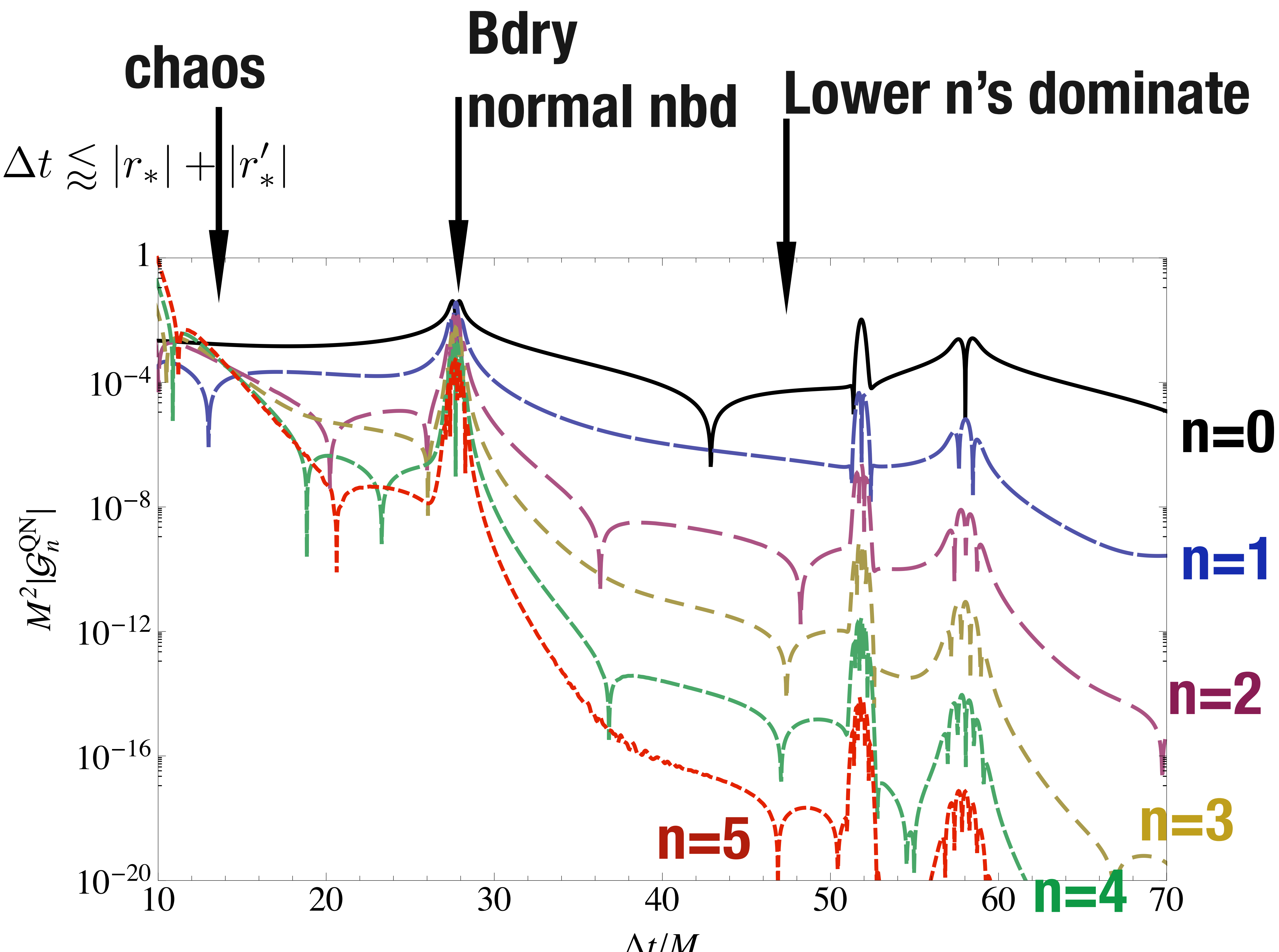
$$\Delta R_\ell^{up} \propto \sum_{n=0}^{\infty} a_n \frac{(-1)^n \Gamma(1+n-2\nu) U(s-n+2\nu, 2s+1, 2\nu r)}{\Gamma(1+s+n-2\nu)\Gamma(1-s+n-2\nu)}$$

this can be evaluated *on* the NIA

- **Large-$|\omega|$ BC** asymptotics by analytic continuation to **complex-r** plane

