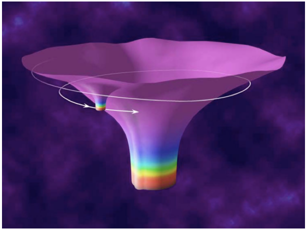
Approaches to Self-Force Calculations on Kerr Spacetime



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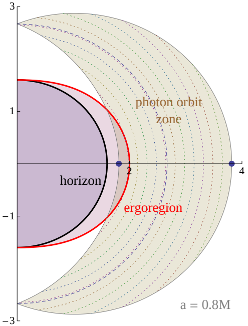
@ Capra 2013, Dublin

Talk Outline

1. **Motivation**
   - **Why** compute GSF on Kerr?
2. **Foundations**
   - **How** do we compute GSF?
3. **#1: Lorenz gauge/time domain**
   - Puncture/effective source schemes
   - 2+1D and 3+1D approaches
   - Mass and angular momentum
   - Linear-in-t gauge modes
4. **#2: Radiation gauge/freq domain**
   - Hertz potential/metric reconstruction
   - Regularization
5. **Results: Circular orbits on Kerr**
   - Gauge-invariant comparison
6. **Prospects**

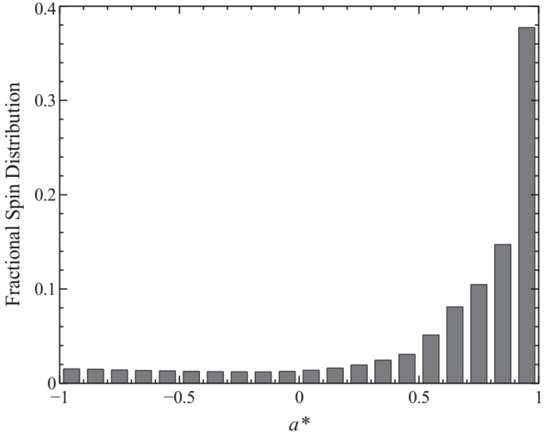
Motivation: Why study GSF on Kerr?

- Galactic BHs are **rotating**, $a/M \sim 0.5 - 0.99$.
- **Structure** : Rotation breaks symmetry leading to, e.g. ergodic geodesics, frame-dragging, light-cone caustics become ‘tubes’, etc.



- **Orbital resonances:** Generic orbits may pass through resonance when $\omega_r/\omega_\theta \sim n_1/n_2$ (Hinderer & Flanagan).
- Orbital evolutions
- Gravitational wave signatures: **eLISA**?

Supermassive BHs appear to be rapidly rotating ...

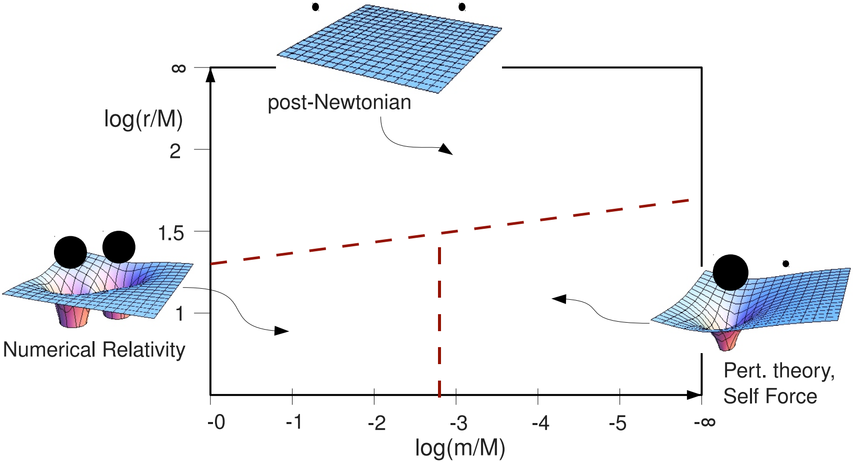


- Fig. 9 in Walton *et al.*, "Observations of 'bare' active galactic nuclei", MNRAS **428**, 2901 (2013), using X-ray reflection spectroscopy.

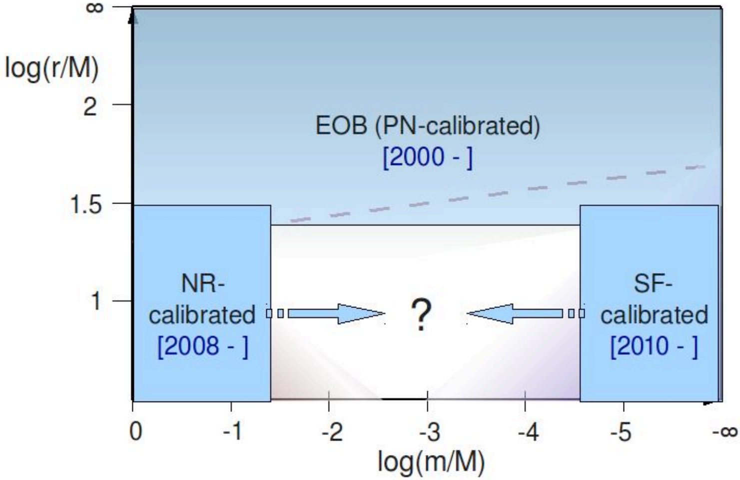
Table 2. Key parameters obtained for the reflection-based models constructed for the compiled sample (see Section 3.2 for details). Parameters in parentheses have not been allowed to vary, and where we were unable to constrain the black hole spin this is indicated with a 'U'.

Source	En. range (keV)	$C_{PIN/XIS}$	Γ	A_{Fe} (solar)	ξ ($erg\,cm\,s^{-1}$)	q	i (°)	a^*	χ^2_ν (χ^2/d.o.f.)
Mrk 509	0.6–44.0	1.17	2.04 ± 0.01	0.5 ± 0.1	170^{+30}_{-80}	>7.4	<18	$0.86^{+0.02}_{-0.01}$	1314/1333
3C 382	0.6–53.0	1.14	1.81 ± 0.01	>5.1	500^{+60}_{-240}	>6.1	(40)	$0.75^{+0.07}_{-0.04}$	1366/1317
Mrk 335	0.6–36.0	1.17	2.16 ± 0.01	1.4 ± 0.2	220 ± 10	>4.9	50^{+8}_{-7}	$0.83^{+0.10}_{-0.13}$	1233/1152
Fairall 9	0.6–39.0	1.16	1.99 ± 0.01	1.1 ± 0.2	140^{+60}_{-30}	>3.5	45^{+13}_{-9}	>0.64	1276/1253
1H 0419−577	0.6–48.0	1.19	$1.98^{+0.01}_{-0.02}$	0.9 ± 0.1	104^{+4}_{-26}	$5.4^{+0.2}_{-1.0}$	51^{+4}_{-6}	>0.88	1384/1294
Ark 564	0.6–22.0	1.20	2.52 ± 0.01	$1.0^{+0.2}_{-0.1}$	540^{+40}_{-20}	>6.2	64^{+1}_{-11}	$0.96^{+0.01}_{-0.06}$	1081/1011
Ark 120	0.6–53.0	1.22	2.13 ± 0.01	2.7 ± 0.6	9^{+2}_{-4}	$7.0^{+2.7}_{-2.2}$	54^{+6}_{-5}	$0.81^{+0.10}_{-0.18}$	1316/1177
3C 390.3	0.6–49.0	1.16	1.66 ± 0.01	$3.1^{+1.2}_{-0.5}$	840^{+490}_{-220}	(3)	(35)	U	1302/1259
PKS 0558−504	0.6–24.0	1.13	$2.30^{+0.02}_{-0.01}$	$0.9^{+0.3}_{-0.1}$	270^{+70}_{-30}	4.0 ± 0.7	(45)	>0.80	990/1022
NGC 7469	0.6–52.0	1.19	$1.84^{+0.03}_{-0.02}$	$2.9^{+1.7}_{-1.0}$	200^{+20}_{-50}	>4.6	<54	>0.96	1262/1139
Mrk 110	0.6–45.0	1.19	$1.96^{+0.02}_{-0.01}$	0.7 ± 0.2	310^{+220}_{-80}	>7.4	31^{+4}_{-6}	>0.99	1184/1115
Swift J0501.9−3239	0.6–36.0	1.22	$2.06^{+0.04}_{-0.03}$	$1.8^{+0.9}_{-0.5}$	200^{+10}_{-40}	>5.1	<48	>0.96	1025/1056
Mrk 841	0.6–53.0	1.19	$1.85^{+0.03}_{-0.01}$	1.0 ± 0.2	210^{+20}_{-70}	$4.1^{+2.8}_{-1.9}$	45^{+7}_{-5}	>0.56	1089/1053
Ton S180	0.6–23.0	1.16	2.36 ± 0.01	$0.9^{+0.2}_{-0.1}$	280^{+50}_{-20}	>8.1	60^{+3}_{-1}	$0.91^{+0.02}_{-0.09}$	876/838
PDS 456	0.6–17.0	1.15	$2.30^{+0.03}_{-0.01}$	>8.4	59^{+17}_{-4}	$5.9^{+1.8}_{-1.5}$	70^{+3}_{-5}	>0.97	829/826
1H 0323+342	0.6–42.0	1.25	$1.91^{+0.03}_{-0.01}$	0.8 ± 0.2	250^{+40}_{-20}	(3)	(45)	>0.48	864/922
UGC 6728	0.6–26.0	1.27	$2.00^{+0.04}_{-0.03}$	$0.7^{+0.6}_{-0.3}$	190^{+80}_{-170}	$6.8^{+2.8}_{-1.4}$	<55	>0.95	877/885
Mrk 359	0.6–21.0	1.15	$1.89^{+0.04}_{-0.03}$	$1.5^{+0.9}_{-0.6}$	21^{+32}_{-16}	>4.1	47 ± 6	$0.66^{+0.30}_{-0.46}$	820/833
MCG−2-14-9	0.6–37.0	1.19	1.89 ± 0.02	(1)	<10	(3)	(45)	U	802/804
ESO 548-G081	0.6–36.0	1.23	1.70 ± 0.03	$3.5^{+4.1}_{-1.5}$	570^{+560}_{-380}	(3)	(45)	U	853/845
Mrk 1018	0.6–41.0	1.21	$1.94^{+0.04}_{-0.03}$	$2.0^{+1.4}_{-0.7}$	5^{+10}_{-4}	>3.9	45^{+14}_{-10}	$0.57^{+0.31}_{-0.82}$	681/721
RBS 1124	0.6–23.0	1.22	$1.86^{+0.04}_{-0.02}$	$2.9^{+1.5}_{-0.9}$	51^{+7}_{-9}	>8.4	66^{+5}_{-15}	>0.98	661/668
IRAS 13224−3809	0.6–7.6	–	(2.7)	(20)	22 ± 3	$6.1^{+0.7}_{-0.6}$	(64)	>0.995	447/412
1H 0707−495	0.6–6.7	–	(2.7)	(10)	53^{+1}_{-2}	$7.6^{+0.4}_{-0.3}$	(58)	>0.994	278/236
IRAS 05262+4432	0.6–7.8	–	$2.18^{+0.13}_{-0.06}$	(1)	<51	(3)	(45)	U	234/231

