# Eccentric motion on a Schwarzschild background:

## Self-force in a modified Regge-Wheeler gauge

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# Outline

- Gauge freedom on Schwarzschild
- Infinitesimal gauge transformations
- Lorenz gauge
- Modified Regge-Wheeler gauge

# Gauge freedom

- In GR, gauge freedom is coordinate freedom
- Zeroth order: use Schwarzschild coordinates
- First-order options:
  - Lorenz
  - Regge-Wheeler
  - Modified Regge-Wheeler?

# Lorenz gauge

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- 10 coupled wave equations
- Locally isotropic solutions
- Regularization procedure in Lorenz gauge
  - Other gauges may be possible ...

# Regge-Wheeler gauge

- Schematically: $$p_{\mu\nu} \to \sum_{\ell,m} h_{\mu\nu}^{\ell m} Y^{\ell m}$$
- Set four components of $h_{\mu\nu}^{\ell m}$ to zero
- Field equations simplify greatly:

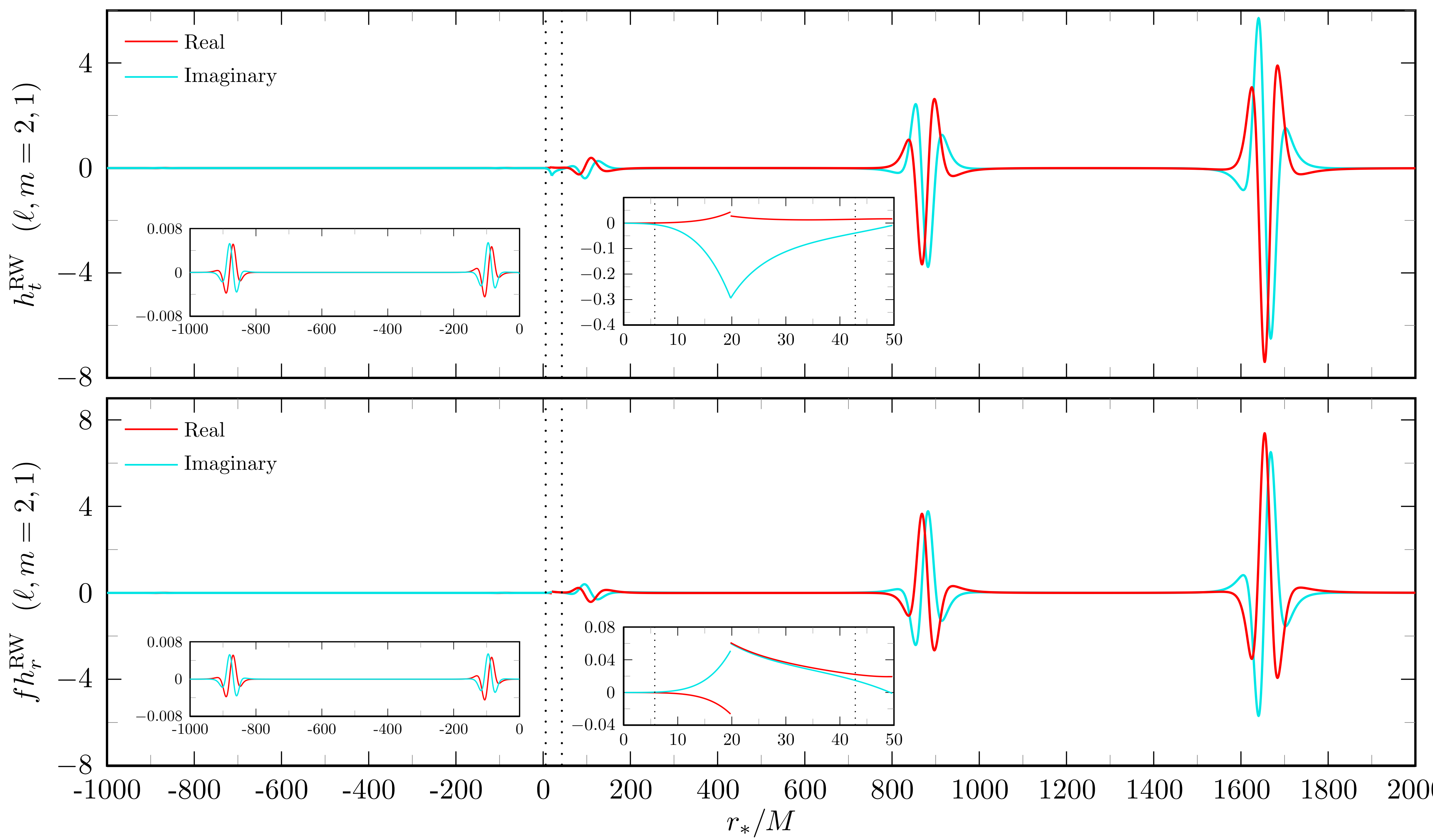
$$\left[-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial r_*^2} - V_\ell(r)\right] \Psi_{\ell m}(t,r) = S_{\ell m}(t)$$