Results: QNMs

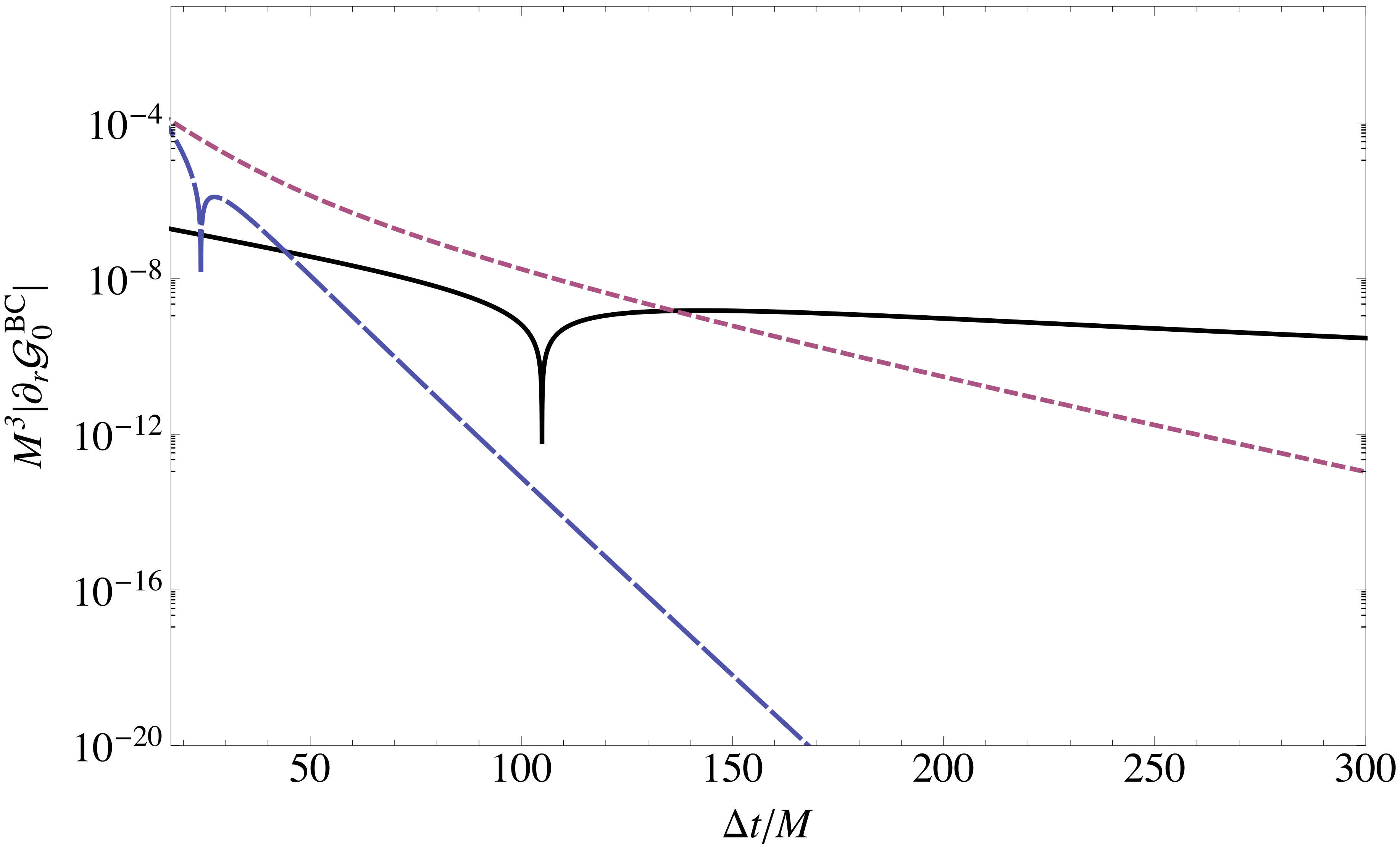
- QNMs for different n's



Results: Branch Cut

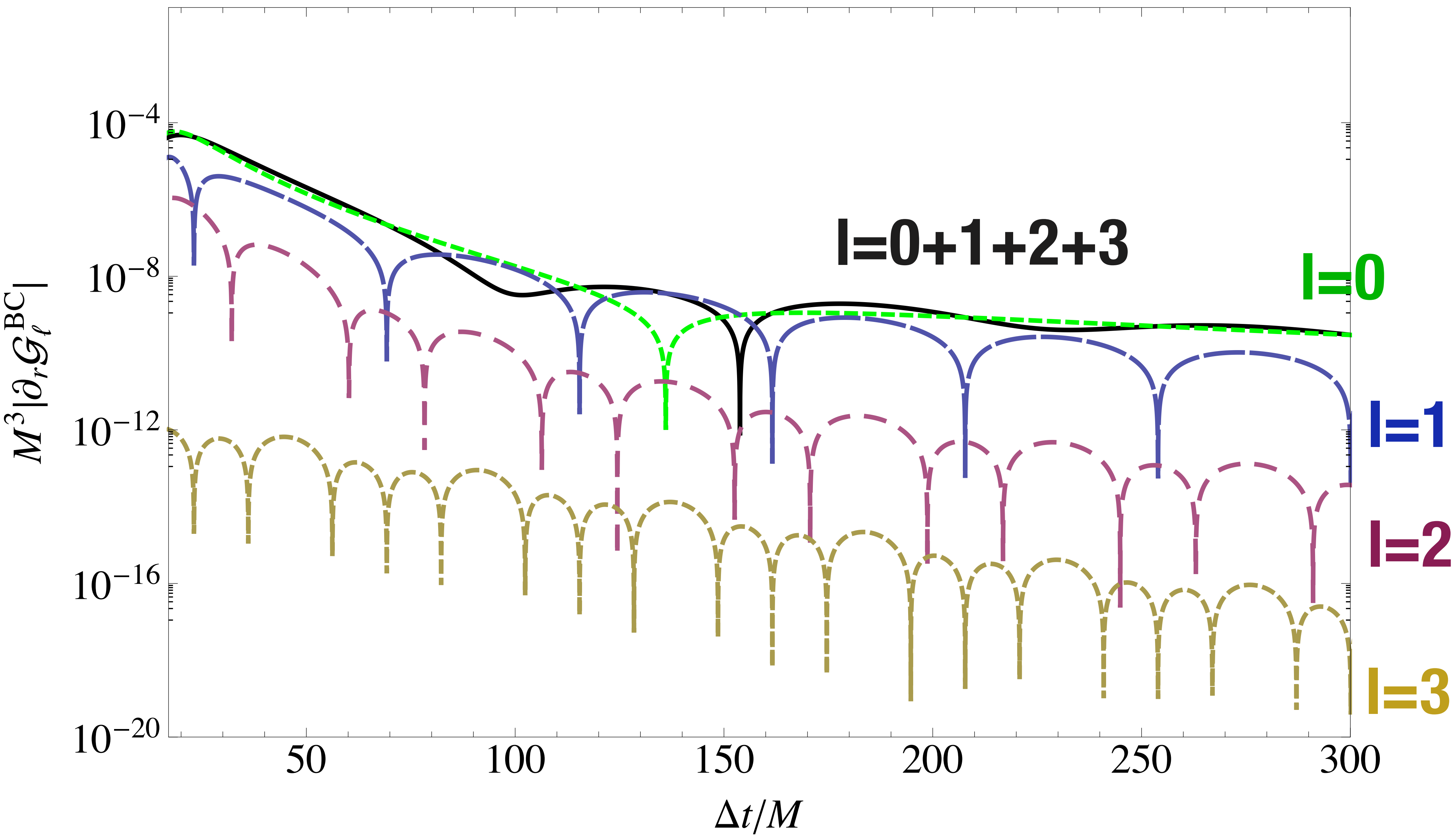
- **Different integration regimes for BC mode $\ell = 0$**

$$G_\ell^{BC} = \frac{-i}{2\pi}\left\{\int_0^{0.05/M} + \int_{0.05/M}^{0.225/M} + \int_{0.225/M}^{9/M}\right\} d\nu\ \Delta G_\ell\ e^{-\nu\Delta t}$$



