Gravitational self-force from curvature scalars

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Motivation

- One of the main sources of gravitational waves is the inspiral of compact objects into massive black holes in galactic nuclei.
- We work in the extreme mass-ratio inspiral (EMRI) regime, where the separation distance is small but the mass ratio of the bodies is large.
- The EMRI problem is amenable to a perturbative treatment, where the perturbation gives rise to the self-force (SF).
- Obtain accurate theoretical templates of EMRI waveforms. These waveforms have to include deviations from geodesic motion due to the SF.
- Current calculations of the SF rely on numerical solutions of the linearised Einstein's equations in the Lorenz gauge. For Kerr the field equations in the Lorenz gauge are not separable.

- The treatment of black-hole perturbations for Kerr is much simpler in the radiation gauge, where it is possible to reconstruct the perturbations from the Weyl scalars.

- In the radiation gauge we don't have a SF formulation. The perturbation due to a point particle is a string-like 2-D singularity.

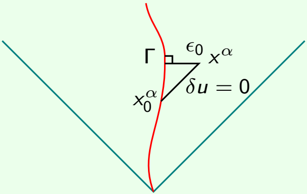
- We work in a gauge where it is "easy" to obtain the metric perturbations and relates through a regular gauge transformation to the Lorenz gauge. We call it *locally Lorenz radiation gauge* (LLR).

- The implementation will give the gravitational SF in the LLR gauge starting from a "force" in the ingoing radiation gauge. We obtain a mode-sum formula for the SF that has the form

$$F^{\alpha}_{self}(x_0) = \sum_{\ell=0}^{\infty} \left(F^{\alpha\ell}_{full\pm}(x_0) \mp A^{\alpha} L - B^{\alpha} - C^{\alpha}/L \right) - D^{\alpha}, \ (L \equiv \ell + 1/2).$$

SF in a locally Lorenz radiation gauge: Schwarzschild

Consider a particle of mass **m** moving along Γ. Let the particle be embedded in the background spacetime of a massive Schwarzschild black hole of mass M.



In LLR the perturbation near the particle has the same leading-order singularity as the Lorenz gauge,

$$h_{\alpha\beta}^{\mathrm{LLR}} = 2\mathbf{m}\epsilon_0^{-1}(g_{\alpha\beta} + 2u_\alpha u_\beta) + O(1).$$

We associate a given field point x^α with a "nearby" point x_0^α on the worldline, at the separation δx^α. The most convenient choice is to take $x_0^\alpha(x)$ to be the point on Γ with the same retarded time as x^α ($\delta u = 0$).

The metric perturbation tensor transforms (from Rad→LLR) according to

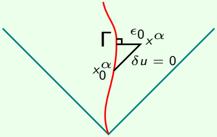
$$h^{\mathrm{LLR}}_{\alpha\beta} = h^{\mathrm{Rad}}_{\alpha\beta} + \xi_{\alpha;\beta} + \xi_{\beta;\alpha}.$$

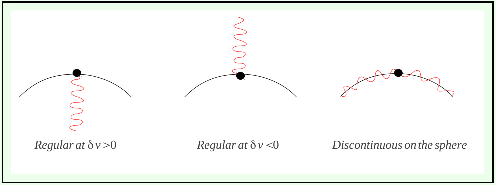
Which admits analytical solutions given by

$$\xi^{\pm}_{\alpha} = \mp 2u_\alpha \ln(\epsilon_0 \mp u_\alpha \delta x^\alpha) + \frac{\delta_\alpha}{\epsilon_0 \mp u_\alpha \delta x^\alpha} + O(\delta x^\alpha),$$

where

$$\delta_\alpha \equiv 2\mathcal{L}\left\{0, -\frac{\delta\varphi}{u^u}, \frac{\delta\theta}{u^\varphi}, \frac{\delta\varphi}{u^\varphi}\right\}.$$





Before calculating the contributions to the SF we decompose $\xi_\alpha^{\pm}$ in ℓ-modes,

$$\xi_{\alpha\perp}^{\pm\ell} = \pm\delta_0^\ell \left(0, \ -\frac{\mathcal{L}^2 f_0}{r_0^2(\mathcal{E}-\dot{r})}, 0, \mathcal{L}\right) \quad \text{(in EF coordinates)}.$$

We compare with the mode sum formula

$$F_\alpha^{\rm LLR} = \sum_{\ell=0}^{\infty}\left[F_\alpha^{{\rm Rad}\,\ell} + \delta F_\alpha^{{\rm Rad}\to{\rm LLR}\,\ell} - A_\alpha L - B_\alpha - C_\alpha/L\right] - D_\alpha.$$