- Reconstruct metric perturbation:

$$\Psi_{\ell m}(t,r) \to p_{\mu\nu}$$

# RW gauge fields: odd-parity

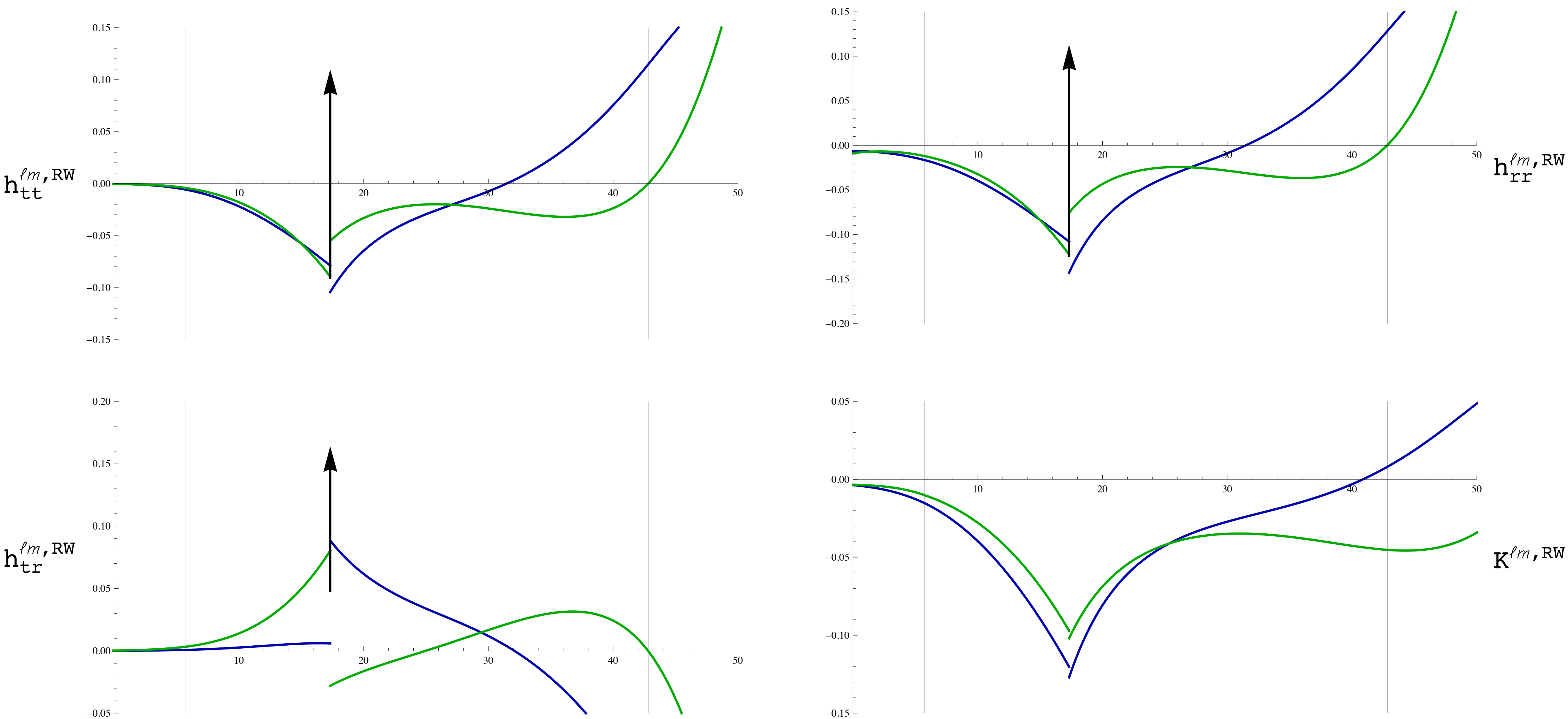
- Two non-vanishing amplitudes: $h_t^{\ell m}$, $h_r^{\ell m}$



# RW gauge fields: even-parity

- Four non-vanishing amplitudes: $h^{\ell m}_{tt},\ h^{\ell m}_{tr},\ h^{\ell m}_{rr},\ K^{\ell m}$

$$(\ell, m) = (2, 2)$$



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- Drawbacks:
  - Only valid for radiative modes, $\ell \geq 2$
  - Singularities/discontinuities at the particle
  - Self-force not well-defined

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- SF is well-defined if and only if $\delta F^\alpha_{\rm self}$ relative to Lorenz gauge is
- If vector $\Xi^\mu$ is well-defined, the SF will be also
- Then, regularization is done with Lorenz gauge parameters $A^\alpha, B^\alpha, C^\alpha, D^\alpha$

# First-order gauge transformations

- Transform from RW to Lorenz gauge:

$$x^\mu_{\rm L} = x^\mu_{\rm RW} + \Xi^\mu_{\rm RW\to L}, \qquad |\Xi^\mu_{\rm RW\to L}| \sim |p_{\mu\nu}| \ll |g^{\rm Schw}_{\mu\nu}|$$

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- Metric perturbation transforms:

$$p^{\mathrm{L}}_{\mu\nu} = p^{\mathrm{RW}}_{\mu\nu} - \Xi^{\mathrm{RW}\to\mathrm{L}}_{\mu|\nu} - \Xi^{\mathrm{RW}\to\mathrm{L}}_{\nu|\mu}$$