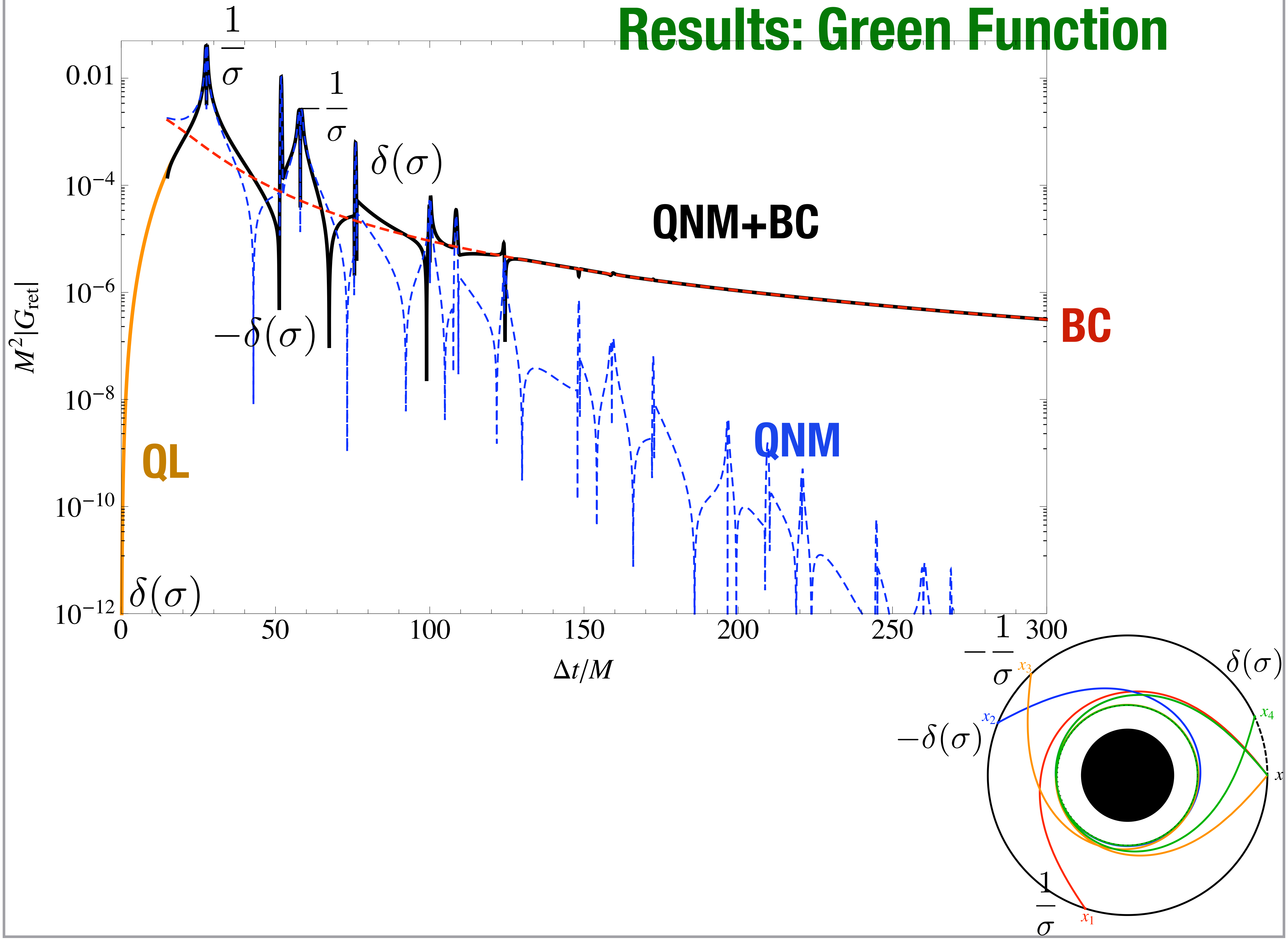
Results: Branch Cut

- Different BC ℓ-modes



- 'Wagging of the tail' due to $\ell = 1$

Results: Green Function

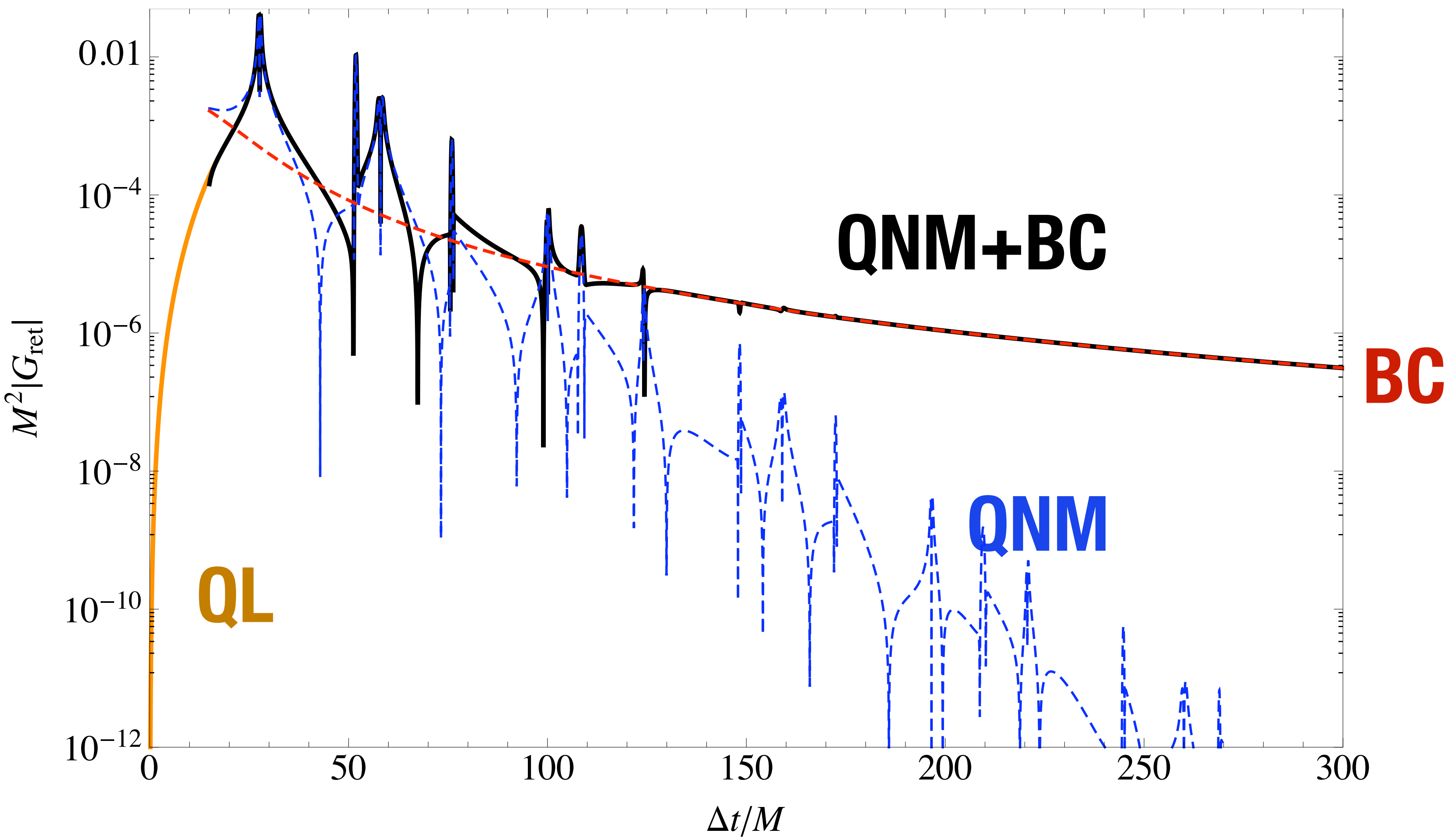


