Self-interaction and extended bodies

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Extended bodies: We all use them

Distributional sources don't work in GR (or even ordinary EM).

Rather, distributional sources (**with special rules** [regularization]) can describe limiting behaviors for classes of extended sources.

Perturbative methods usually apply only on scales much larger than the body's. What happens a little closer in?

Approaches to motion problems

Consider a compact clump of matter interacting with long-range fields (charged solid in Maxwell EM, star in GR, . . .)

1. Either compute "everything" (numerics)
   - *Many inputs*: detailed matter model, initial and boundary conditions
   - *Complicated output*: detailed density, velocity, temperature fields
   - "Complete"
   - **Describes only very specific systems**
2. ...or focus only on a few "bulk" or "external" quantities (CM etc.)
   - *Simple input*
   - *Simple output*: center of mass, spin, ...
   - Not complete
   - **Can describe large classes of systems simultaneously**

Internal and external variables in celestial mechanics

Ordinary celestial mechanics makes "PDEs $\to$ ODEs:"

External (or bulk) variables

Center of mass positions
Linear momenta
Angular momenta

Internal variables

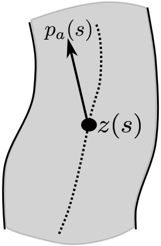
Density distributions
Internal velocities
Thermodynamic variables

Focus on the external variables.

Effective point particles

An **extended** mass can partially be replaced by an **effective point particle** where bulk variables are evolved on a worldline.

1. Choose a "representative" worldline
2. Define momenta
3. Find force and torque as integrals
4. **Expand these integrals in multipole series**



These steps aren't entirely independent.

Self-force causes trouble mostly in step 4.

Representative worldlines

There are several approaches.

1. Various perturbative constructions (see Pound)
   - Limiting worldtubes
   - Parameter in a metric perturbation
2. Look at structure of null geodesic congruences far away and define a worldline in an auxiliary space (Newman, Adamo, Kozameh)
3. **Fix a genuine worldline in the physical spacetime**

All of these arise from setting a "mass dipole moment" to zero.

Flat spacetime centroids

Defining a mass dipole is subtle even for a free object in flat spacetime (!).

$$S^{\mu\nu}(z) = 2\int (x-z)^{[\mu} T^{\mu]\lambda}(x) dS_\lambda$$
$$\sim \text{(spin} \oplus \text{mass dipole moment)}$$

Mass dipole vanishes wrt a timelike observer field v^a if

$$S_{ab}(z) v^b(z) = 0.$$

"Spin supplementary condition" (actually a choice of centroid)

Which mass dipole?

Which v^a to use in $S_{ab} v^b = 0$?

Do you want to describe lone objects, collisions, ...?

p^a or $\dot{z}^a$ both seem reasonable.

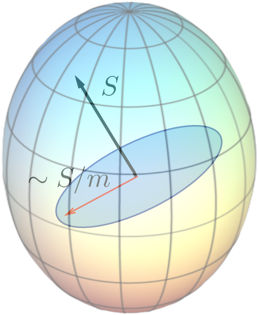
1. $v^a = \dot{z}^a$ gives an **infinite number** of (mostly) accelerated worldlines!
2. $v^a = p^a$ gives a unique geodesic worldline

Tube of centroids

The centroids formed from *all* possible observer fields v^a form a disk of radius $\sim S/m$.

$S/m < r$, so maybe you don't care.

1. $S/m \lesssim 10^{-6} r$ for everyday objects, the Earth and Sun
2. But LLR measurements get to this level...



All of this is well-understood only for a freely-falling mass in flat spacetime!

Momenta

Something like $S_{ab}p^b = 0$ is probably a good choice for a center of mass.

But what are p_a, S_{ab} anyway?

In the presence of Killing fields, a linear combination of momenta should be conserved:

$$P_\xi = (p_a \xi^a + \frac{1}{2} S^{ab} \nabla_a \xi_b)|_{z(s)} = \int_\Sigma T^a{}_b \xi^b \mathrm{d}S_a = (\text{constant})$$

A generalized version of this can be imposed in general.

Linear and angular momenta are treated on equal footing.