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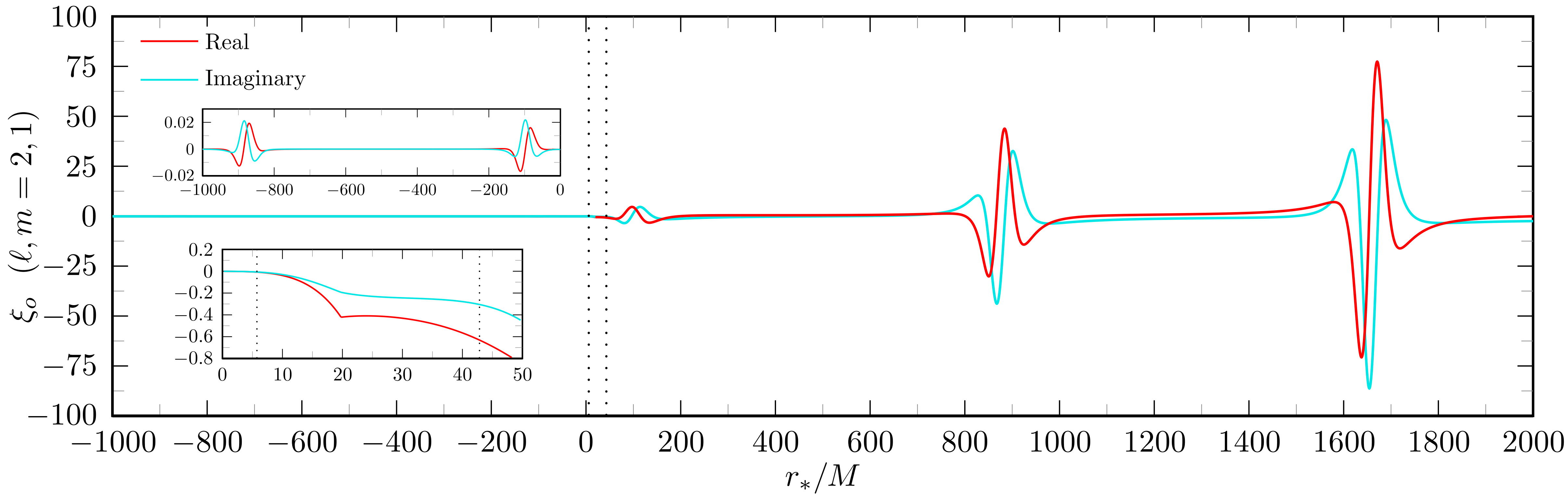
- Gauge vector satisfies a wave equation:

$$\Box \Xi^\mu_{\rm RW\to L} = \bar{p}^{\mu\nu}_{{\rm RW}|\nu}$$

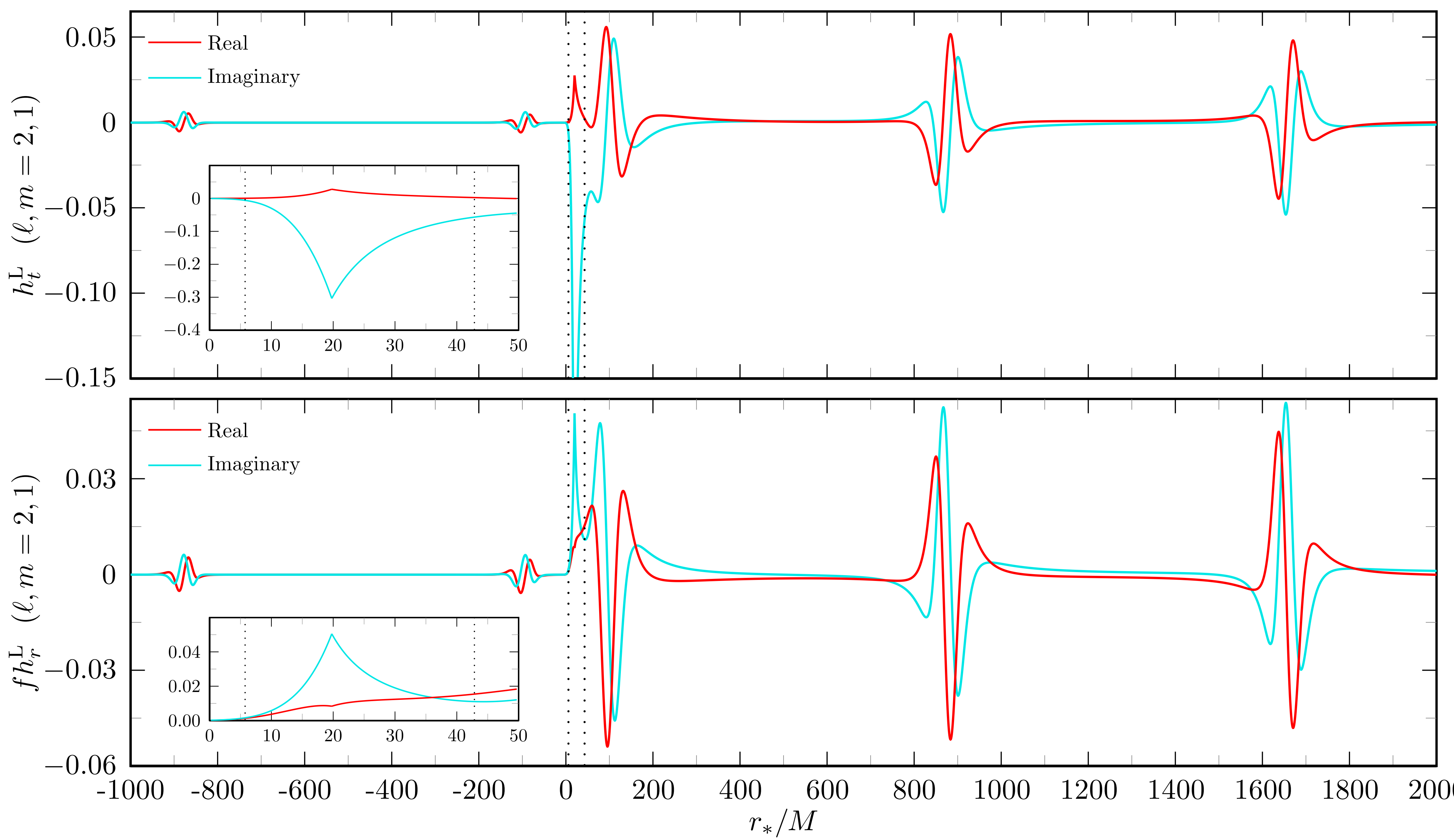
# RW->Lorenz gauge vector: odd-parity

- Transform the global solution, mode-by-mode



# Lorenz gauge fields: odd-parity

- Amplitudes are now $C^0$ and asymptotically flat



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- Benefits:
  - Gives the solution, everywhere in Lorenz gauge
  - Gives solution to low-order modes
- Drawbacks:
  - Computationally difficult and expensive
  - Discontinuous, extended source terms
  - Excessive, if you just want the self-force