Results: Green Function

- **QNMs**: singularities at light-crossings

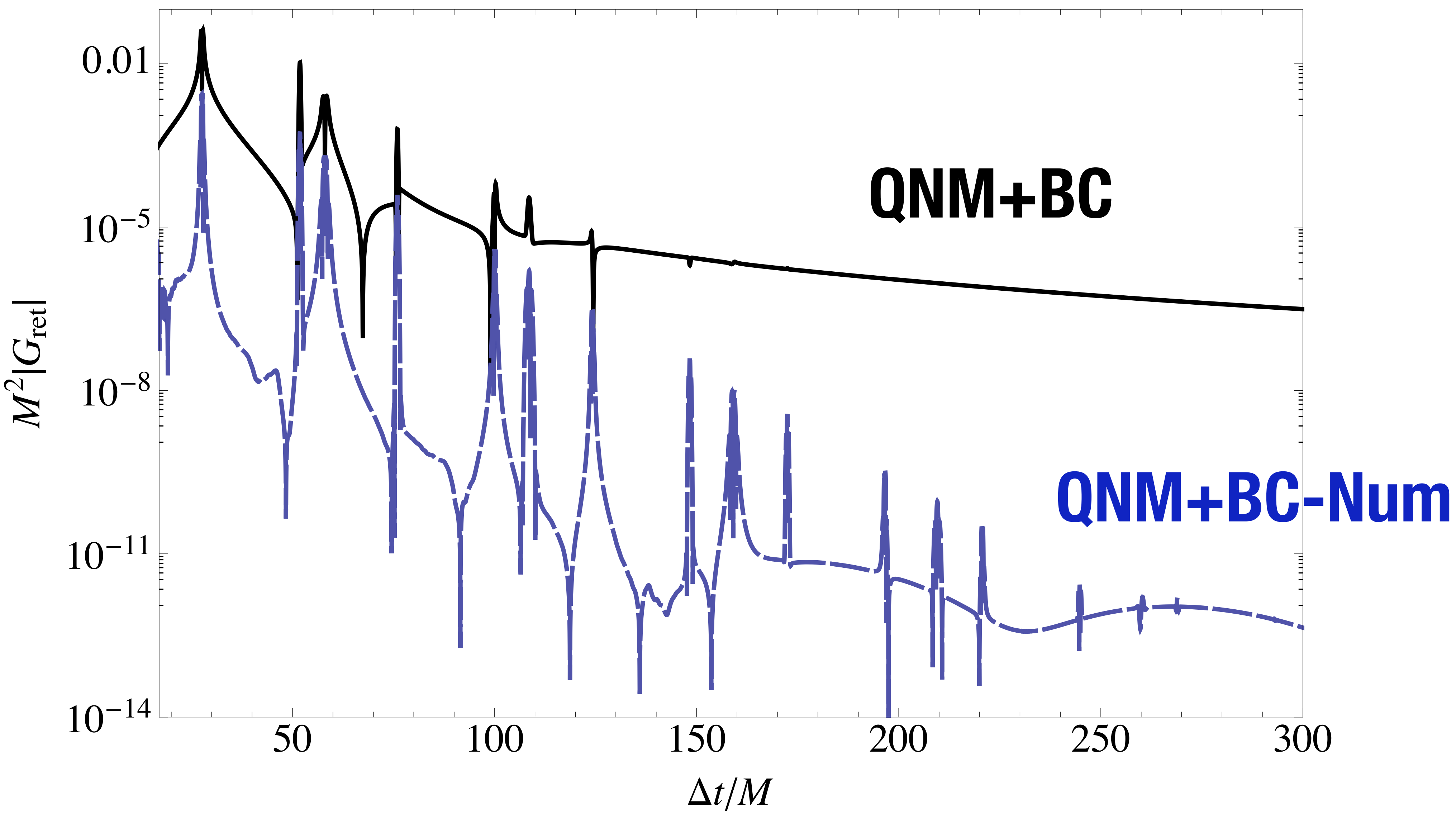
$$G_{ret} \sim \delta(\sigma),\ \frac{1}{\sigma},\ -\delta(\sigma),\ -\frac{1}{\sigma}$$