Motivation: the general 2-body problem in relativity



Motivation: the general 2-body problem in relativity



Motivation: Orbital resonances

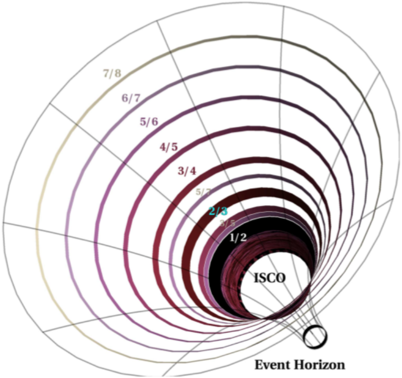


FIG. 1: (Color online) The location of low order resonances around a black hole superimposed on an embedding diagram. The line width of each resonance is inversely proportional to the order of the resonance to give an indication of the relative importance of a particular resonance.

Fig. 1 in Brink, Geyer & Hinderer, arXiv:1304.0330.

Motivation: Orbital resonances

Kolmogorov-Arnold-Moser theorem

For Hamiltonian system, perturbed dynamics will be a smooth and 'small' distortion if frequencies are sufficiently irrational:

$$|m\omega_r - n\omega_\theta| > K(\epsilon)/(n+m)^3$$

cf van de Meent

Motivation: Orbital resonances

- **Two timescales**: orbital period $\sim M$, radiation reaction μ^{-1}.
- Hinderer & Flanagan (2010) made two-timescale expansion for EMRIs, using **action-angle variables**:
  - **Action :** 'constants' of motion : $J_\nu = \left(E/\mu, L_z/\mu, Q/\mu^2\right)$
  - **Angle :** 'phase' variables $q_\alpha = (q_t, q_r, q_\theta, q_\phi)$.
- $q_r \to q_r + 2\pi$ as orbit goes $r = r_{\min} \to r_{\max} \to r_{\min}$ with period $\tau_r = 2\pi/\omega_r$.
- Frequencies $\omega_\alpha(J) = (\omega_r, \omega_\theta, \omega_\phi)$
- Isometries of Kerr $\Rightarrow$ (q_t, q_ϕ) 'irrelevant', (q_r, q_θ) 'relevant' params.

Motivation: Orbital resonances

1. **Geodesic** approximation ($\eta = 0$):

$$\begin{aligned} \frac{dq_\alpha}{d\tau} &= \omega_\alpha(J) \\ \frac{dJ_\nu}{d\tau} &= 0 \end{aligned}$$

Solution :

$$\begin{aligned} q_\alpha(\tau, \eta = 0) &= \omega_\alpha\, \tau && (1) \\ J_\nu(\tau, \eta = 0) &= \text{const.} && (2) \end{aligned}$$

Timescale : unchanging

Motivation: Orbital resonances

2. **Adiabatic** approximation:

$$\begin{aligned} \frac{dq_\alpha}{d\tau} &= \omega_\alpha(J) \\ \frac{dJ_\nu}{d\tau} &= \eta \left\langle G^{(1)}_\nu(q_r, q_\theta, J) \right\rangle_{\text{average}} \end{aligned}$$

Solution :

$$\begin{aligned} q_\alpha(\tau, \eta) &= \eta^{-1}\hat{q}(\eta\tau) \\ J_\nu(\tau, \eta) &= \hat{J}(\eta\tau) \end{aligned}$$

Timescale : $\tau_{rad.reac.} \sim \eta^{-1}$

Motivation: Orbital resonances

3. **Post-adiabatic** approximation:

$$\begin{aligned} \frac{dq_\alpha}{d\tau} &= \omega_\alpha(J) + \eta g^{(1)}_\alpha(q_r, q_\theta, J) + \mathcal{O}(\eta^2) \\ \frac{dJ_\nu}{d\tau} &= \eta G^{(1)}_\nu(q_r, q_\theta, J) + \eta^2 G^{(2)}_\nu(q_r, q_\theta, J) + \mathcal{O}(\eta^3). \end{aligned}$$

Two timescales : $\sim \eta^{-1}$ (secular) and ~ 1 (oscillatory).

Motivation: Orbital resonances

- **Is adiabatic approximation justified?** i.e. is it OK to neglect fast-oscillating parts?
- Consider Fourier decomposition

$$G_\nu^{(1)}(q_r, q_\theta, J) = \sum_{k_r, k_\theta} G_{\nu k_r, k_\theta}^{(1)}(J) e^{i(k_r q_r + k_\theta q_\theta)}$$

and $q_r = \omega_r \tau + \dot{\omega}_r \tau^2 + \ldots, \quad q_\theta = \omega_\theta \tau + \dot{\omega}_\theta \tau^2 + \ldots$

$$k_r q_r + k_\theta q_\theta = (k_r \omega_r + k_\theta \omega_\theta)\, \tau + (k_r \dot{\omega}_r + k_\theta \dot{\omega}_\theta)\, \tau^2 + \ldots$$

- Cannot neglect higher Fourier components if **resonance condition**

$$k_r \omega_r + k_\theta \omega_\theta = 0$$

is satisfied! i.e. when ω_r / ω_θ passes through low-order integer ratio.

Motivation: Orbital resonances

- Duration of resonance set by $(k_r \dot{\omega}_r + k_\theta \dot{\omega}_\theta)\, \tau^2 \sim 1$, i.e.

$$\tau_{\text{res}} \sim 1/\sqrt{p\eta}$$

  where $p \equiv |k_r| + |k_\theta|, \quad \eta = \mu/M$.
- Change in ‘constants’ of motion:

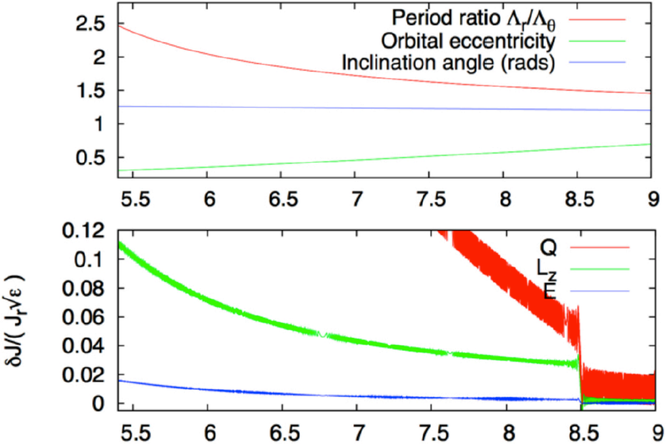
$$\Delta J \sim \sqrt{\eta/p}$$

- Change in phase:

$$\Delta q \sim 1/\sqrt{\eta p}$$

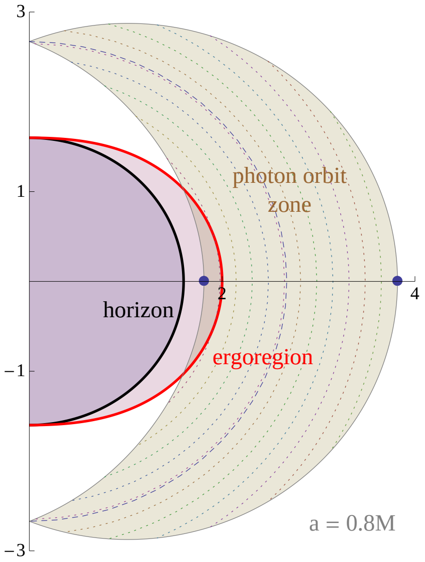
- Need to know precise first-order SF and (possibly) dissipative part of 2nd-order SF to model resonance accurately.
- Without complete knowledge, a resonance effectively resets the phase and ‘kicks’ the orbital parameters.

Motivation: Orbital resonances

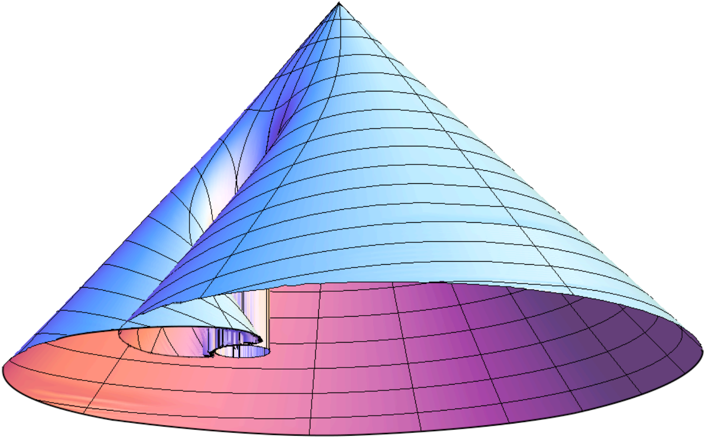


- from Hinderer & Flanagan, arXiv:1009.4923.