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- The metric perturbation in MRW gauge is

$$p_{\mu\nu}^{\mathrm{MRW}} = p_{\mu\nu}^{\mathrm{RW}} - 2\Xi_{(\mu|\nu)}^{\mathrm{RW}\to\mathrm{MRW}}$$

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- Decompose into spherical harmonics, e.g.

$$h_t^{\ell m,\mathrm{MRW}} = h_t^{\ell m,\mathrm{RW}} - \partial_t \xi_{\mathrm{odd}}^{\ell m}$$

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- Restrictions on $[\![\xi^{\ell m}_{\rm odd}]\!],\ [\![\partial_t \xi^{\ell m}_{\rm odd}]\!],\ [\![\partial_r \xi^{\ell m}_{\rm odd}]\!]$

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- A possible vector:

$$\xi^{\ell m}_{\mathrm{odd}}(t,r) = (r-r_p)[\![h_r^{\ell m,\mathrm{RW}}]\!]\theta\left[r-r_p\right]$$

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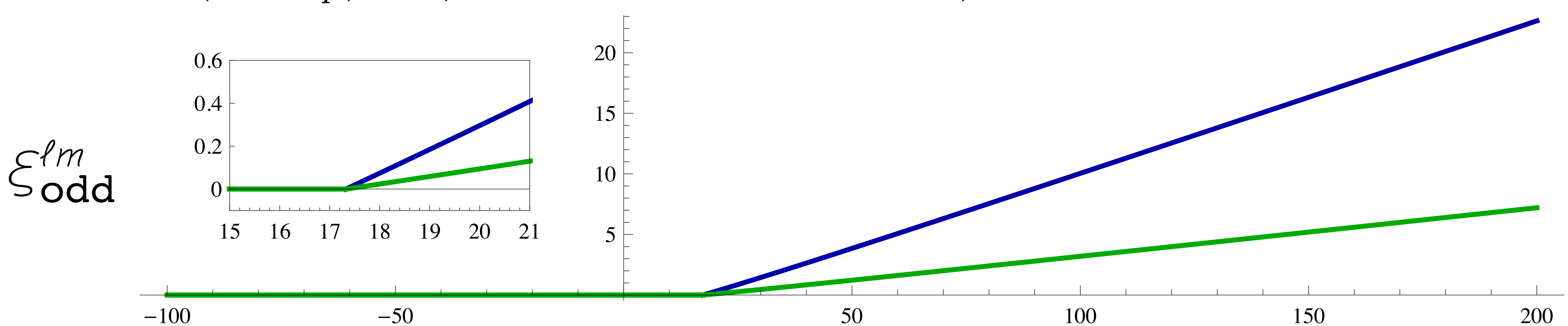

# Metric perturbation in modified RW gauge

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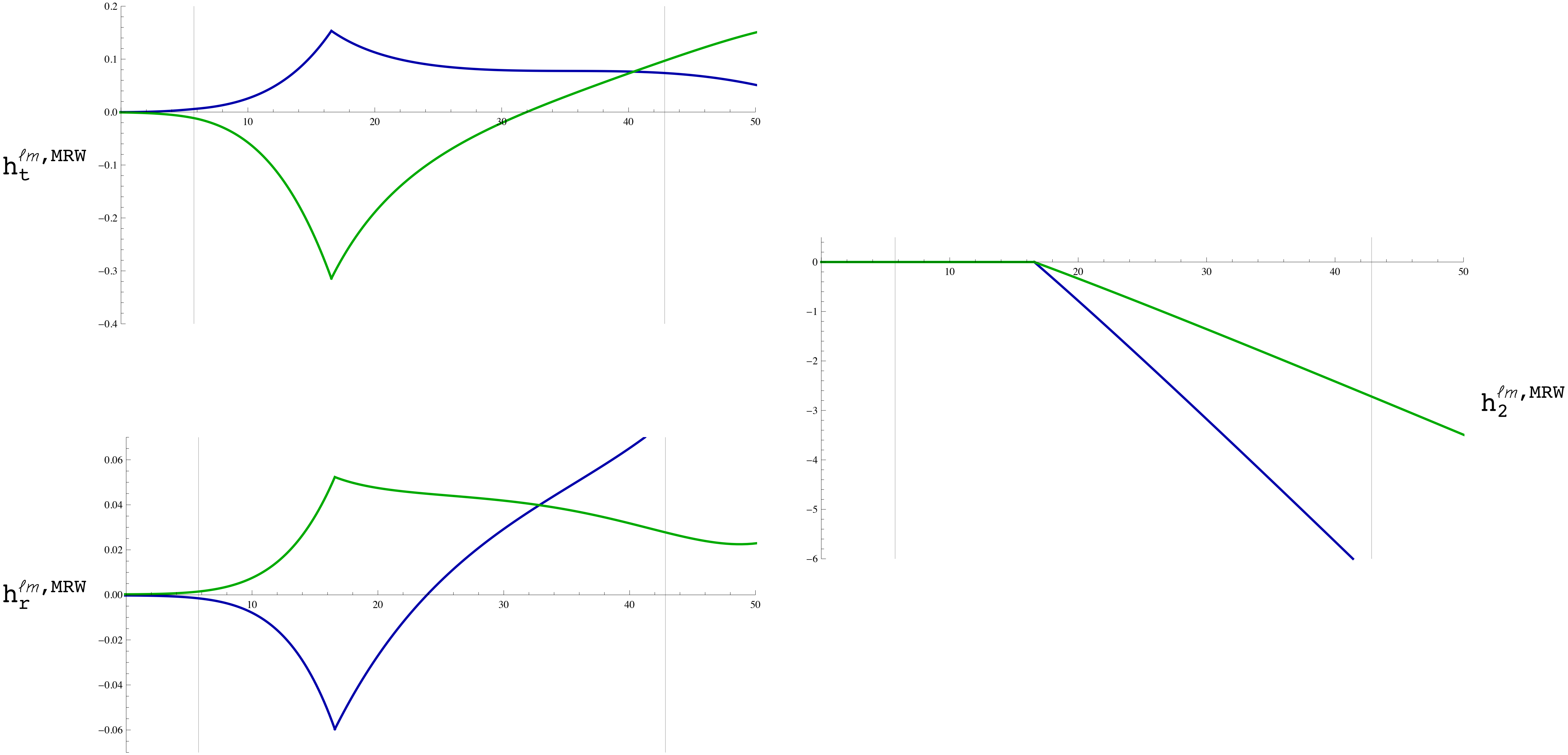