- **BC**: late-time tail
- Other times: sometimes QNM dominates, sometimes BC

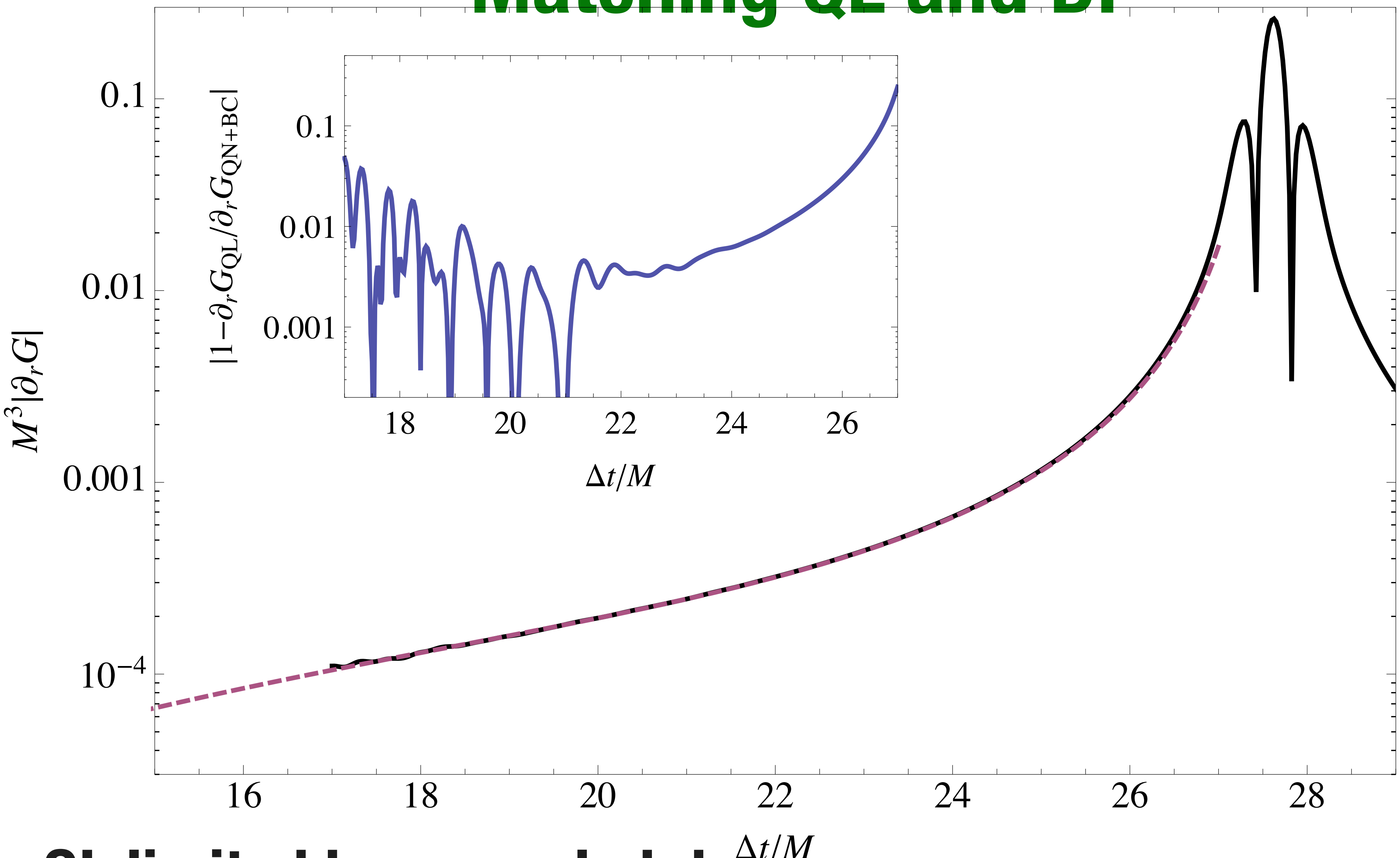


Green Function Validation

- **Validation of QNM+BC against an 'exact' numerical GF**



Matching QL and DP

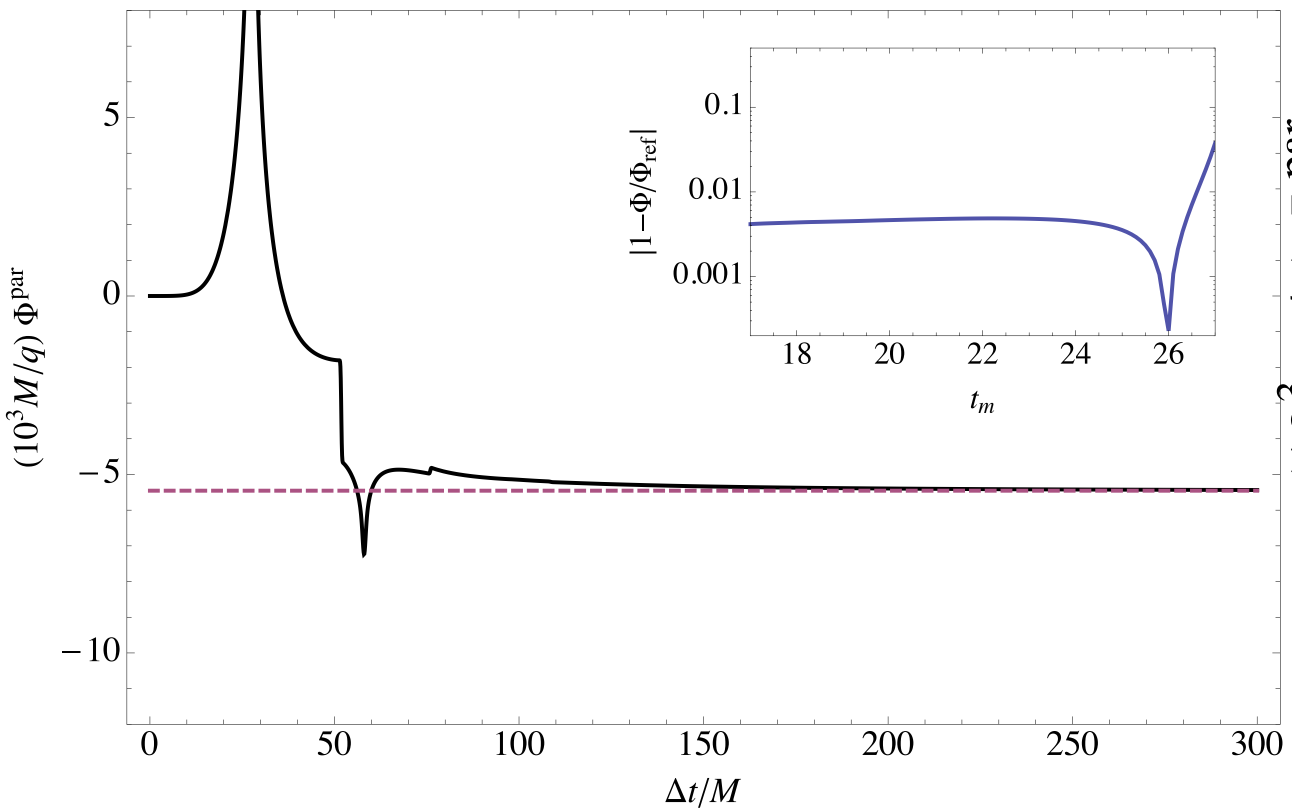


- **QL limited by normal nbd**