Because ξ has only an $\ell = 0$ contribution, we can see that

$$\delta A_\alpha = \delta B_\alpha = \delta C_\alpha = 0, \quad \delta D_\alpha = \delta_\xi F_\alpha^{{\rm Rad}\to{\rm LLR}\,\ell=0}.$$

Finally we calculate the change in the SF with

$$\delta F^{\alpha\,\ell}_{grav} = -\mathbf{m}\left[(g^{\alpha\lambda} + u^\alpha u^\lambda)\frac{D^2\xi^\ell_\lambda}{D\tau^2} + R^\alpha{}_{\mu\lambda\nu}u^\mu \xi^{\ell\,\lambda} u^\nu\right].$$

We obtain the explicit value of δD_α:

$$\delta D^\pm_\alpha = \left\{\pm\frac{\mathbf{m}^2\mathcal{L}^2 C_t(\mathcal{E},r,\dot r)}{r^7(\mathcal{E}-\dot r)^3}, \frac{\mathbf{m}^2\mathcal{L}^2 C_r(\mathcal{E},r,\dot r)}{r^7 f(\mathcal{E}-\dot r)^3}, 0, \pm\frac{2\mathbf{m}^2\mathcal{L} C_\varphi(\mathcal{E},r,\dot r)}{r^4(\mathcal{E}-\dot r)^2}\right\}.$$

For circular orbits they reduce to

$$\delta D^\pm_\alpha = \left\{0, \pm\frac{3\mathbf{m}^2 M^2}{r^{5/2}(r-3M)^{3/2}}, 0, 0\right\}.$$

Metric reconstruction in the IRG.

The procedure to obtain the metric perturbations in the radiation gauge starting from the curvature scalars ψ_0 and ψ_4 was first proposed by Chrzanowski and also by Cohen and Kegeles. The CCK reconstruction can be computed from the expression

$$h_{\alpha\beta}^{\mathrm{IRG}} = \{-\ell_\alpha \ell_\beta (\boldsymbol{\delta} + 2\beta)(\boldsymbol{\delta} + 4\beta) - m_\alpha m_\beta (\mathbf{D} - 2\varrho)(\mathbf{D} + 3\varrho) + \ell_{(\alpha} m_{\beta)} \left[(\boldsymbol{\delta} + 4\beta)(\mathbf{D} + 3\varrho) + \mathbf{D}(\boldsymbol{\delta} + 4\beta)\right]\} \Psi^{\mathrm{IRG}} + \mathrm{c.c.},$$

where Ψ^{IRG} is found from ψ_0 or $\varrho^{-4}\psi_4$ inverting a radial equation or an angular equation:

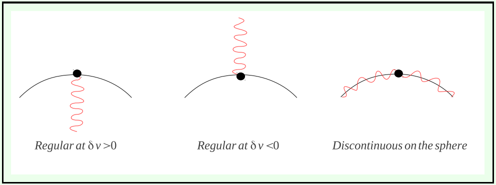
$$\psi_0 = \frac{1}{2}\mathbf{D}^4 \Psi^{\mathrm{IRG}}$$

$$\varrho^{-4}\psi_4 = \frac{1}{8}\left[\bar{\mathfrak{L}}^4 \bar{\Psi}^{\mathrm{IRG}} - 12M\partial_t \Psi^{\mathrm{IRG}}\right]$$

- What happens to the string singularity when implementing CCK reconstruction?
- How do we deal with the $\ell = 0, 1$ modes that are not included in the reconstruction?

Example: We performed the metric reconstruction for the static flat-spacetime mode by mode, starting from ψ_0

- The Hertz potential is continuous at the particle mode by mode.
- The reconstruction procedure gives regular MP on both sides of the sphere.
- The modes of the MP are in general discontinuous but without string singularities.



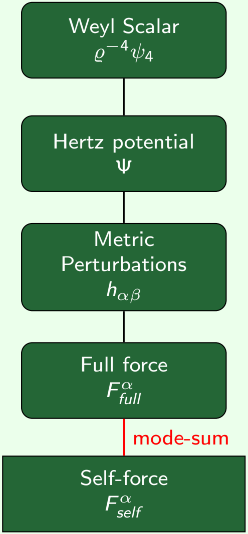
Wald showed (1973) that the only things we can add to the metric reconstruction are:

- Mass and angular momentum perturbations (δM and δJ).
- C-metric and Kerr-NUT perturbations.
- Gauge perturbations.

Our current understanding is:

- C-metric and Kerr-NUT are physically unacceptable.
- For the flat reconstruction: Mass and Mass dipole outside the sphere and gauge inside.
- In Kerr we expect: Mass and Angular momentum outside the orbit and gauge inside.