# Even-parity gauge vector

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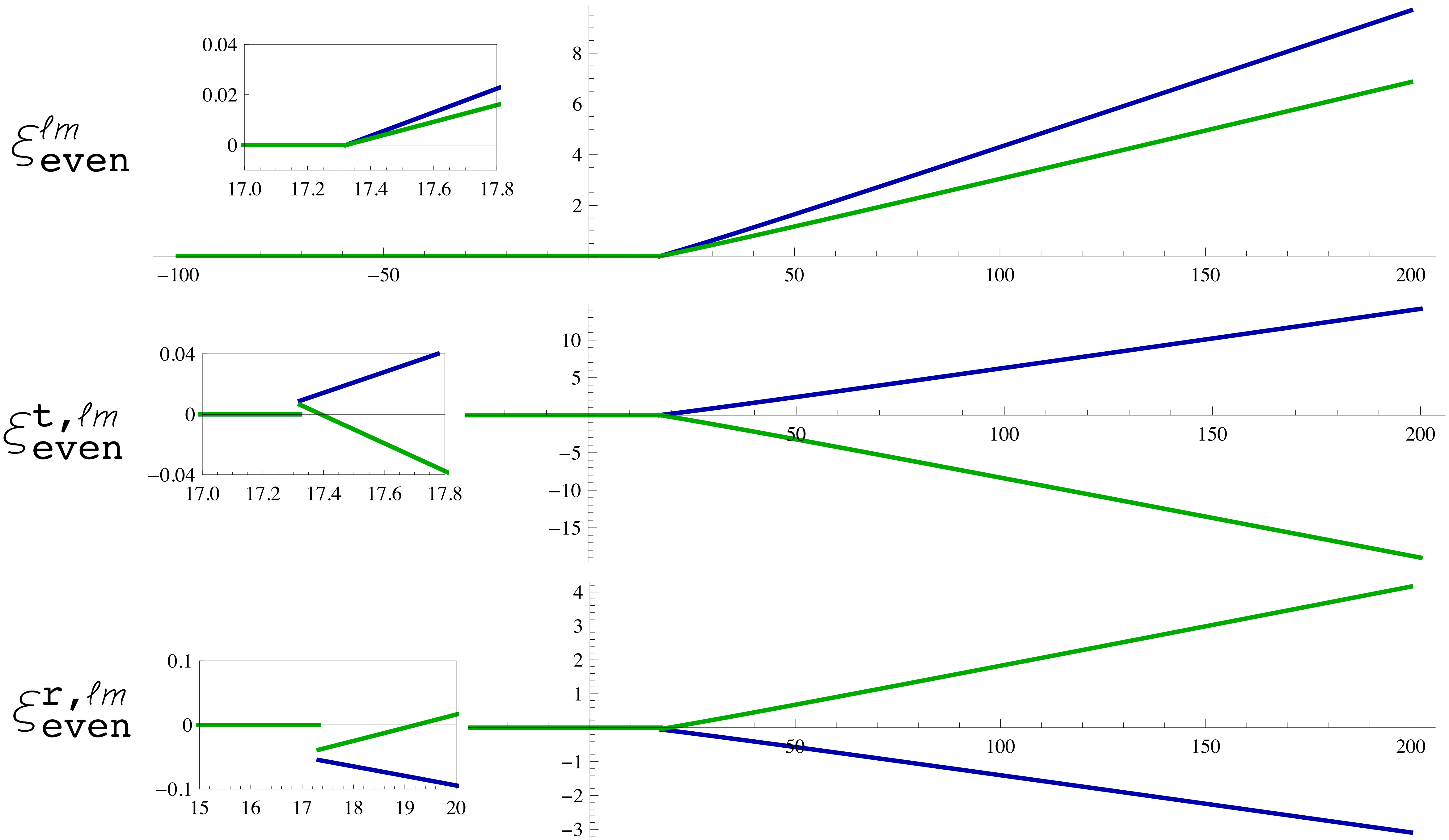