- **QNM+BC limited by divergences at early times**
- **QNM+BC limited by finite l-sum near 1st light-crossing (alternative: Nolan's talk)**

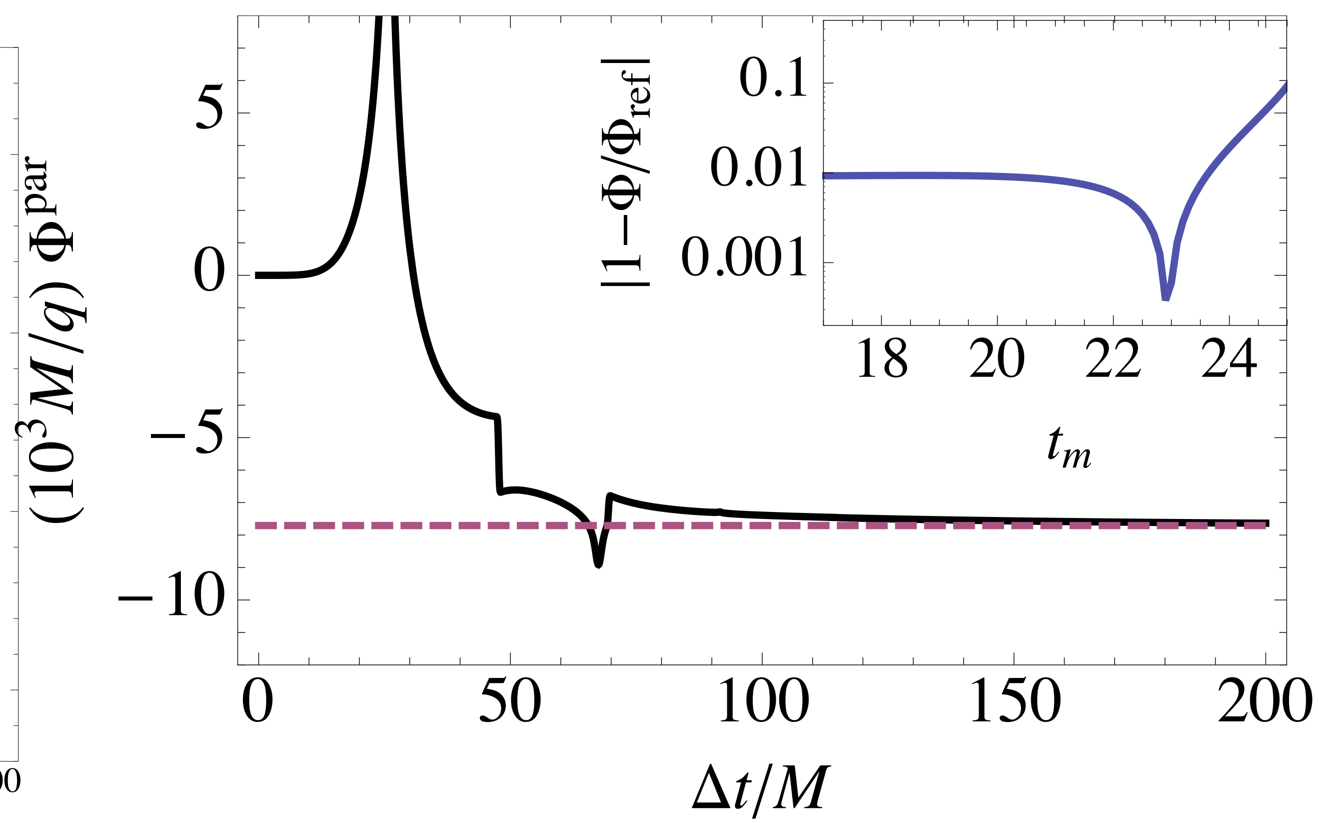
Results: Field

- ‘Exact’ values from mode-sum regularization (Warburton&Barack’11; Diaz-Rivera et al’04)