Motivation: Structure of spacetime

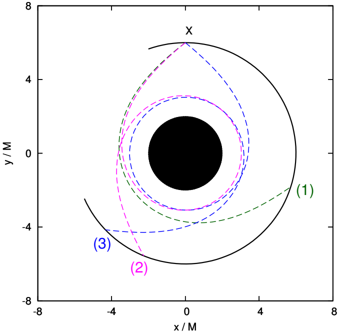


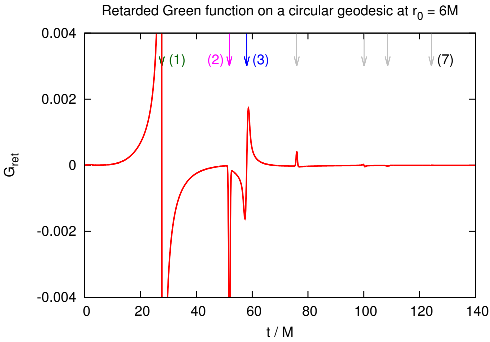
Motivation: Structure of spacetime



Light cone in Schwarzschild. See e.g. V. Perlick's Living Review on lensing.

Singular structure of Green function





MiSaTaQuWa: SF from worldline integral:

$$f_\mu^{(SF)} \sim q \int_{-\infty}^{\tau^-} \nabla_\mu G(z(\tau), z(\tau'))d\tau'.$$

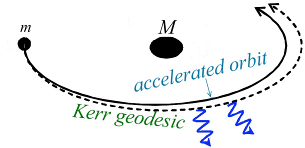
(scalar field case) see e.g. Casals *et al.* (2013), arXiv:1306.0884.

Foundations

Gravitational Self-Force (GSF)

accelerated motion on a background spacetime

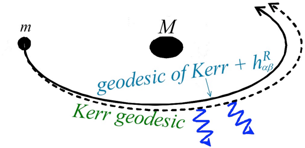
$$\mu\vec{a}_g = \vec{F}_{\rm self} = \vec{F}_{\rm diss} + \vec{F}_{\rm cons}$$



$$\Updownarrow$$

geodesic motion in a perturbed spacetime

$$\mu\vec{a}_{g+h} = 0$$



$$g_{\mu\nu} = \tilde{g}_{\mu\nu} + \mu h_{\mu\nu} + \ldots \quad \text{and} \quad F^\alpha_{{\rm ret}/S} \equiv \mu \nabla^{\alpha\mu\nu} h^{{\rm ret}/S}_{\mu\nu}$$

Three (related) methods for GSF calculations

1. Worldline integral (MiSaTaQuWa equation, schematically):

$$F_\alpha^{\text{self}} = \text{local terms} + \mu^2 u^\mu u^\nu \int_{-\infty}^{\tau^-} \nabla_{[\alpha} \bar{G}_{\mu]\nu\mu'\nu'}(z(\tau), z(\tau')u^{\mu'} u^{\nu'} d\tau'$$

2. Mode sum regularization: $h_{\mu\nu} = \sum_{ilm} h_{\mu\nu}^{(i)lm} Y_{lm}^{(i)}(\theta, \phi)$

$$F_{\text{self}}^\alpha = \sum_{\ell=0}^{\infty} \left[F_{\text{ret}}^\ell(p) - AL - B - C/L\right] - D$$

where $L = l + 1/2$.

3. Effective source / puncture schemes: $h = h^R + h^S$ split (Detweiler-Whiting '03)

$$F_{\text{self}}^\alpha = -\frac{\mu}{2}\left(g^{\alpha\beta} + u^\alpha u^\beta\right)\left(2h^R_{\beta\gamma;\delta} - h^R_{\gamma\delta;\beta}\right) u^\gamma u^\delta.$$

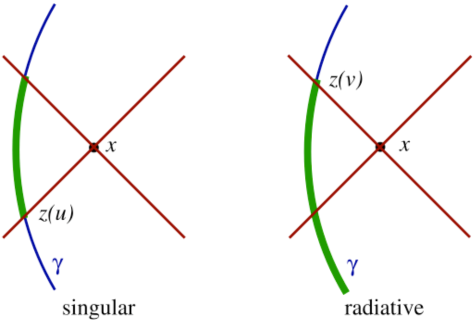
(2): ℓ-mode regularization

Define $F^{\alpha}_{\mathrm{ret}/S} \equiv \mu \nabla^{\alpha\mu\nu} h^{\mathrm{ret}/S}_{\mu\nu}$ (as fields), then write

$$
\begin{aligned}
F_{\mathrm{self}} &= \left.(F_{\mathrm{ret}} - F_{\mathrm{S}})\right|_{\mathrm{p}} \\
&= \sum_{\ell=0}^{\infty} \left.\left(F^{\ell}_{\mathrm{ret}} - F^{\ell}_{S}\right)\right|_{\mathrm{p}} \quad (\ell\text{-mode contributions are finite}) \\
&= \sum_{\ell=0}^{\infty} \left[F^{\ell}_{\mathrm{ret}}(p) - AL - B - C/L\right] - \sum_{\ell=0}^{\infty} \left[F^{\ell}_{S}(p) - AL - B - C/L\right] \\
&= \sum_{\ell=0}^{\infty} \left[F^{\ell}_{\mathrm{ret}}(p) - AL - B - C/L\right] - D \quad (\text{where } L = \ell + 1/2)
\end{aligned}
$$

- Regularization Parameters A, B, C, D calculated analytically for generic orbits in Kerr in Lorenz gauge $\bar{h}_{\mu\nu}{}^{;\nu} = 0$.

(3): Detweiler-Whiting split



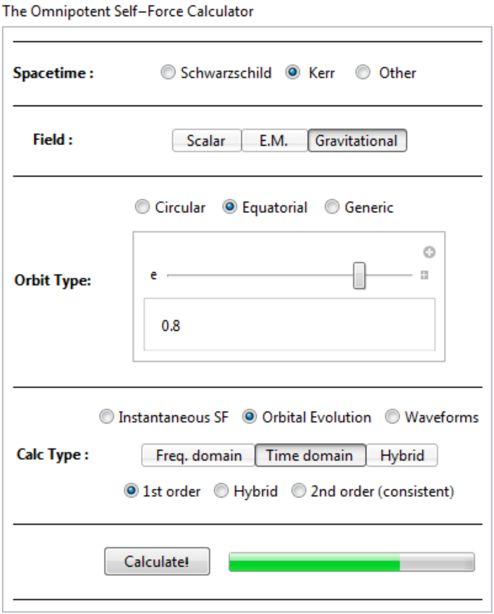
- Dirac's split into singular and radiative fields is **acausal** in curved spacetime
- Detweiler & Whiting ('03) made causal split into S and R fields
- Correct SF recovered from R part.
- S part not known exactly, but can be computed in vicinity of worldline via series expansions.

Dissipative/Conservative part of GSF

- Retarded and advanced fields $h_{\rm ret}$ and $h_{\rm adv}(t)$
- Ret. and adv. 'R' fields, $h^R_{\rm ret} = h_{\rm ret} - h_S$, $h^R_{\rm adv} = h_{\rm adv} - h_S$
- Define conservative and dissipative parts of field

$$
\begin{aligned}
h^{\rm cons} &= \frac{1}{2}\left(h^R_{\rm ret} + h^R_{\rm adv}\right) = \frac{1}{2}\left(h_{\rm ret} + h_{\rm adv} - 2h_S\right) \\
h^{\rm diss} &= \frac{1}{2}\left(h^R_{\rm ret} - h^R_{\rm adv}\right) = \frac{1}{2}\left(h_{\rm ret} - h_{\rm adv}\right)
\end{aligned}
$$

- Dissipative part does not need regularization!
- Conservative part needs knowledge of S field.
- Dissipative part $\Rightarrow$ secular loss of energy and angular momentum.
- Conservative part $\Rightarrow$ shift in orbital parameters, periodic.



This is what we need ...

Approach #1: Lorenz gauge / time domain

Approach #1: Lorenz-gauge time-domain

$$\Box \bar{h}_{\mu\nu} + 2R^{\alpha}{}_{\mu}{}^{\beta}{}_{\nu}\bar{h}_{\alpha\beta} = -16\pi G T_{\mu\nu}, \qquad \bar{h}_{\mu\nu}{}^{;\nu} = 0.$$

Q1. Why work in Lorenz-gauge $\bar{h}_{\mu\nu}{}^{;\nu} = 0$?

- Hyperbolic (wave-like) formulation of equations for metric perturbation
- S-field has 'symmetric' singular part $\bar{h}_{ab} \sim 1/r$
  $\Rightarrow$ regularization is well-understood.

Q2. Why work in time-domain?

- Lorenz-gauge metric perturbation is **not separable** on Kerr
  $\Rightarrow$ no *ordinary* differential equation formulation in freq. domain.
- **Self-consistent evolutions** are most naturally handled within a time-domain scheme.

2+1D vs 3+1D methods

Two related approaches:

- 3+1D effective source method, developed by Vega, Detweiler, Diener, Wardell *et al.*
- 2+1D m-mode regularization scheme, developed by Barack, Sago, Golbourn, Thornburg, Dolan, Wardell.