Forces and torques

$\nabla_a T^{ab} = 0$ implies that

$$\frac{\mathrm{d}P_\xi}{\mathrm{d}s} = \frac{\mathrm{d}}{\mathrm{d}s}(p^a \xi_a + \frac{1}{2} S^{ab} \nabla_a \xi_b) = \frac{1}{2} \int_\Sigma T^{ab} \mathcal{L}_\xi g_{ab} \mathrm{d}S.$$

- Demanding that $\mathcal{L}_\xi g_{ab}|_z = \nabla_a \mathcal{L}_\xi g_{bc}|_z = 0$ recovers Papapetrou terms from the LHS.
- RHS measures the degree to which the ξ^a fails to be Killing inside the body.

Papapetrou terms (monopole and dipole) are just kinematics

$$\dot{p}^a - \frac{1}{2} R_{bcd}{}^a S^{bc} \dot{z}^d = (\ldots)$$
$$\dot{S}^{ab} - 2p^{[a} \dot{z}^{b]} = (\ldots)$$

RHSs here depend only on $\mathcal{L}_\xi g_{ab}$ (Always zero in de Sitter, Minkowski!)

Deviations from the Papapetrou equations measure the lack of symmetry inside a body.

Multipole expansions

Writing a force as an integral isn't useful in practice. Expand this:

If an object doesn't backreact at all (small test body),

$$\begin{aligned} \int_\Sigma T^{ab}\mathcal{L}_\xi g_{ab}\mathrm{d}S &\sim \int_\Sigma T^{ab}\sum X\cdots X(\partial\cdots\partial\mathcal{L}_\xi g_{ab})|_z \mathrm{d}S \\ &= \sum_{n=2}^{\infty}\frac{1}{n!}I^{c_1\cdots c_n ab}\mathcal{L}_\xi g_{ab,c_1\cdots c_n}. \end{aligned}$$

Quadrupole term: $\mathcal{L}_\xi g_{ab,cd} \sim \mathcal{L}_\xi R_{abcd}(z)$,

Octupole: $\mathcal{L}_\xi \nabla_a R_{bcdf}(z)$.

Right answer, wrong reason

Writing $\mathcal{L}_\xi g_{ab}$ in a power series with $g \to g_{\text{background}}$ is a ridiculously strong assumption.

Curvature inside a rock due to itself is comparable to the curvature produced by the entire Earth.

The integral form for the force *must* be manipulated before anything can be said about contributions from individual moments. One needs an analog of Detweiler-Whiting subtraction (even in Newtonian gravity!).

Newtonian gravity

Total force acting on a Newtonian mass:

$$\frac{\mathrm{d}p_i}{\mathrm{d}t} = -\int_{\mathcal{B}} \rho \nabla_i \phi d^3 x =: F_i[\phi]$$

This is hard to use as-is.

First show that $F[\phi] = F[\hat{\phi}]$ with $\nabla^2 \hat{\phi}|_{\mathcal{B}} = 0$. *Only then*,

$$\frac{\mathrm{d}p_i}{\mathrm{d}t} = F_i[\hat{\phi}] \approx -m\nabla_i \hat{\phi} \neq -m\nabla_i \phi.$$

Self-field subtraction

The natural field with which to compute motion in Newtonian gravity is

$$\hat{\phi}(x) := \phi(x) - \int_{\mathcal{B}} \rho(x') G_S(x, x') \mathrm{d}V'.$$

$\hat{\phi}$ is fictitious but useful.

In more complicated theories, **reasonably-defined self-forces don't vanish**: $F[\phi] \neq F[\hat{\phi}]$.

$F[\phi] - F[\hat{\phi}]$ is "ignorable"

Consider a small charged particle in flat spacetime:

$$m\dot{u}_a = qF^{\text{ext}}_{ab}u^b + \frac{2}{3}q^2 h_{ab}\ddot{u}^b - \delta m \dot{u}_a$$

$$(m + \delta m)\dot{u}_a = q(F^{\text{ext}}_{ab} + \frac{4}{3}qu_{[a}\ddot{u}_{b]})u^b = q\hat{F}_{ab}u^b$$

So self-field subtractions can still be useful if all of their effects may be interpreted as renormalizations:

$$\begin{aligned}\frac{\mathrm{d}P_\xi}{\mathrm{d}s} &= \mathcal{F}_\xi[\phi; q, q^a, \ldots] \\ &= \mathcal{F}_\xi[\hat{\phi}; \hat{q}, \hat{q}^a, \ldots] - \frac{\mathrm{d}\mathcal{E}_\xi}{\mathrm{d}s}\end{aligned}$$

Effective test bodies

An effective metric $\hat{g}_{ab}[g]$ may be defined around the body such that **if** $\hat{g}_{ab}$ varies slowly inside the body,

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(P_\xi + \mathcal{E}_\xi\right) = \frac{1}{2}\sum_{n=2}^{\infty}\frac{1}{n!}\hat{I}^{c_1\cdots c_n ab}\mathcal{L}_\xi \hat{g}_{ab,c_1\cdots c_n}.$$

1. This looks like a test body moving in $\hat{g}_{ab} \neq g_{ab}$.
2. Forces and torques all at once.
3. All multipole moments are (finitely) renormalized.
4. $\hat{g}_{ab}$ is a dynamically selected (rather than chosen) "background"
5. Under the usual assumptions, $\hat{g}_{ab}$ is the DW R-metric.