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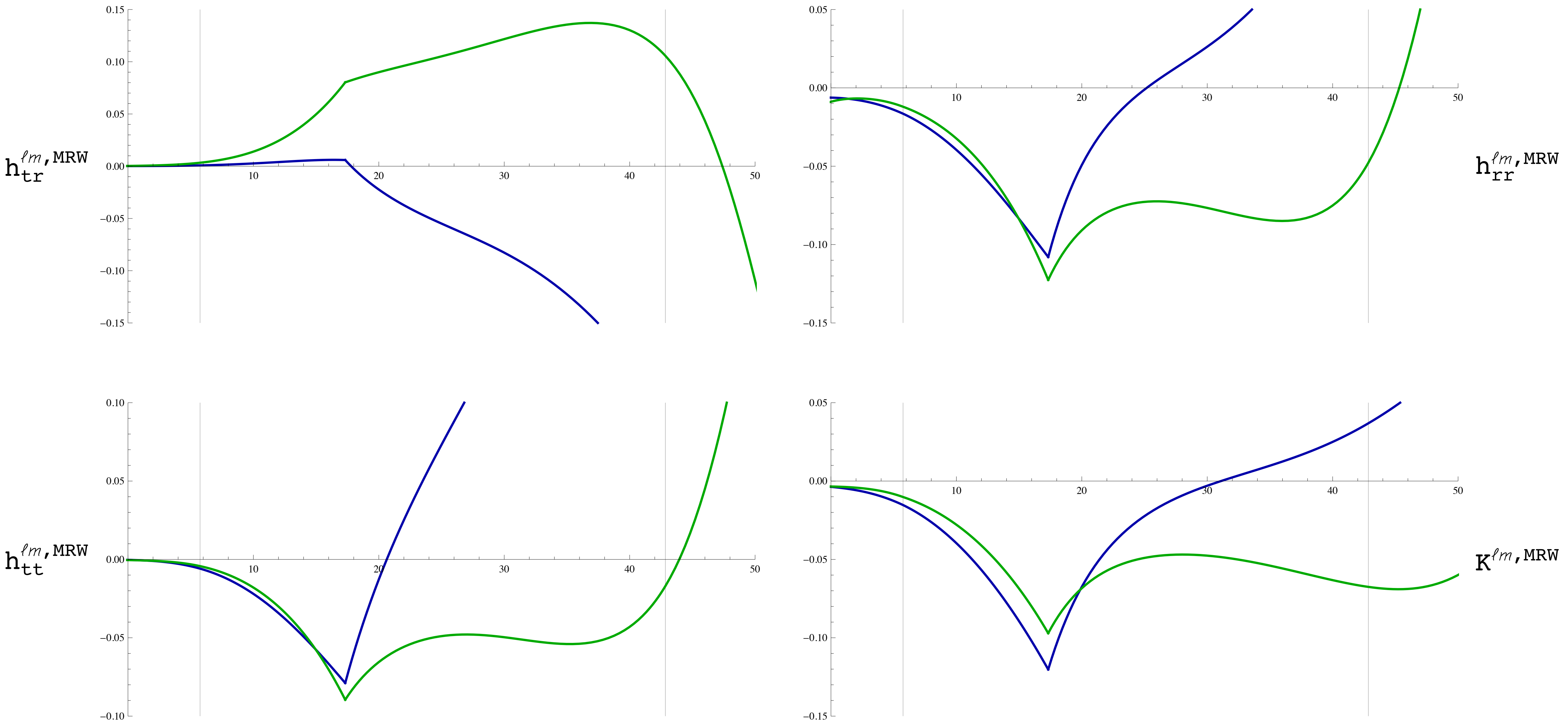

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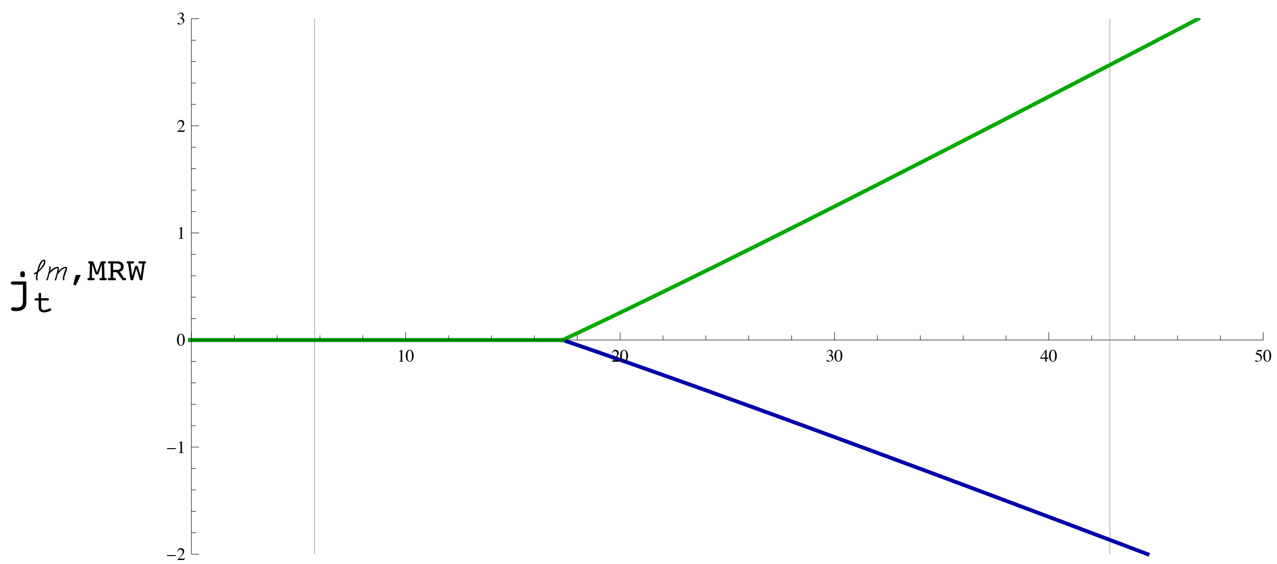