- ‘Partial field’ $\Phi^{par} \equiv q \int_{\tau-\Delta\tau}^{\tau^-} d\tau' \, G_{ret}$



Circular

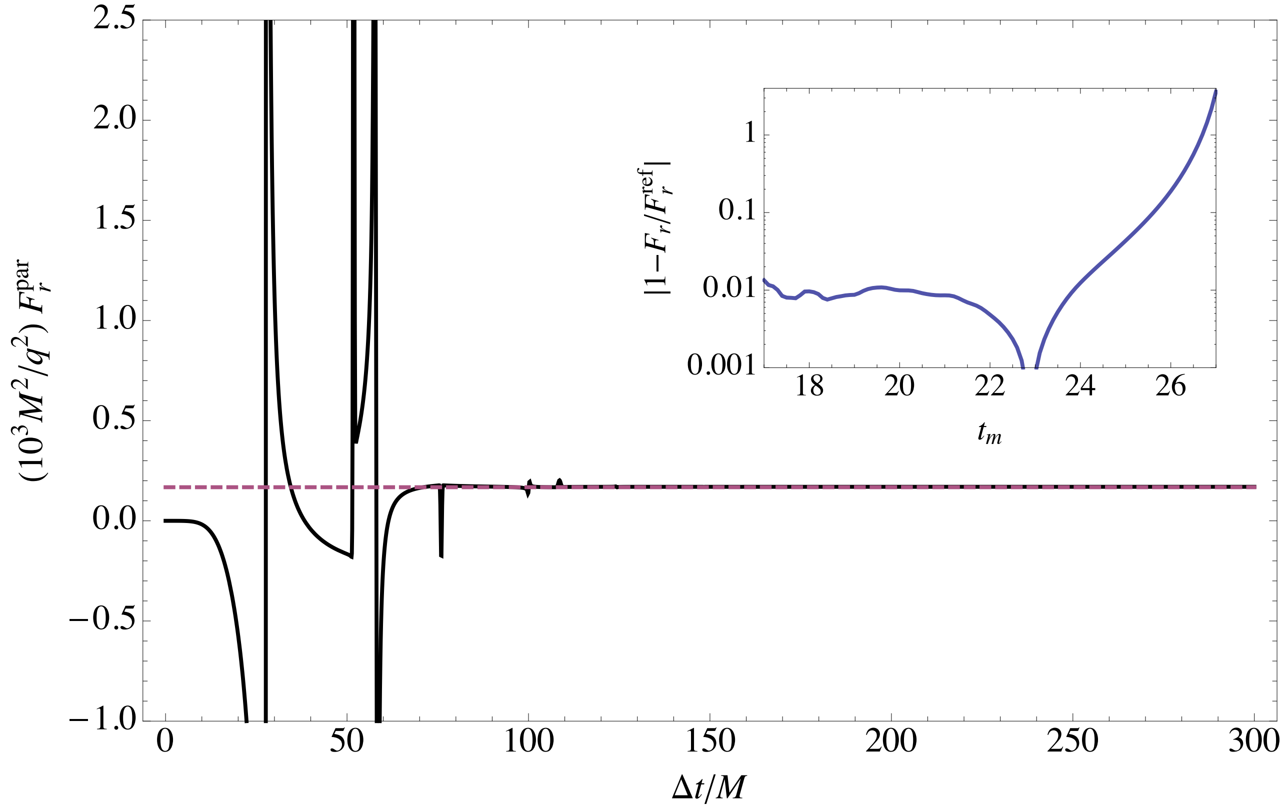


Eccentric

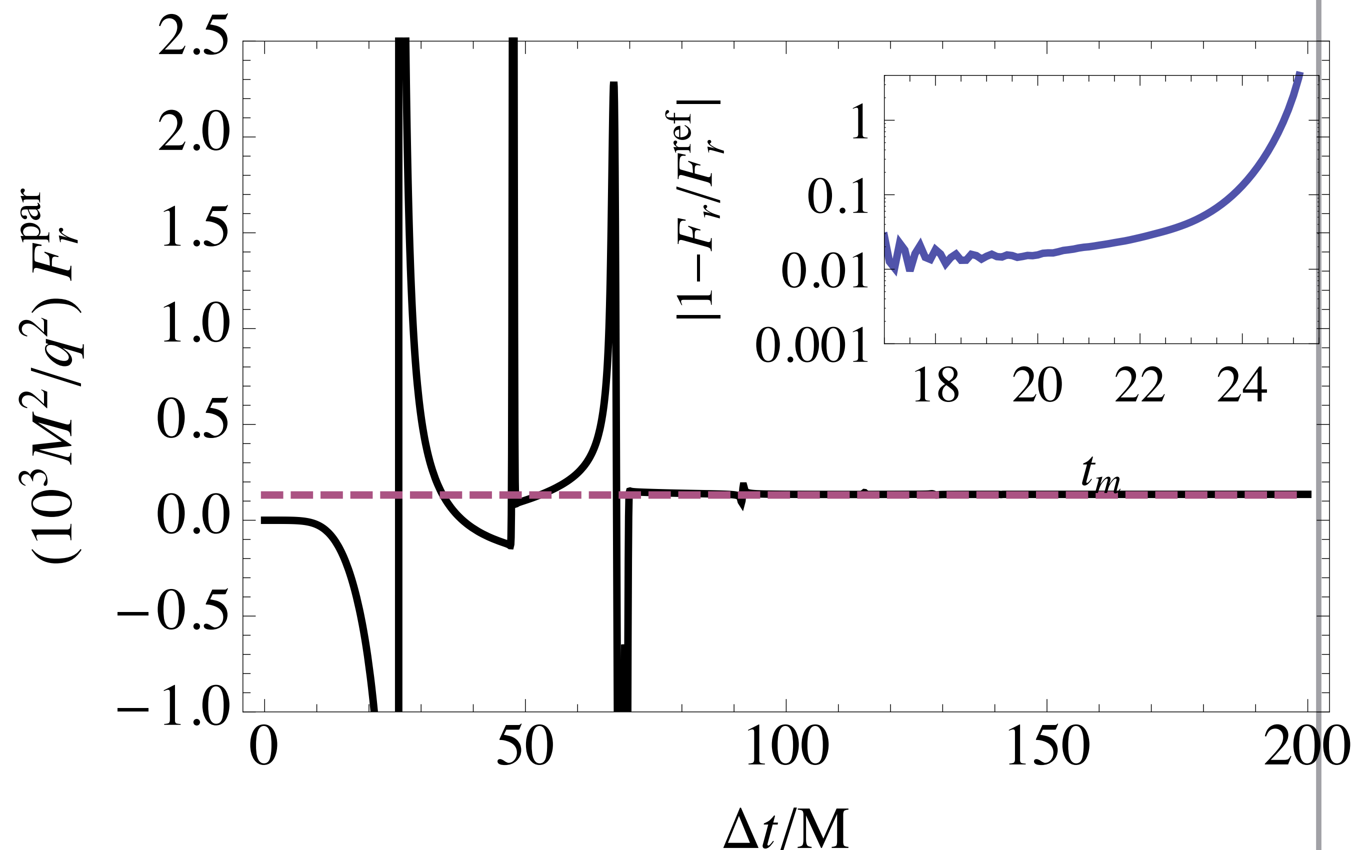
Results: Self-Force

- **'Partial S-F'** $F_\mu^{par} \equiv q\nabla_\mu \Phi^{par}$

Circular $F_r^{circ} = 1.67728 \cdot 10^{-4} q^2 M^{-2}$

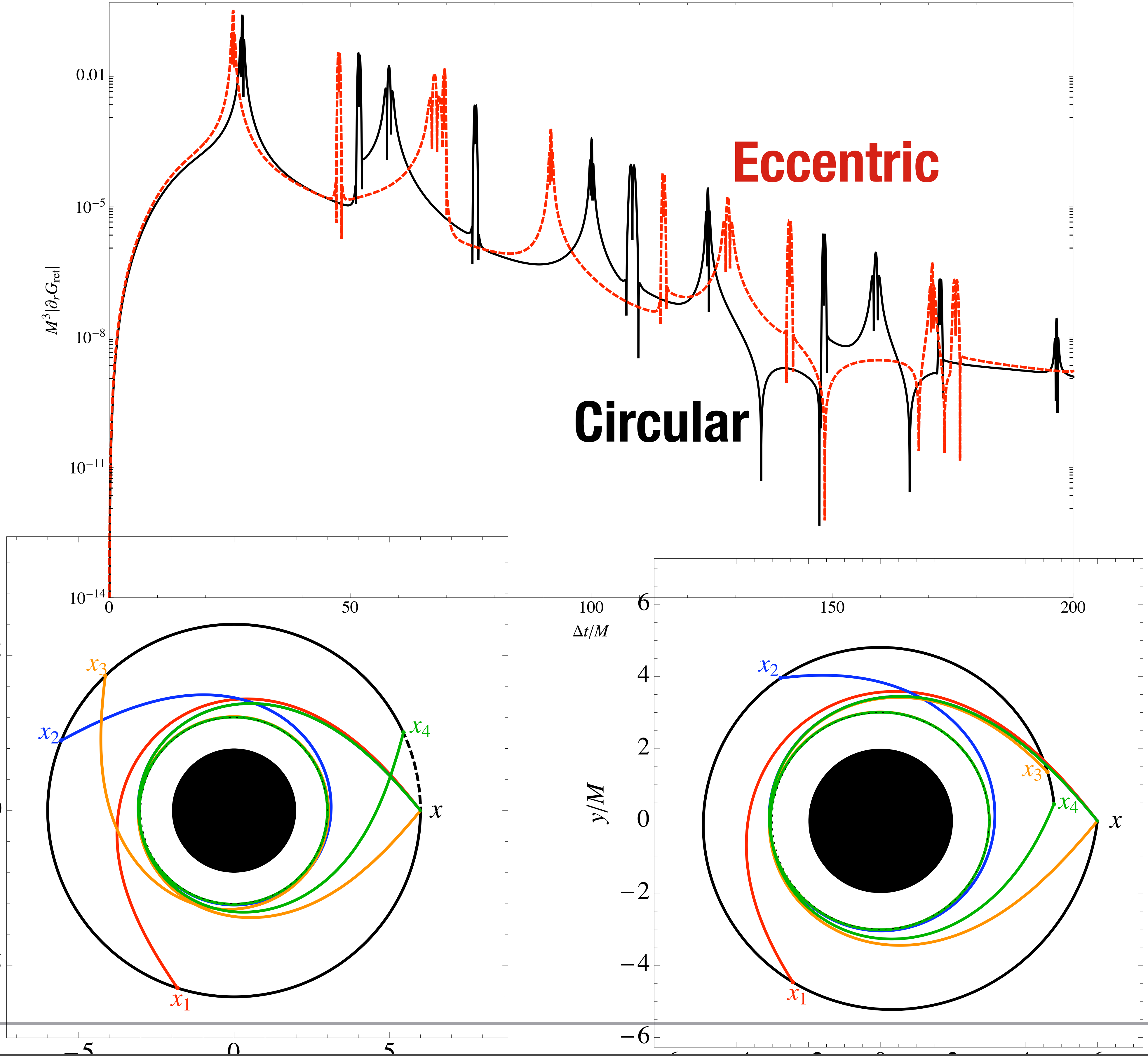


Eccentric $F_r^{ecc} = 1.31717 \cdot 10^{-4} q^2 M^{-2}$



- **Value 'settles' after 3rd light-crossing**
- **Rel.err.** $\approx 1\%$ **for** $t_m \in (17M, 23M)$

Circular vs Eccentric



Summary

- **Method of matched expansions successful in Schwarzschild!**
- **Advantages:**

- Trivial regularization

- Physical insight

- How good an approx. using n=0 for QNM and l=0 for BC ?

- Once r-indep quantities are calculated, only requires solving radial ODE

- Once GF calculated for all pairs of points, SF can be obtained for any orbit