Effective test bodies II

Equivalently,

$$\frac{\hat{\mathrm{D}}\hat{p}^a}{\mathrm{d}s} = \frac{1}{2}\hat{R}_{bcd}{}^a(z)\hat{S}^{bc}\dot{z}^d + \dots$$

$$\frac{\hat{\mathrm{D}}\hat{S}^{ab}}{\mathrm{d}s} = 2\hat{p}^{[a}\dot{z}^{b]} + \dots$$

$\hat{g}_{ab}$, $\hat{p}^a$, $\hat{S}^{ab}$, ... can be computed from g_{ab} and T^{ab}.

Appropriately interpreted, test-body equations also work with self-interaction (to all multipole orders)

A simple application

The simplest freely-falling test masses move on geodesics and have constant mass.

$\Rightarrow$ The simplest self-interacting masses satisfy

$$\frac{\hat{\mathrm{D}}\dot{z}^a}{\mathrm{d}s} = 0$$

and $\hat{m} = \mathrm{const.}$

Using the definition for $\hat{g}_{ab}$, this implies the standard MiSaTaQuWa equation used to describe 1st-order gravitational self-force.

Spin effects

The simplest freely-falling test bodies parallel-transport their angular momentum.

$\Rightarrow$ Spins of simple self-interacting masses satisfy

$$\frac{\hat{\mathrm{D}}\hat{S}_a}{\mathrm{d}s} = 0.$$

This can be interpreted as a "precession-inducing self-torque."

Some comments

1. Directions of momenta are renormalized as well as magnitudes.

2. Incorporating self-field inertia into momenta introduces some "temporal fuzziness."

3. Detweiler-Whiting S-type Green functions play a central role.

4. Renormalizations of higher moments depend on $\mathcal{L}_\xi \hat{G}_S^{aba'b'}$ ("Violations of Newton's 3rd law")

A problem

The current definition for $\hat{g}_{ab}$ doesn't satisfy $\hat{R}_{ab} = 0$ exactly.

- This makes it less likely that $\hat{g}_{ab}$ is well-behaved "generically enough."
- It also means that more than the usual number of multipole moments enter the laws of motion.

A proposal: geometric flows and effective metrics

1. Don't define $\hat{g}_{ab}$ in one large step.
2. *Continuously deform* $g_{ab} \to \hat{g}_{ab}$.
3. Every infinitesimal step is a linear perturbation.
4. So apply the DW-type subtraction at every step:

$$g_{ab}(\lambda + \mathrm{d}\lambda) = g_{ab}(\lambda) - \mathrm{d}\lambda \left(\int T(\lambda) G_S(\lambda) \right)$$

$G_S(\lambda)$ is a Green function for the Einstein eqn. linearized off of $g(\lambda)$.

This converts nonvacuum solutions to vacuum solutions, but I also want to relate forces exerted by g_{ab} to forces exerted by $\hat{g}_{ab}$.

Maybe

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}(\text{Force}) = \frac{\mathrm{d}}{\mathrm{d}\lambda} \int T^{ab}(\lambda) \mathcal{L}_\xi \hat{g}_{ab}(\lambda) = 0$$

can be used to derive

$$\frac{\mathrm{d}T^{ab}(\lambda)}{\mathrm{d}\lambda} = (\cdots), \qquad \frac{\mathrm{d}g_{ab}(\lambda)}{\mathrm{d}\lambda} = (\cdots)$$

such that $R_{ab}(\lambda) \to 0$ and (physical force) = (force in $\hat{g}$).

Next Capra...

Conclusions

1. Theory of motion is well-developed for arbitrarily-structured relativistic objects.
2. Self-interaction just gives effective test bodies with renormalized moments falling in an effective metric.
3. The Detweiler-Whiting subtraction is very general. It has nothing to do with perturbation theory or point particles.

You save work and gain insight by doing non-perturbative things before applying perturbation theory.