3+1D approach

- Window function W:

$$S_{\text{eff}} = S - \Box(W\Phi^S)$$

- No mode sum required
- Methods of Num. Relativity
- Only scalar field so far

2+1D approach

- Puncture + worldtube:

$$\Phi_{\mathcal{R}} = \Phi - \Phi_{\mathcal{P}}$$

- Mode sum reconstruction
- Isolate $m = 0$, $m = 1$ parts
- Scalar & gravitational cases

Formulation: Linearized equations

Linearized Einstein Eqs for Ricci-flat background:

$$\Box \bar{h}_{ab} + 2R^{c}{}_{a}{}^{d}{}_{b}\bar{h}_{cd} + Z^{c}{}_{;c} - Z_{a;b} - Z_{b;a} = -16\pi T_{ab},$$

$Z_a \equiv \bar{h}_{ab}{}^{;b}$, where $\bar{h}_{ab}$ is the trace-reversed metric perturbation:

$$\bar{h}_{ab} = h_{ab} - \tfrac{1}{2} g_{ab} h, \quad \text{and} \quad h = h^{a}{}_{a}.$$

Z4 system and gauge choice

Introduce Generalized Lorenz gauge with gauge-driver $H_a(h_{bc}, x)$

$$Z_a = H_a(x, h_{bc}) \quad (= 0 \quad \text{for Lor. gauge})$$

Z4 system: 10 eqns with 4 constraints,

$$\begin{aligned} \Box \bar{h}_{ab} + 2R^{c}{}_{a}{}^{d}{}_{b}\bar{h}_{cd} + H^{c}{}_{;c} - H_{a;b} - H_{b;a} &= -16\pi T_{ab}, \\ c_a \equiv Z_a - H_a &= 0 \end{aligned}$$

Formulation: Linearized equations

Z4 with constraint damping

$$\Box\bar{h}_{ab} + 2R^{c\ \ d}_{\ a\ \ b}\bar{h}_{cd} + H^{c}{}_{;c} - H_{a;b} - H_{b;a} + \kappa\left(n_a c_b + n_b c_a\right) = -16\pi T_{ab},$$

where $\kappa(x)$ is a scalar function and n_a is a vector, and

$$c_a = Z_a - H_a.$$

- Choose κ, n_a so that constraints are damped, under

$$\Box c_a = -\left(\kappa(n_a c_b + n_b c_a)\right)^{;b}.$$

- Good choice: n_a = ingoing principal null direction, with $\kappa < 0$.
- h_{ab} is a solution of linearized Einstein eqns iff $c_a = 0$.

Formulation: Regularization

- **Problem:** $\bar{h}_{ab}$ is divergent $\sim 1/\epsilon$ towards worldline
- **Solution:** Introduce **puncture** $\bar{h}^{P}_{ab}$: a local approximation to Detweiler-Whiting singular field $\bar{h}^{S}_{ab}$.
- Covariant expansion of $\bar{h}^{S}_{ab} \Rightarrow$ power-series in coordinate differences,

$$\delta x^a = x^a - \bar{x}^a, \quad \text{where} \quad x = \text{field pt}, \quad \bar{x} = \text{worldline pt}$$

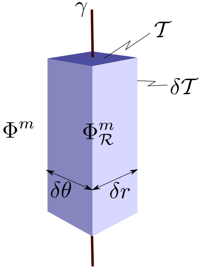
- **Classification:** nth order puncture iff

$$h^{P}_{ab} - h^{S}_{ab} \sim \mathcal{O}\left(|\delta x|\, \delta x^{n-2}\right)$$

- 2nd-order in Barack *et al* '07, 4th+ order from Wardell.
- Local $\rightarrow$ Global definition: let $\bar{x}$ become a function of x, e.g. set $\bar{t} = t$, $\bar{\mathbf{x}} = \mathbf{x}_p(t)$.
- Global continuation is arbitrary, but should be smooth around circle, except at worldline, for m-mode scheme
- Use a periodic definition φ, e.g. $\delta\varphi^2 \rightarrow 2(1 - \cos\delta\varphi) = \delta\varphi^2 + \mathcal{O}(\delta\varphi^4)$

Formulation: Puncture scheme

Introduce a **worldtube $\mathcal{T}$** surrounding the worldline:



- Outside worldtube $\mathcal{T}$, evolve *retarded* field $\bar{h}_{ab}$.
- Inside worldtube $\mathcal{T}$, evolve *residual* field $\bar{h}^{\mathcal{R}}_{ab}$, i.e.

$$\begin{cases} \hat{\mathcal{D}} h_{ab} = 0, & \text{outside } \mathcal{T}, \\ \hat{\mathcal{D}} h^{\mathcal{R}}_{ab} = -16\pi T^{\text{eff}}_{ab}, & \text{inside } \mathcal{T}, \\ h^{\mathcal{R}}_{ab} = h_{ab} - h^{\mathcal{P}}_{ab}, & \text{across } \partial\mathcal{T}. \end{cases}$$

where $T^{\text{eff}}_{ab} \equiv T_{ab} - (-16\pi)^{-1} \hat{\mathcal{D}} h^{\mathcal{P}}_{ab}$, and $\hat{\mathcal{D}}$ is wave operator.

Formulation: m-mode decomposition

- Exploit the **axial symmetry**: decompose MP in m-modes $\Rightarrow$ 2+1D eqns:

$$\bar{h}_{ab} = \sum_m \bar{h}^{(m)}_{ab} e^{im\varphi}.$$

- Real field $\Rightarrow \bar{h}^{(m)*}_{ab} = \bar{h}^{(-m)}_{ab}$
- Reconstruct self-force, field, etc. from mode sums, e.g.

$$\bar{h}^{R}_{ab} = \lim_{x\to z}\left(\bar{h}^{\mathcal{R}(m=0)}_{ab} + 2\sum_{m=0}^{\infty} \mathrm{Re}\left[\bar{h}^{\mathcal{R}(m)}_{ab} e^{im\varphi_0(t)}\right]\right)$$

- Convergence-with-m depends on order of puncture

Formulation: Mode sums and convergence

For circular orbits, F_r is conservative and F_φ is dissipative.

punc. order	$\bar{h}^{\mathcal{R}}_{\mu\nu}$	C	$S_{\rm eff}$	$\bar{h}^{\mathcal{R},m}_{\mu\nu}$	F^m_r	F^m_φ
1	$\delta x/\lvert\delta x\rvert$	C^{-1}	$1/\delta x^2$	m^{-2}	—	—
2	$\lvert\delta x\rvert$	C^0	$1/\lvert\delta x\rvert$	m^{-2}	m^{-2}	$e^{-\lambda m}$
3	$\lvert\delta x\rvert\,\delta x$	C^1	$\delta x/\lvert\delta x\rvert$	m^{-4}	m^{-2}	$e^{-\lambda m}$
4	$\lvert\delta x\rvert\,\delta x^2$	C^2	$\lvert\delta x\rvert$	m^{-4}	m^{-4}	$e^{-\lambda m}$

Formulation: Mass and angular momentum

- Combine Killing vector X^a and stress-energy T_{ab} to form

$$\textbf{conserved current:} \quad j_a \equiv T_{ab} X^b, \qquad {j_a}^{;a} = 0.$$

- Poincaré lemma: $\delta j = 0 \Rightarrow j = \delta F$ (where $\delta = {}^*d^*$), i.e.

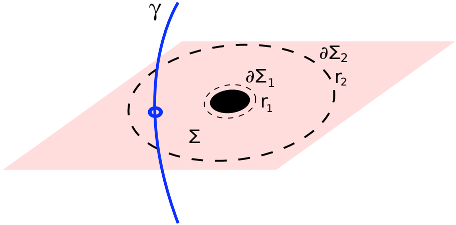
$$j_a = {F_{ab}}^{;b}, \quad \text{where} \quad F_{ab} = F_{[ab]},$$

(locally at least).

Abbott & Deser (1982): Conserved two-form

$$F_{ab} \equiv -(8\pi)^{-1} \left(X^c \bar{h}_{c[a;b]} + {X^c}_{;[a} \bar{h}_{b]c} + X_{[a} Z_{b]} \right),$$

Formulation: Mass and angular momentum



Apply Stokes' theorem to get 'quasi-local' definitions:

$$\begin{aligned}
\int_\Sigma j^a d\Sigma_a &= \int_\Sigma F^{ab}{}_{;b}\, d\Sigma_a \\
&= \frac{1}{2}\left[\int_{\partial\Sigma} F^{ab} dS_{ab}\right]_{r_1}^{r_2} \\
&= \begin{cases} \mu X^a u_a, & r_1 < r_0 < r_2, \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

Formulation: Mass and angular momentum

Quasi-local quantity: $\mathcal{F}(X, \partial\Sigma) \equiv \frac{1}{2} \int_{\partial\Sigma} F^{ab} dS_{ab}.$

Is $\mathcal{F}$ a useful definition of the mass/ang.mom. in a given homogeneous metric perturbation h_{ab}?

Property 1: $\mathcal{F}$ is gauge-invariant

- If $h_{ab} = 2\xi_{(a;b)}$ then $F_{ab} \propto \eta_{abc}{}^{;c}$, where

$$\eta_{abc} \propto X_{[a}\xi_{b;c]} + X_{[a;b}\xi_{c]}.$$

- It follows that $\mathcal{F} \propto \int (b_{\phi,\theta} - b_{\theta,\phi}) d\theta d\phi = [b_\phi]_0^\pi = 0$, where $b = {}^*\eta$.

Formulation: Mass and angular momentum

Quasi-local quantity: $\mathcal{F}(X, \partial\Sigma) \equiv \frac{1}{2}\int_{\partial\Sigma} F^{ab} dS_{ab}.$

Is $\mathcal{F}$ a useful definition of the mass/ang.mom. in a given homogeneous metric perturbation h_{ab}?

Property 2: $\mathcal{F}$ gives correct mass/ang. mom. for Kerr pert.

