Resonances in orbital dynamics

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Outline

1. Resonant Orbits
2. Self-force
3. Resonant evolution
4. Sustained resonances

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Geodesic motion in Kerr has 3 constants of motion E, L, and Q.

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- The specific angular momentum L.
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Frequencies as constants of motion

Simpler coordinates on phase space:

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$$\dot{w}_\theta = \Upsilon_\theta \qquad \dot{\Upsilon}_\theta = 0,$$
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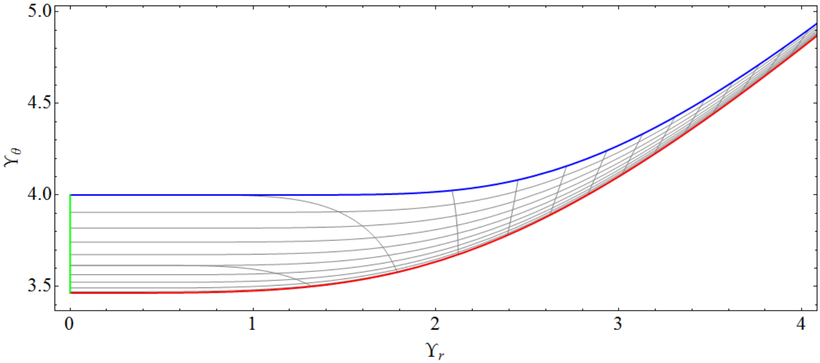
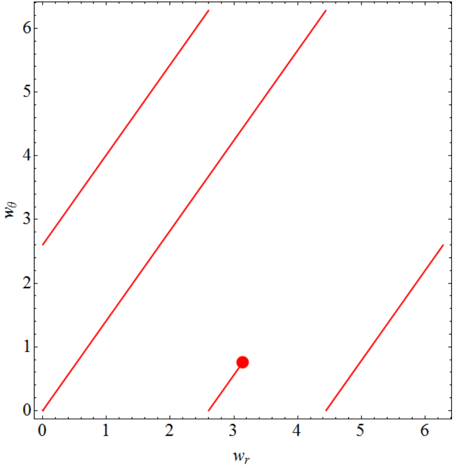
Frequency space



Generic orbits



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- Orbit is uniquely determined by constants of motion. (E, L, Q) or $(\Upsilon_r, \Upsilon_\theta, \Upsilon_\phi)$

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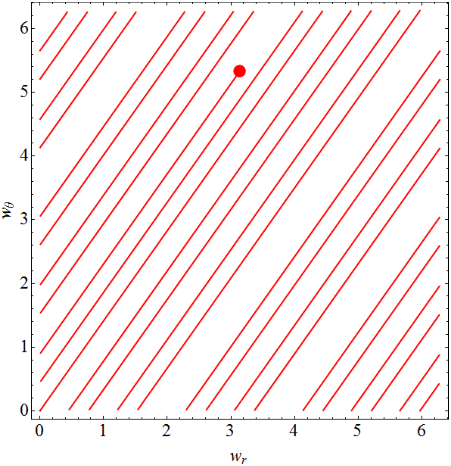

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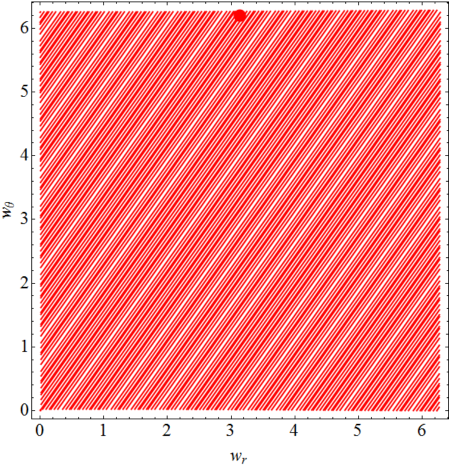