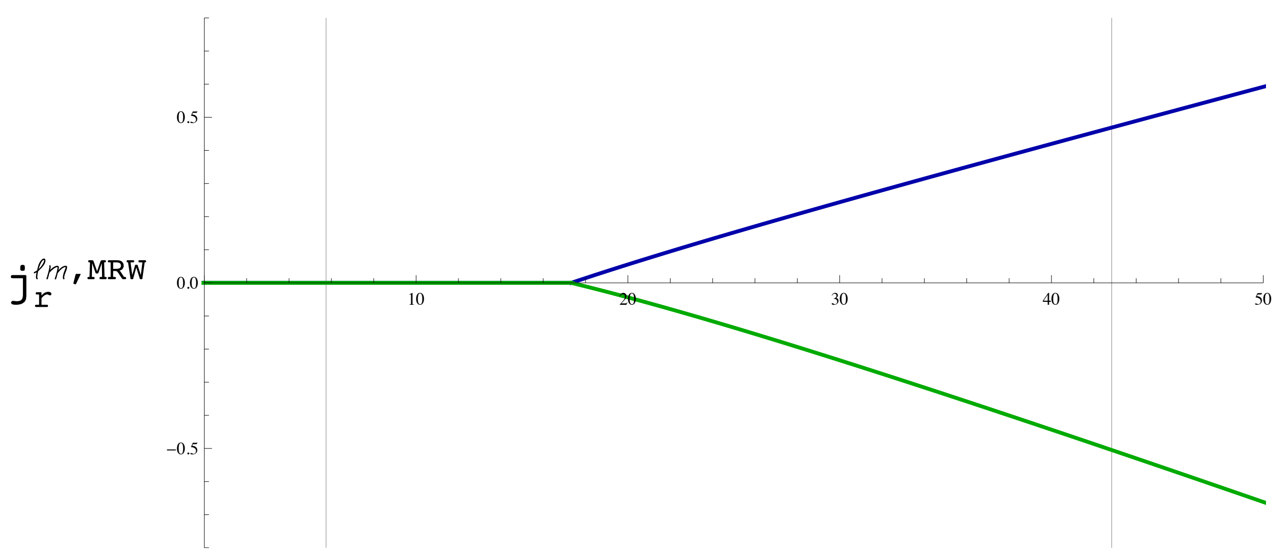


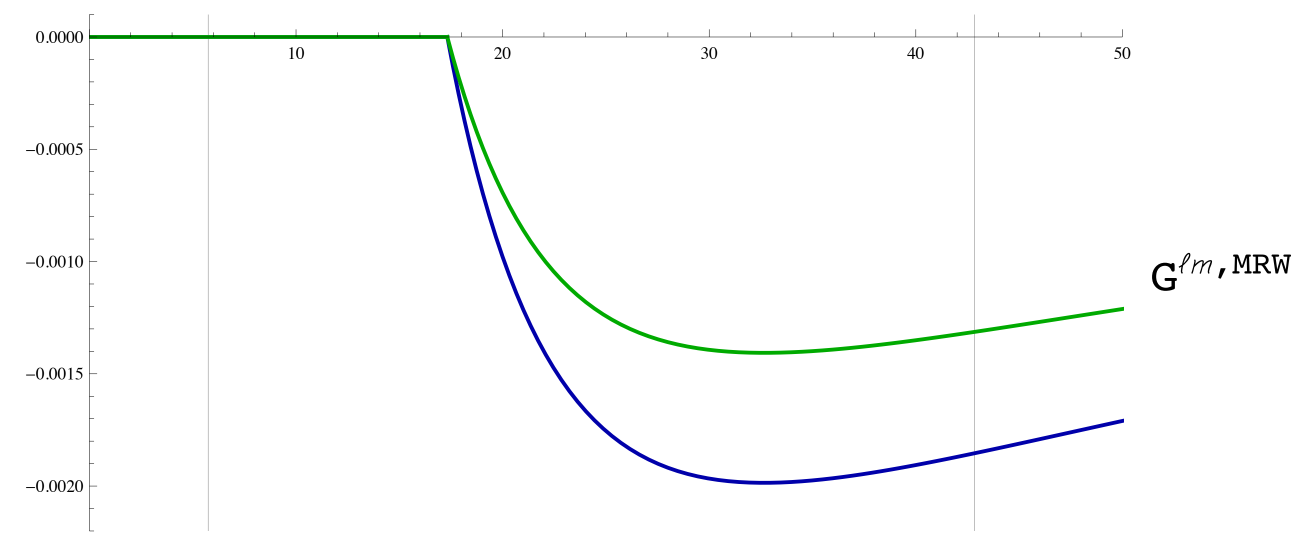

# Is this good enough?

- Split gauge transformation into two steps

$$\Xi^{\mu}_{\mathrm{RW}\to\mathrm{L}} = \underline{\Xi^{\mu}_{\mathrm{RW}\to\mathrm{MRW}}} + \underline{\Xi^{\mu}_{\mathrm{MRW}\to\mathrm{L}}}$$

Remove major discontinuities

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- But we can do better

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Restriction on the gauge vector