- $X^a_{(t)} = [1, 0, 0, 0] \Leftrightarrow \mathcal{F}_{(t)}$ and $X^a_{(\phi)} = [0, 0, 0, 1] \Leftrightarrow \mathcal{F}_{(\phi)}$
- Mass (M) and ang. mom ($J \equiv aM$) perturbations:

$$h_{ab} = \mu\mathcal{E}\, \frac{\partial}{\partial M} g^{\text{Kerr}}_{ab}\bigg|_J \quad \Rightarrow \quad \mathcal{F}_{(t)} = \mu\mathcal{E}, \quad \mathcal{F}_{(\phi)} = 0.$$

$$h_{ab} = \mu\mathcal{L}\, \frac{\partial}{\partial J} g^{\text{Kerr}}_{ab}\bigg|_M \quad \Rightarrow \quad \mathcal{F}_{(t)} = 0, \quad \mathcal{F}_{(\phi)} = \mu\mathcal{L}.$$

Implementation: Circular orbits on Kerr

- Particle on circular orbit with frequency $\omega = \sqrt{M}/(r_0^{3/2} + a\sqrt{M})$
- Define $\bar{h}_{ab}$ w.r.t. Boyer-Lindquist coordinate system (t, r, θ, ϕ)
- Introduce tortoise coords: $r_* = \int \frac{r^2+a^2}{\Delta} dr$, $\varphi = \phi + \int \frac{a}{\Delta} dr$
- Second-order puncture $\bar{h}^{\mathcal{P}}_{ab} \sim 4\mu\chi_{ab}/\epsilon$ [Barack et al.'07], with

$$\chi_{ab} = \begin{cases} u_a u_b + C_{ab}\delta r & \text{for } ab = tt, t\phi, \phi\phi \\ C_{ab}\sin\delta\phi & \text{for } ab = tr, t\phi. \end{cases}$$

- m-mode decomposition:

$$\bar{h}^{\mathcal{P}(m)}_{ab} = \frac{e^{-im(\omega t + \Delta\phi)}}{2\pi} \int_{-\pi}^{\pi} \bar{h}^{\mathcal{P}}_{ab}(\delta r, \delta\theta, \delta\phi) e^{-im\delta\phi} d(\delta\phi)$$

  Integrals have an elliptic integral representation.
- Use scaled evolution variables $u^{(m)}_{ab}$,

$$\bar{h}^{(m)}_{ab} = \frac{1}{r}\Xi_a \Xi_b u^{(m)}_{ab}(t, r, \theta) \qquad \text{(no sum)}$$

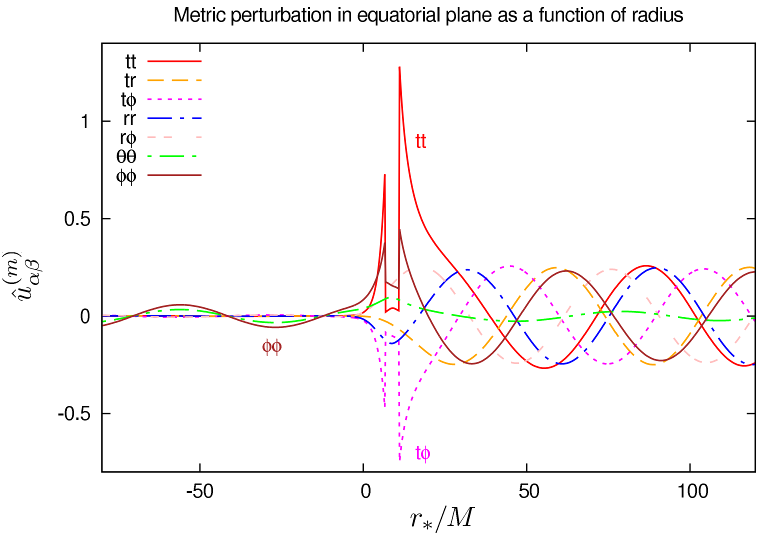
  where $\Xi_a = [1, 1/(r - r_h), r, r\sin\theta]$.

Implementation: Circular orbits on Kerr

- I used Lorenz-gauge $Z4$ system with constraint damping.
- Cauchy evolution in (t, r_*, φ), with worldtube and effective source.
- Fourth-order-accurate finite-differencing ... except at worldline where residual field is not smooth.
- Boundary conditions:
  1. Regular MP at the poles
  2. Regular MP on the future horizon
  3. $u_{ab}^{(m)} \sim \mathcal{O}(1)$ as $r \to \infty$
- Trivial initial conditions, $u_{ab}^{(m)} = 0$ ... wait long enough and
- 'Junk' dissipates with time (in radiative sector).
- Gauge-violation is driven to zero.

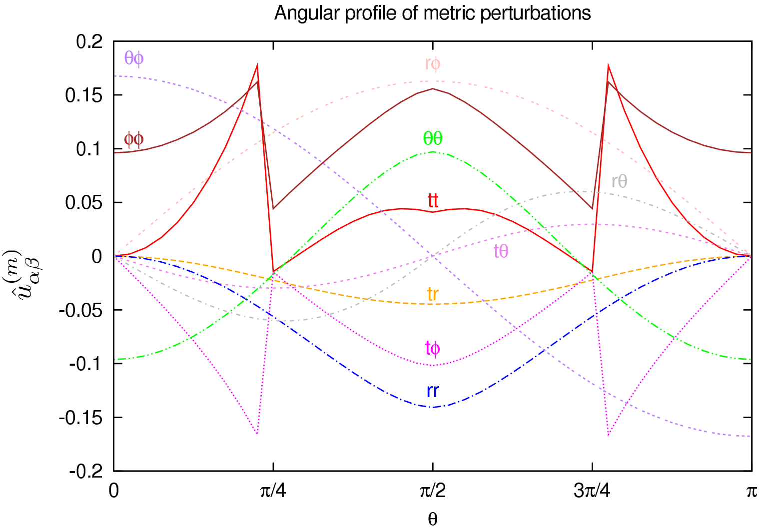
Results: Modal profiles

Slice 1: $t = 250M, \theta = \pi/2$ (and $r_0 = 7M$, $m = 2$)



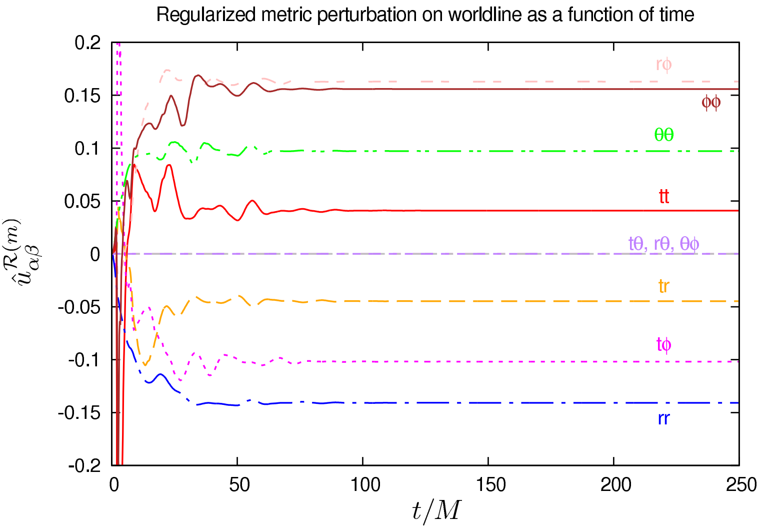
Results: Modal profiles

Slice 2: $t = 250M, r = r_0$ (and $r_0 = 7M$, $m = 2$)

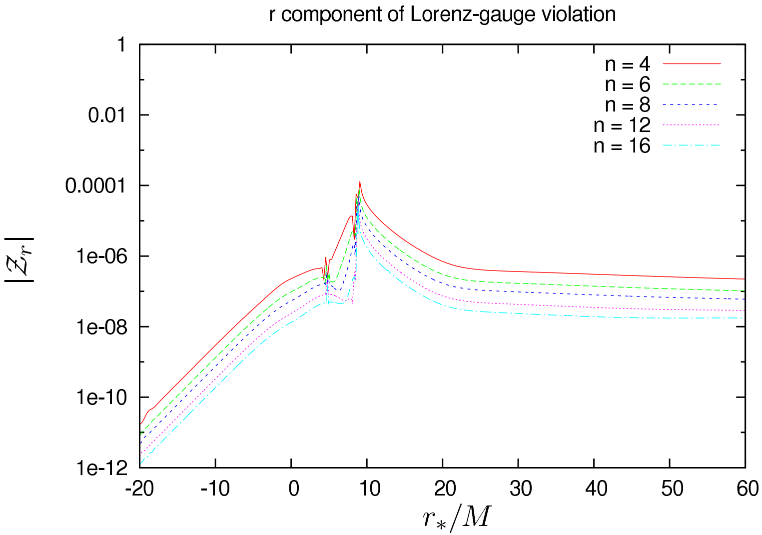


Results: Modal profiles ($r_0 = 7M$, $m = 2$)

Slice 3: $\theta = \pi/2$, $r = r_0$ (and $r_0 = 7M$, $m = 2$)

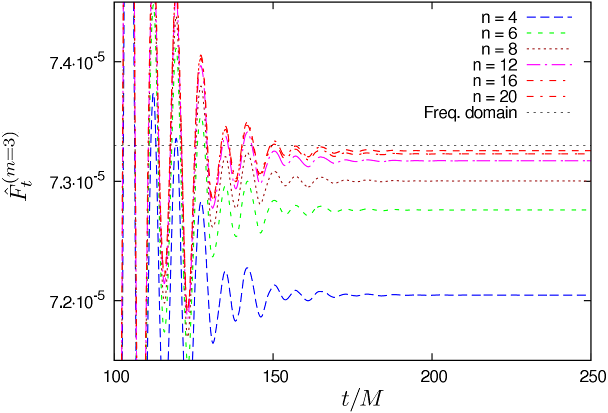


Results: Gauge-constraint violation



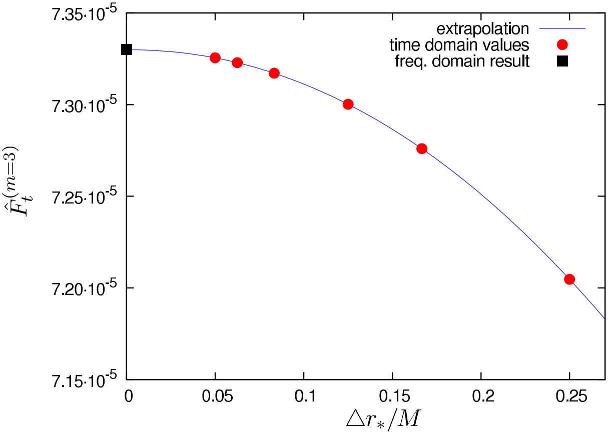
- Constraint violation diminishes with increasing grid resolution

Results: F_t and energy balance



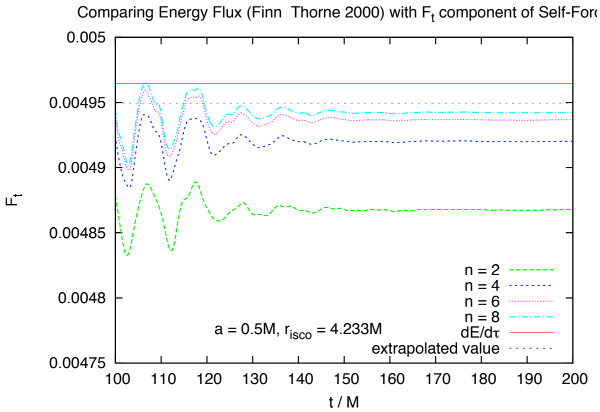
- Showing time-domain value of F_t for various grid resolutions $dr_* = M/n$.
- In principle, $F_t = u_0^t \dot{E}$, where $\dot{E}$ is energy loss rate (from Teuk. ψ_0, ψ_4).