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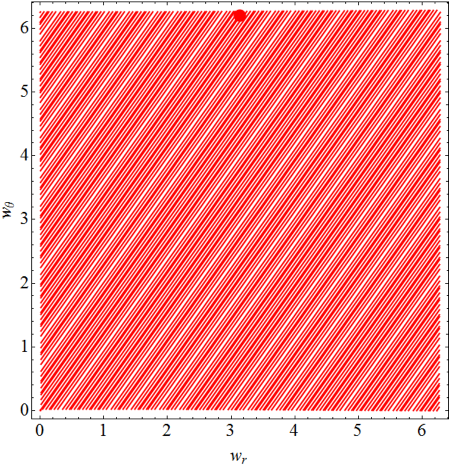

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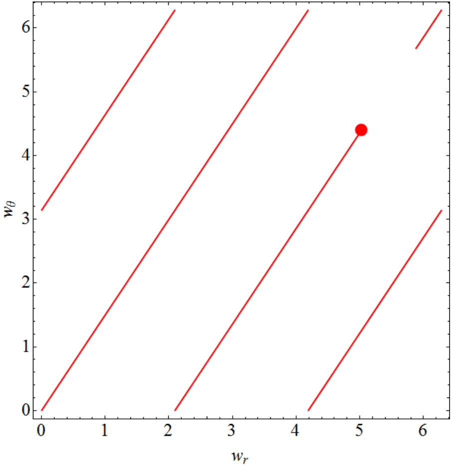
Resonant orbits

- Resonant orbits close
- Invariant torus is foliated by resonant orbits
- Need phase difference δ in addition to constants of motion to determine orbit.



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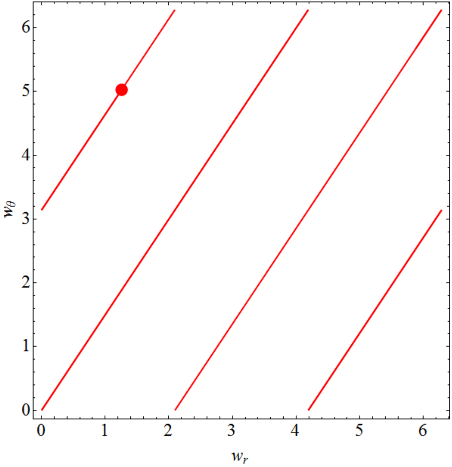

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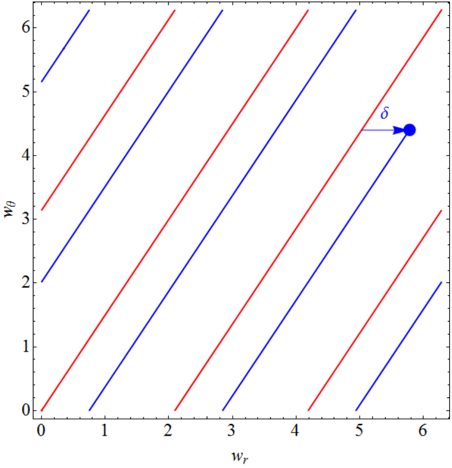

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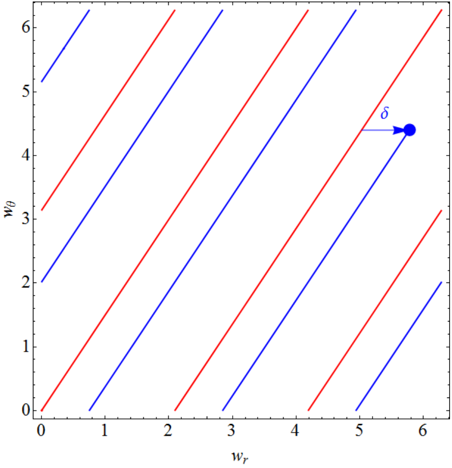
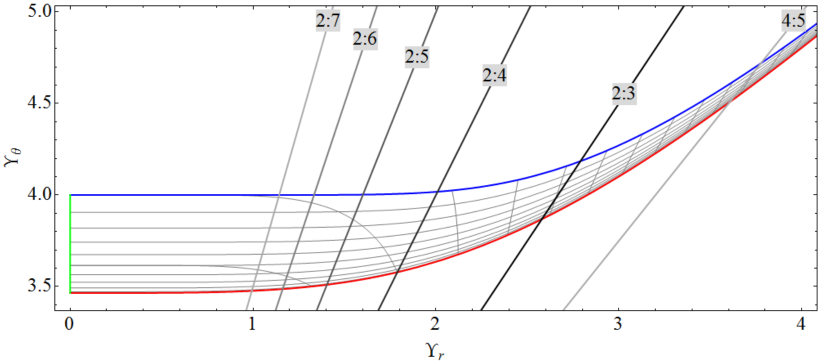