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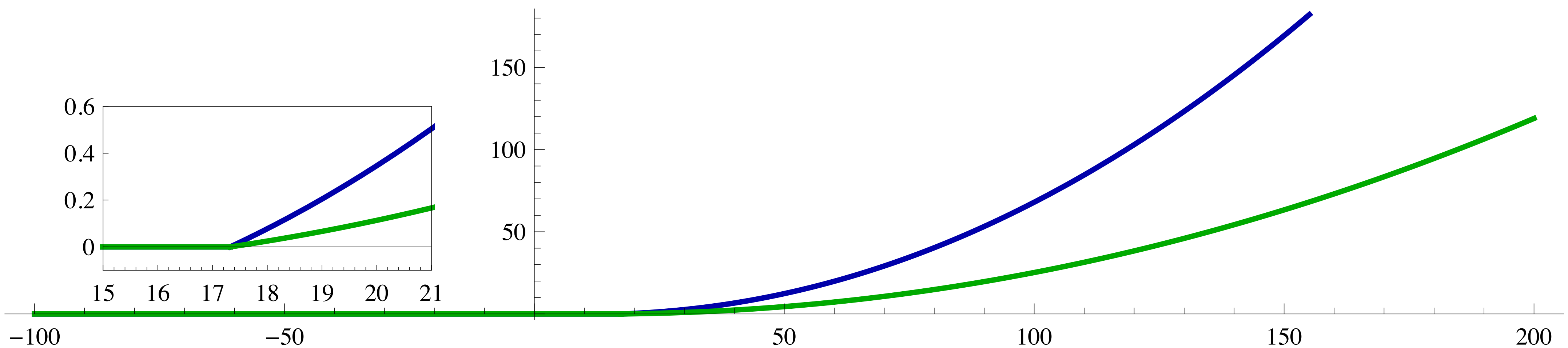

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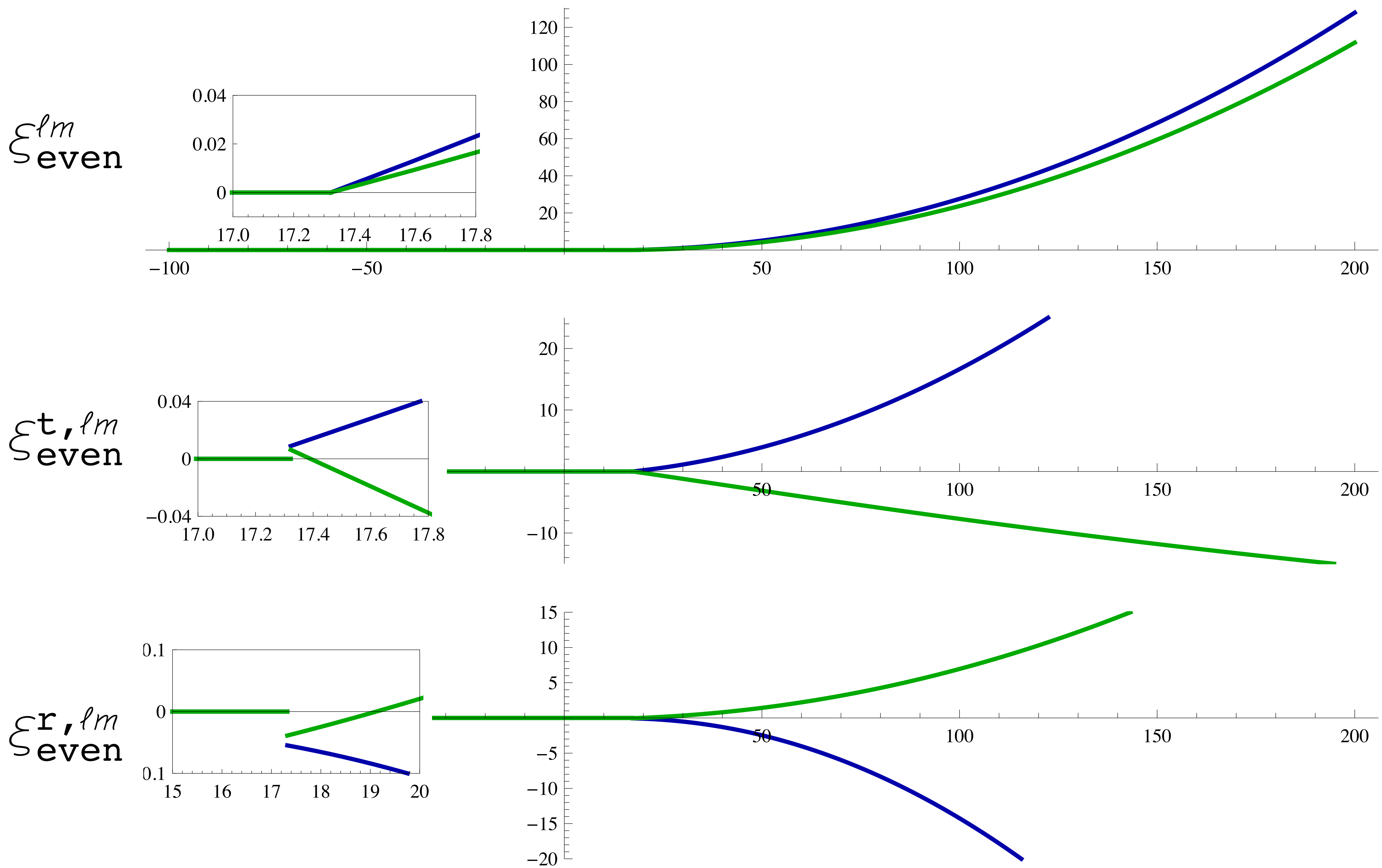

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- Modified RW gauge will have the same discontinuities as Lorenz gauge, to arbitrary orders of discontinuity

# Where does this leave us?

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- If vector $\Xi^{\mu}$ is well-defined, the SF will be also
- We can make the gauge vector $\Xi^{\mu}_{\mathrm{MRW}\to\mathrm{L}}$ as smooth as necessary

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- Working on the conservative SF
  - Want same value on both sides of particle
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- Non-radiative modes should follow from a "Modified Zerilli gauge"
- Solving field equations in Mathematica yields high (theoretically arbitrary) accuracy

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- Local singularities make a “well-defined” self-force impossible in this gauge
- Global gauge transformations to Lorenz are possible but difficult
- Modified RW gauge (hopefully) yields a way to find the self-force with no extra computational cost