Results: F_t and energy balance



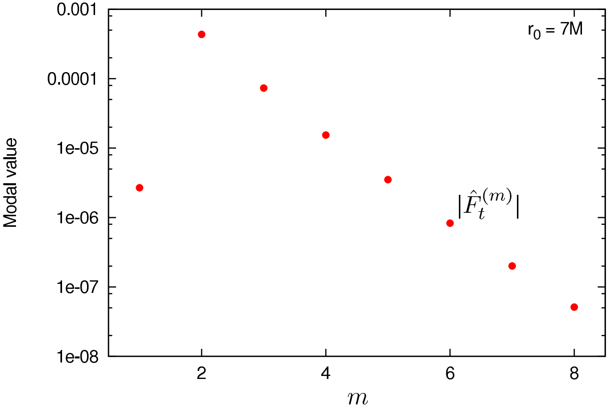
- Extrapolate over grid resolution to obtain best estimate
- Convergence rate only $x^2 \ln x$ with 2nd-order puncture

Results: F_t validation at $a = 0.5M$ ($m = 2$ mode)



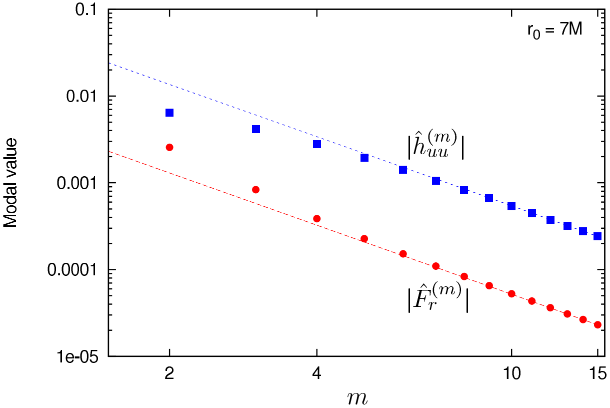
- For each m-mode, validate $\dot{E} = F_t/u^t_0$ against results of Finn & Thorne.
- 0.3% disagreement here because Finn & Thorne give $\dot{E}_\infty$, whereas $\dot{E} = \dot{E}_\infty + \dot{E}_{hor}$.

Results: m-mode convergence: dissipative



- Modes of dissipative component of GSF, F_t, converge exponentially, $F_t^m \sim \exp(-\lambda|m|)$.

Results: m-mode convergence: conservative

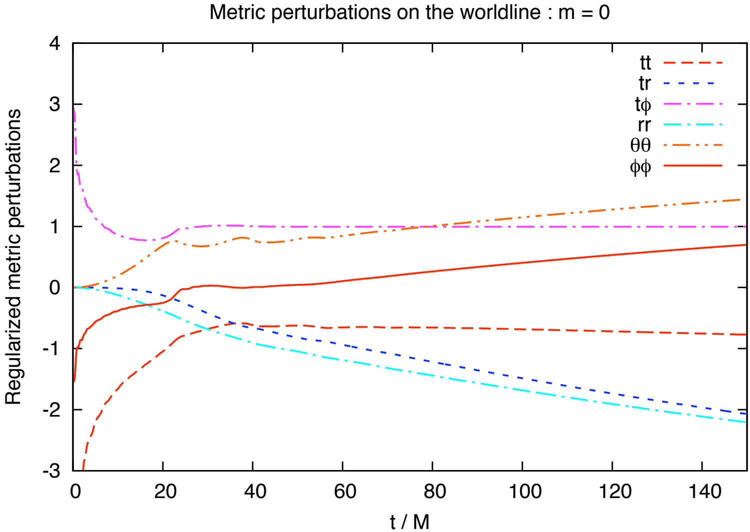


- Modes of conservative component, F_r (and h^R_{uu}) converge with power-law, $F^m_t \sim m^{-2}$ (for 2nd-order puncture).

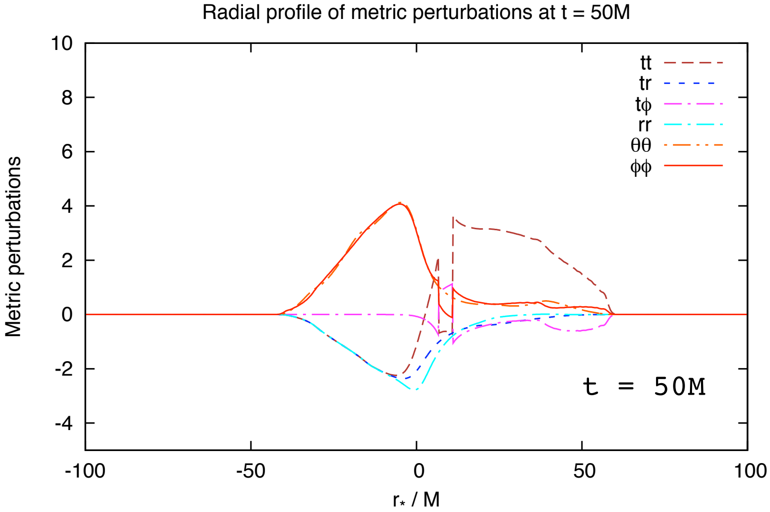
Problem: Linear-in-t modes in Lorenz gauge

- **Problem:** Modes $m = 0, 1$ suffer from **linear-in-t** instabilities!
- Linear-in-t modes are homogeneous, pure-Lorenz-gauge solutions
- Linear-in-t modes are regular on future horizon and asymp-flat.
- Linear-in-t modes are excited by generic initial data.
- In Schw., these modes are in $l = 0$, $l = 1$ sectors only.
- Analytic solutions of these modes in Dolan & Barack (2013)
- N.B. No l-mode time-domain scheme has successfully evolved Schw. $l = 0, 1$ modes in Lorenz gauge.

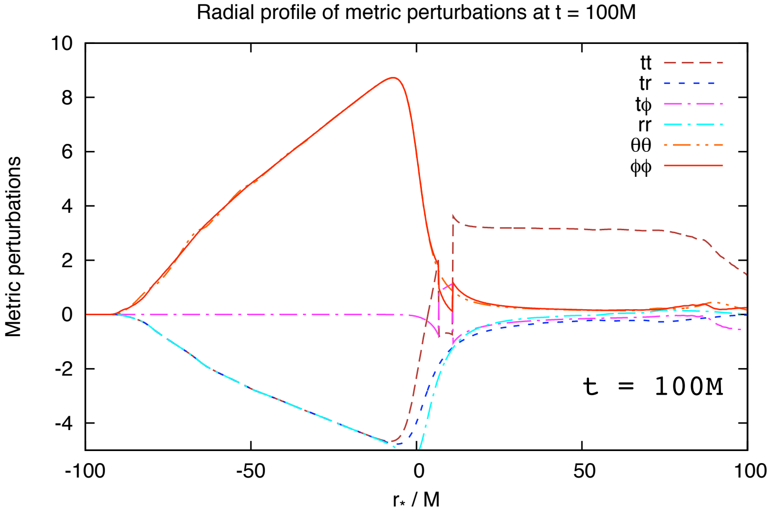
Problem: Time Evolution of $m = 0$ mode



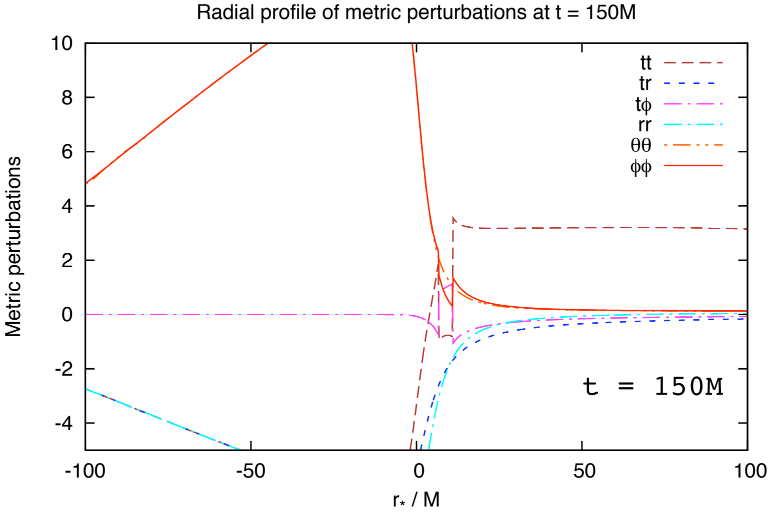
Radial Profile : $m = 0$ mode



Radial Profile : $m = 0$ mode



Radial Profile : $m = 0$ mode



Detail: Monopole $l = 0$ mode

Consider circular orbit on Schwarzschild (cf Detweiler & Poisson '04):