Resonance locations



Include self-force

In the extreme mass ratio limit ($\eta \equiv \mu/M << 1$) corrections due to the finite mass of the object can be added order by order:

$$\dot{\vec{w}} = \vec{\Upsilon} + \eta \vec{g}(\vec{\Upsilon}, \vec{w}) + O(\eta^2)$$
$$\dot{\vec{\Upsilon}} = \eta \vec{G}^{(1)}(\vec{\Upsilon}, \vec{w}) + \eta^2 \vec{G}^{(2)}(\vec{\Upsilon}, \vec{w}) + O(\eta^3),$$

where $\vec{w} = (w_r, w_\theta, w_\phi)$, $\vec{\Upsilon} = (\Upsilon_r, \Upsilon_\theta, \Upsilon_\phi)$, etc.

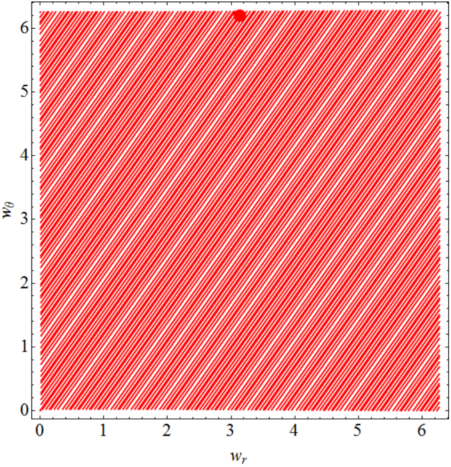
Fourier expand

Focussing on $\eta\vec{G}^{(1)}(\vec{\Upsilon},\vec{w})$ we can Fourier expand the dependence on the phases w_r and w_θ.

$$\dot{\vec{w}} = \vec{\Upsilon}$$

$$\dot{\vec{\Upsilon}} = \eta\vec{v}(\vec{\Upsilon}) + \eta\sum_{n,m}\vec{k}(\vec{\Upsilon})\cos(nw_r + mw_\theta) + \vec{\kappa}(\vec{\Upsilon})\sin(nw_r + mw_\theta)$$

Generic (non-resonant) orbits



- Inspiral time scale ($O(\eta^{-1})$) is much longer than orbital time scale ($O(1)$).
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- Consequently, the oscillatory terms average to zero.

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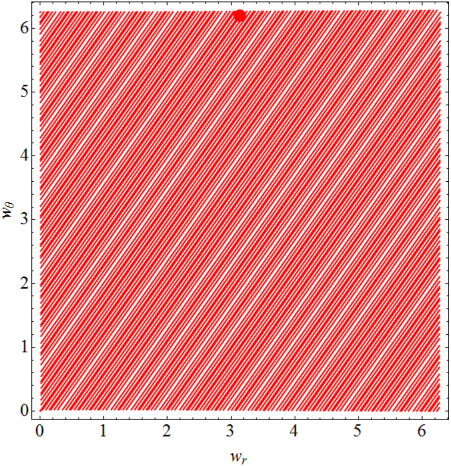

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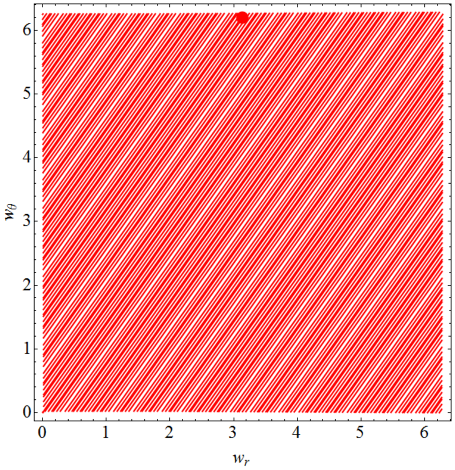

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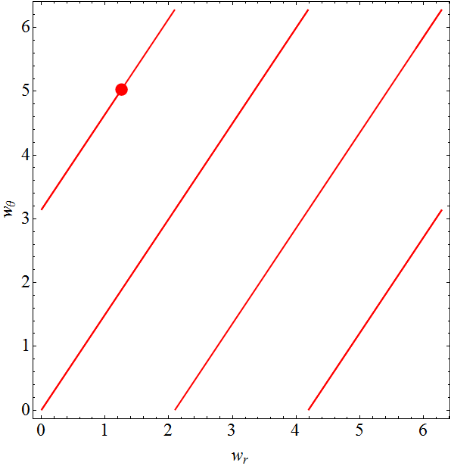
- Suppose system evolves through a resonant orbit with $n\Upsilon_r + m\Upsilon_\theta = 0$. (Happens generically!)
- Adiabatic approximation fails.
- The $nw_r + mw_\theta = 0$ harmonics remain relevant near the harmonic surface.



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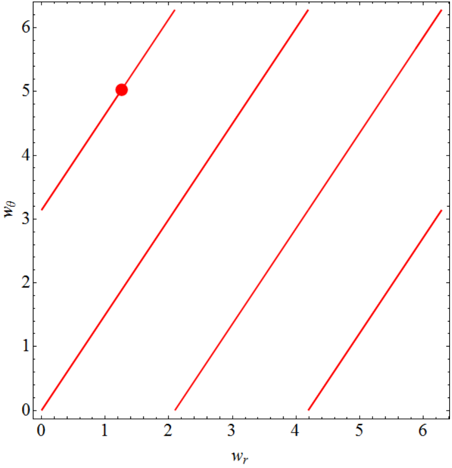

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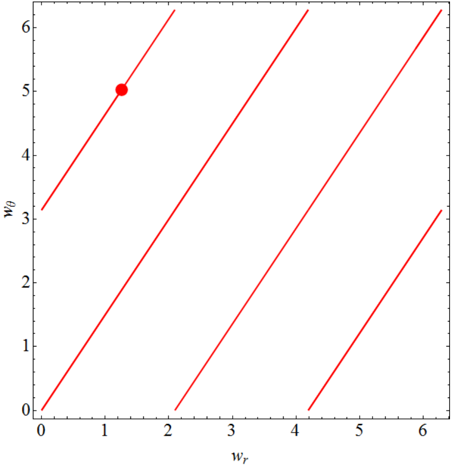

Resonant evolution

Suppose there is just one resonant harmonic. Then the equations of motion become: [Gair et al. '12]

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$$\ddot{w}_\theta = \dot{\Upsilon}_\theta = v_\theta(\vec{\Upsilon}) + k_\theta(\vec{\Upsilon})\cos(nw_r + mw_r)$$

Introduce convenient coordinates (and drop dependence on $\vec{\Upsilon}$):

$$\ddot{w}_\perp = v_\perp + k_\perp \cos(w_\perp)$$
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with $X_\perp \equiv nX_r + mX_\theta$ and $X_\parallel \equiv nX_r - mX_\theta$.