- Write down a basis of **four** linearly independent static homogeneous monopole solutions
- Construct a unique physical monopole solution for circular orbit, with following properties:
    1. Solution of inhomogeneous eqn
    2. Lorenz gauge
    3. **Static**: $\partial_t h_{\alpha\beta} = 0$ and $h_{ti} = 0$.
    4. Continuous across $r = r_0$.
    5. Regular on future horizon $\mathcal{H}^+$
    6. Regular at infinity, $h_{\mu\nu}/g_{\mu\nu} \sim \mathcal{O}(1/r)$
    7. Has correct mass-energy
- But can't satisfy **all** these properties simultaneously ...
- Relax condition (6). Then $h_{tt} \sim -2\mu\alpha$ where $\alpha = \mathcal{E}/r_0 f_0$.
- Move to asymptotically-regular but non-Lorenz gauge with simple gauge transformation.

$$\xi^\nu = -\mu\alpha(t + r_* - r)\delta_t^\nu .$$

- Resulting solution is not static, $h_{t\cdot} \neq 0$

Detail: Monopole $l = 0$ mode

- **Unique solution?** There are stationary but not static ($h_{tr} \neq 0$) homogeneous gauge modes which satisfy all other conditions
- For example, a **scalar** gauge mode

$$h_{\alpha\beta} = 2\xi_{(\alpha;\beta)}, \quad \xi_\alpha = [1/2, 2/(r^2 f), 0, 0] = \Phi_{;\alpha}, \quad \Phi = \frac{1}{2}t + \ln(f).$$

- There is a **linearly-growing gauge mode** which satisfies all conditions, except (i) it is not stationary, and (ii) it is not asymptotically-regular in tt component

$$\xi_t^{\text{lin}} = \ln(2f) + t/2 + \frac{13}{6}, \qquad \xi_r^{\text{lin}} = \frac{2t}{r^2 f} + \frac{r^3 + 3r^2 + 12r + 24\ln(fr)}{6r^2 f} - \frac{r}{6f}.$$

Detail: Monopole $l = 0$ mode

The linearly-growing mode homogeneous gauge mode is $(M = 1)$

$$
\begin{aligned}
h_{tt}^{\rm lin} &= -\frac{-r^4 + 4t + r^2 + 4r + 8\ln(rf)}{r^4}, \\
h_{tr}^{\rm lin} &= -\frac{t + \frac{1}{3} + 2\ln(2f)}{r^2 f}, \\
h_{rr}^{\rm lin} &= -\frac{4t(2r-3) + 5r^2 - 12r + 8(2r-3)\ln(rf)}{r^4 f^2}, \\
r^{-2} h_{\theta\theta}^{\rm lin} &= \frac{4t + r^2 + 4r + 8\ln(rf)}{r^3} = (r\sin\theta)^{-2} h_{\phi\phi}. \qquad (3)
\end{aligned}
$$

Note that $h_{tt} \sim 1 + \mathcal{O}(1/r)$

This mode is generically excited in our initial-value formulation.

Solution : Generalized Lorenz gauge

- To recover stability, I experimented with using generalized Lorenz gauges, $\bar{h}_{ab}{}^{;b} = H_a$
- I found an explicit gauge driver of the form:

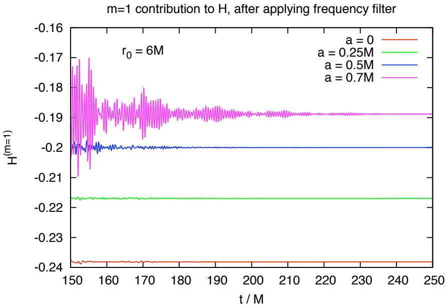
$$H_a \propto n_a \times h_{tr}^{(m=0)}/r^k, \quad \text{where} \quad n_a \text{ is ingoing null vector}$$

  restores stability to $m = 0$ sector.
- For circular orbits, $h_{tr}^{S} = 0$, so this gauge is non-singular.
- But leads to non-unique stationary solution which depends on initial condition.
- The *static* solution ($h_{ti} = 0$) is also in Lorenz gauge ($H_a = 0$).
- Take linear combination of solutions to find static soln with $h_{tr} = 0$.
  1. Schw.: combine **two** solns in monopole ($l = 0$) sector.
  2. Kerr: combine **three** solns, as mass & ang. mom. pert. are no longer decoupled.
- Unnecessary if we are only interested in gauge-invariant (e.g. ΔU).

Solution : $m = 1$ mode?

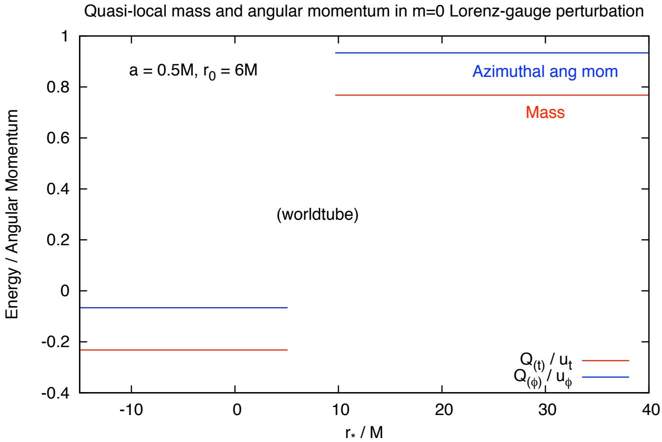
- I have **not** found a generalized Lorenz gauge that stabilizes the $m = 1$ sector.
- Instead, I apply a frequency-filter to eliminate stationary and linear-in-t modes:

$$h_{ab} \to -\frac{1}{\omega^2}\frac{\partial^2}{\partial t^2} h_{ab}$$



- This trick will *not* work for general orbits

Correcting the mass and angular momentum



- Take integrals over two-spheres to find 'quasi-local' mass $\mathcal{F}_{(t)}$ and angular momentum $Q_{(\phi)}$ in numerical solution $\mathcal{F}_{ab}^{(m=0)}$.

Correcting the mass and angular momentum

- To correct the mass and ang.mom. I add homogeneous Lorenz-gauge solutions which are regular on the future horizon,

$$h_{ab}^{(\partial M)} = \frac{\partial}{\partial M} g_{ab}\Big|_J + \text{gauge}, \quad h_{ab}^{(\partial J)} = \frac{\partial}{\partial J} g_{ab}\Big|_M + \text{gauge}.$$

- . . . but, once again, these solutions are *not* asymp-flat.
- Recall that in Schw., the static Lorenz-gauge solution with correct mass is *not* asymp-flat: $h_{tt} \to -2\mu\alpha$ [Sago et al. '08].
- In Kerr, I find that Lorenz-gauge static solution with correct mass and ang.mom. is not asymp-flat in **two** components:

$$h_{tt} \sim O(1) \quad \text{and} \quad h_{t\phi} \sim O(r^2).$$

- In Schw., $\partial g_{ab}/\partial J(a = 0)$ is already in Lorenz-gauge – this is not the case in Kerr.

Approach #2: Radiation gauge / frequency domain

(developed by Friedman, Shah, Keidl *et al.*)

Method #2: Radiation-gauge frequency-domain

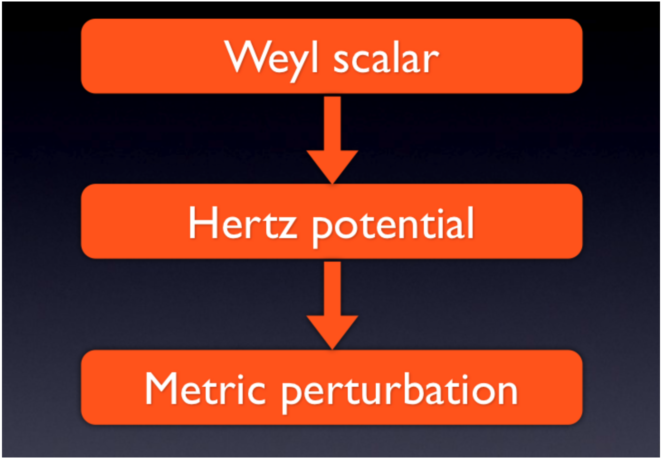
$$\mathcal{O}\psi_0 = \mathcal{T}, \quad \psi_0 \to \Psi, \quad h_{\mu\nu} = \mathcal{S}^\dagger \Psi, \quad h_{\mu\nu} n^\nu = 0 = h_\mu{}^\mu.$$

Q1. Why work in radiation gauge?

- Components of Weyl tensor satisfy decoupled, separable equation.
- Can recover metric perturbation via Hertz potential.
- Frequency domain $\Rightarrow$ ODEs

Q2. What are the drawbacks?

- Not obvious how to regularization in radiation gauge $\Rightarrow$ hybrid gauges?
- Add non-radiative perturbations (mass + angular momentum) ‘by hand’
- Suited to self-consistent evolutions?



cf. Shah, Friedman & Keidl.

Metric reconstruction in radiation gauge on Kerr

- Teukolsky ('73) showed that extreme-helicity components of Weyl tensor, ψ_0 and $\rho^{-4}\phi_4$, satisfy decoupled, separable equations.
- Cohen & Kegeles ('74) showed how to reconstruct vector potential A_μ from Hertz potential satisfying decoupled equation.
- Chrzanowksi ('75) showed how to get $h_{\mu\nu}$ in radiation gauge from twice-differentiating Teukolsky functions.
- Wald ('78) showed the connection between Teukolsky potential, Hertz potential and metric reconstruction.

Ingoing RG

$$h_{\mu\nu}l^\nu = 0 = h^\mu{}_\mu$$

Outgoing RG

$$h_{\mu\nu}n^\nu = 0 = h^\mu{}_\mu$$

Hertz and Debye potentials

Cohen and Kegeles (1974) analyzed EM using forms:

- Electromagnetism in vacuum:

$$\begin{aligned} dF &= 0, \qquad \Rightarrow F = dA, \\ \delta F &= 0, \end{aligned}$$

  i.e. F is closed and co-closed.