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$k_\perp << v_\perp$

Solution for late times:

$$\Upsilon_\perp = \dot{w}_\perp = t v_\perp + \frac{\sqrt{\pi} k_\perp}{\sqrt{2|v_\perp|}} \cos(w_\perp(0) \pm \pi/4) + O(t^{-1}, \frac{k_\perp^2}{v_\perp^2})$$

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Higher harmonics

Easy to include other harmonic terms

$$\Upsilon_\perp = tv_\perp + \sum_i \frac{\sqrt{\pi} k_{\perp,i}}{\sqrt{2i|v_\perp|}} \cos(iw_\perp(0) \pm \pi/4) + \sum_i \frac{\sqrt{\pi} \kappa_{\perp,i}}{\sqrt{2i|v_\perp|}} \sin(iw_\perp(0) \pm \pi/4) + O(t^{-1}, \frac{k_\perp^2}{v_\perp^2})$$

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Large $k_\perp$

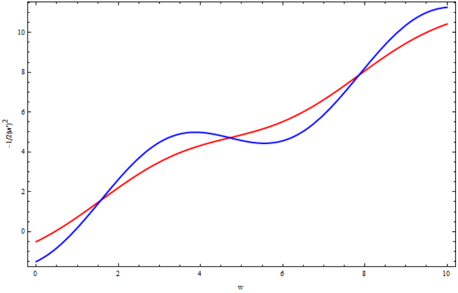
The equation of motion

$$\ddot{w}_\perp = v_\perp + k_\perp \cos(w_\perp)$$

Allows a first integral:

$$\frac{1}{2}(\dot{w}_\perp)^2 = v_\perp w_\perp + k_\perp \sin w_\perp + \dot{w}_\perp(0)$$

Potential



- If $k_\perp < v_\perp$ the potential is monotonic.
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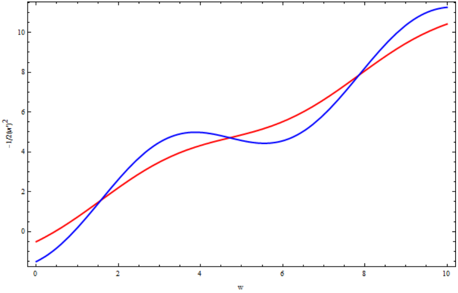

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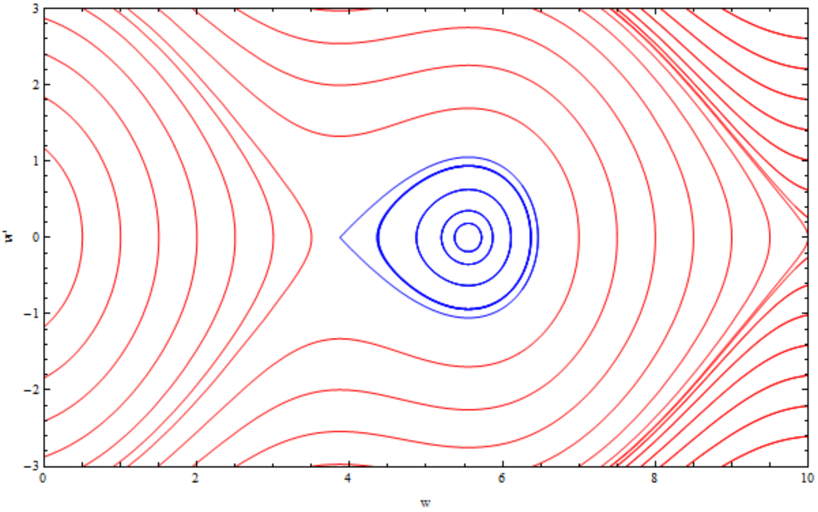
Phase portrait



Do sustained resonances occur?

- Not generically. $k_\perp$ is typically much smaller than $v_\perp$.
- e.g. [Flanagan, Hughes Ruangrsi, '12] find variations no larger than a few tenth of percent.
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