- Suppose $\Delta P = 0$ where $\Delta = d\delta + \delta d$ and P is a two-form. Then

$$F = d\delta P = -\delta d P,$$

  so F is closed and co-closed.

- The vector potential can be constructed from P,

$$A = \delta P$$

Debye potential

- Most likely, $\Delta P = 0$ is not separable. Instead, consider

  $$\Delta P = dG + \delta({}^*W)$$

  where G and W are gauge one-forms. Then let

  $$A = \delta P - G$$

  so

  $$F = d(\delta P - G) = -\delta(dP - {}^*W)$$

  is again closed and co-closed.

- **Debye potential**: Judicious choice of gauge one-forms G, W to obtain separable equation for P in terms of scalar field.

More on Debye potential

- Type-D spacetime ⇒ principle null directions, null tetrad l_μ,n_μ,m_μ,$\bar{m}_\mu$.
- **Killing-Yano tensor**: $f_{\mu\nu} = f_{[\mu\nu]}$ and $f_{\mu(\nu;\gamma)} = 0$,

$$f_{\mu\nu} \propto r\, i\bar{m}_{[a} m_{b]} + a\cos\theta\, l_{[a} n_{b]}.$$

- Dual of KY is closed conformal Killing-Yano tensor, $dh = 0$.
- May use a CKY tensor h to achieve separation:

$$\begin{aligned} P &= \psi_E h, \qquad G = 2\psi_E \delta h, \quad W = 0, \quad \text{and} \\ P &= \psi_B ({}^* h), \quad W = 2\psi_B \delta h, \quad G = 0. \end{aligned}$$

- Other choices possible (c.f. Teukolsky eqn. for extreme-helicity component; Cohen & Kegeles approach).

Wald/CCK approach

(suppressing indices, and denoting linear differential operators with calligraphic letters e.g. $\mathcal{E}$, $\mathcal{S}$, etc.):

- Linearized equations:

$$\mathcal{E}(h) = 8\pi G T = 0. \tag{4}$$

- *A la* Teukolsky, take linear combinations ($\mathcal{S}$) to find a separable, decoupled equation $\mathcal{O}\psi$ in terms of new variable, $\psi = \mathcal{T}(h)$

$$\mathcal{S}\mathcal{E}(h) = \mathcal{O}\psi = \mathcal{O}\mathcal{T}(h)$$

- How to recover h from 'Debye potential' ψ? Find Hertz potential Ψ which satisfies

$$\mathcal{O}^{\dagger}\Psi = 0$$

where † denotes the **adjoint**, defined by

$$\Phi\mathcal{L}\Phi - (\mathcal{L}^{\dagger}\Phi)\Phi = s^{\mu}_{;\mu}$$

Wald/CCK approach

- Summary

$$\begin{aligned} \textbf{Teukolsky eqn:} \quad & \mathcal{S}\mathcal{E}(h) = \mathcal{O}\mathcal{T}(h) \\ \textbf{Hertz potential:} \quad & \mathcal{O}^\dagger \Psi = 0 \\ \textbf{Self-adjoint :} \quad & \mathcal{E}^\dagger = \mathcal{E} \end{aligned}$$

- Take adjoint of operators in first equation, $\mathcal{E}\mathcal{S}^\dagger = \mathcal{T}^\dagger \mathcal{O}^\dagger$.
- So

$$\mathcal{E}\mathcal{S}^\dagger \Psi = 0.$$

  and therefore $h = \mathcal{S}^\dagger \Psi$ is a solution of original equations.
- **Q.** How to find Hertz potential Ψ from 'Debye' potential ψ (i.e. Teukolsky variables)?
- **A.** Use $\psi = \mathcal{T}\mathcal{S}^\dagger \Psi$, because

$$0 = \mathcal{S}\mathcal{E}\mathcal{S}^\dagger \Psi = \mathcal{O}\left[\mathcal{T}\mathcal{S}^\dagger \Psi\right]$$

Metric reconstruction

- Separation of variables: $\psi_0 = \sum \psi_{lm\omega}$ where

$$\psi_{lm\omega} = {}_2R_{lm\omega\,2}S_{lm\omega}e^{i(m\phi-\omega t)}$$

- ψ_0 satisfies Teukolsky equation, with δ, δ' and δ'' source terms. Solve with Green function methods.
- Relate Weyl scalar to Hertz potential:

$$\psi_0 = \frac{1}{8}\left(\mathcal{L}^4\bar{\psi} + 12M\partial_t\psi\right)$$

- Invert this relationship:

$$\Psi_{lm\omega} = 8\frac{(-1)^m D\bar{\psi}_{l-m-\omega} + 12iM\omega\psi_{lm\omega}}{D^2 + 144M^2\omega^2}$$

where D is the constant in Teukolsky-Starobinskii identity.
- Obtain metric in IRG/ORG

$$h_{\mu\nu} = \mathcal{S}^{\dagger}_{\mu\nu}(l,n,m)\Phi$$

Mode sum regularization

- Expand spheroidal harmonics in spherical harmonics ($S \to Y$)
- Mode sum regularization is understood in Lorenz gauge:

$$F^\alpha_{\rm self} = \sum_{\ell=0}^{\infty} \left[F^\ell_{\rm lor}(p) - A^\alpha L - B^\alpha - C^\alpha / L \right] - D^\alpha$$

- cf. Barack, Friedman *et al.*, Linz, talk later by Merlin.
- Idea: Make gauge transformation to move to a locally-Lorenz gauge,

$$h^{\rm Mrad}_{\mu\nu} = h^{\rm rad}_{\mu\nu} + \xi_{\mu;\nu} + \xi_{\nu;\mu}.$$

- Does this change A^α, B^α, C^α? (no)
- Does this change D^α? (yes)

Comparison of gauge-invariant quantities

Gauge invariant comparison

- Dissipative GSF has an obvious gauge invariant effect (loss of energy, ang momentum), so is easy to validate.
- Conservative GSF is more subtle and dependent on choice of gauge.

For circular orbits:

- Two physically-observable were quantities identified by Detweiler: $U = u^t$ and Ω.
- First-order variations ΔU and $\Delta\Omega$ are invariant under helically-symmetric gauge transformations
- First comparison of ΔU and $\Delta\Omega$ in Schwarzschild made in 2007/8: Sago, Barack, Detweiler.
- First comparison in Kerr made last year: (RG) Friedman, Shah & Keidl **vs** Dolan, Barack & Wardell (LG)

Gauge invariant comparison

Variation at lowest order in μ:

$$\Delta U = \mu \frac{\partial}{\partial \mu} U(\mu, \Omega)|_{\mu=0} \quad (5)$$

$$\Delta \Omega = \mu \frac{\partial}{\partial \mu} \Omega(\mu, U)|_{\mu=0} \quad (6)$$

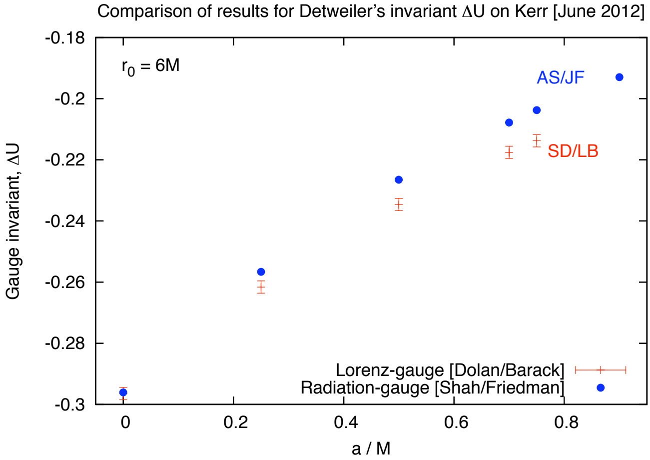
These quantities depend on the renormalized metric perturbation, e.g.

$$\Delta U = -u^t H$$

where

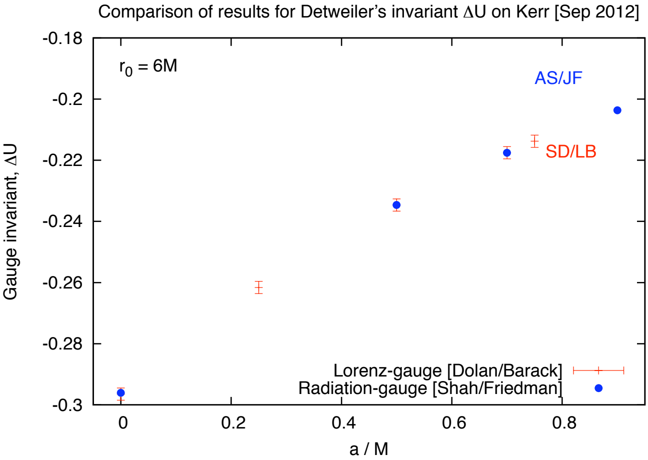
$$H \equiv \frac{1}{2} h^R_{\alpha\beta} u^\alpha u^\beta.$$

Gauge invariant comparison : ΔU for circular orbits



- Preliminary comparison: June 2012 (at Capra).

Gauge invariant comparison : ΔU for circular orbits



- Second comparison in Sep 2012. **Much better!**

Prospects & Conclusion

- Beneficial to have an **ecosystem** of methods for Kerr GSF
- Time domain priorities:
  - Mitigate gauge mode instabilities w. generalized Lorenz gauge
  - Improve accuracy (1 part in 10^6, cf Thornburg)
  - Apply machinery of Numerical Relativity
- Frequency domain priorities:
  - Regularization in (modified) radiation gauge (cf Merlin)
  - Compare with PN & EOB theory (cf Shah)
- Next steps:
  - Gauge-invariant comparisons
  - Compute GSF on generic orbits & study orbital resonances
  - Orbital evolutions