Numerical Implementation



- Analytically solve for the $m = 0$ modes for $\ell > 2$.
- We integrate numerically the homogeneous Teukolsky equation (with $s = -2$) with ingoing boundary conditions for each ℓ, m.
- We obtain the corresponding Weyl curvature scalar $\varrho^{-4}\psi_4$ at x_0^{α} by imposing junction conditions at x_0^{α} given by the source.

Weyl Scalar
$\varrho^{-4}\psi_4$

Obtained from Teukolsky equation for $s=-2$

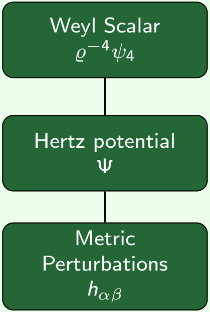
$$(r^2-2Mr)(\varrho^{-4}\psi_4)''-2(r-M)(\varrho^{-4}\psi_4)'-\left[\frac{\omega^2 r^4}{r^2-2Mr}-\frac{4ir^2\omega(r-3M)}{r^2-2Mr}+\bar{\eth}_{-1}\eth_{-2}\right](\varrho^{-4}\psi_4)=-4\pi r^2 T_{-2}.$$

Weyl Scalar
$\varrho^{-4}\psi_4$

Hertz potential
Ψ

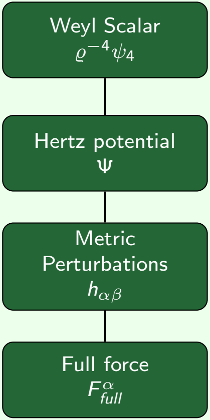
For circular orbits it can be obtained algebraically in terms of $\psi_{-2} \equiv \varrho^{-4}\psi_4$

$$\Psi_{\ell m} = 8\frac{(-1)^m(\ell+2)(\ell+1)\ell(\ell-1)\overline{\psi}_{-2\,\ell,-m} - 12imM\Omega\psi_{-2\,\ell m}}{[(\ell+2)(\ell+1)\ell(\ell-1)]^2 + 144m^2M^2\Omega^2}.$$



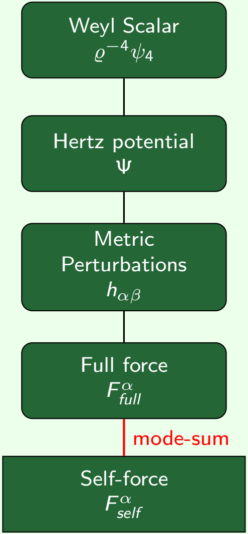
In terms of the Hertz potential

$$h_{\alpha\beta} = \{-\ell_\alpha \ell_\beta (\boldsymbol{\delta} + 2\beta)(\boldsymbol{\delta} + 4\beta) - m_\alpha m_\beta (\mathbf{D} - 2\varrho) (\mathbf{D} + 3\varrho) + \ell_{(\alpha} m_{\beta)} \left[(\boldsymbol{\delta} + 4\beta)(\mathbf{D} + 3\varrho) + \mathbf{D}(\boldsymbol{\delta} + 4\beta) \right] \} \Psi + \text{c.c.}$$



Each tensor harmonic of the full force is obtained with the equation of motion

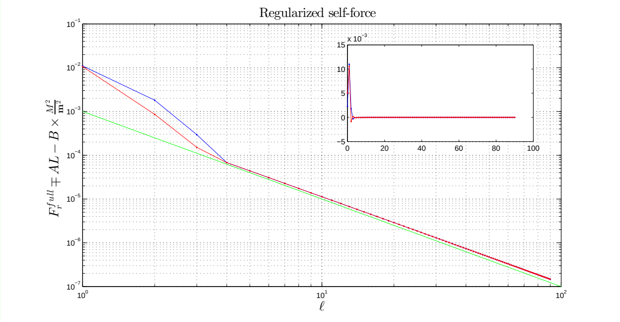
$$F^{\alpha}_{full} \equiv -\mathbf{m}(g^{\alpha\beta} + u^{\alpha}u^{\beta})\left(\nabla_{\mu}h_{\nu\beta} - \frac{1}{2}\nabla_{\beta}h_{\mu\nu}\right)u^{\mu}u^{\nu}.$$



We regularize each mode using the mode-sum formula:

$$F_{\alpha}^{\mathrm{LLR}} = \sum_{\ell=0}^{\infty} \left[F_{\alpha}^{\mathrm{Rad}\,\ell} - A_{\alpha} L - B_{\alpha} \right] + \delta D_{\alpha}.$$

Self-force in ℓ-modes



ℓ-modes in log-log scale of the SF after regularization. Taken from the limit $r \to r_0^+$ (red) and the limit $r \to r_0^-$ (blue). The small graph is in linear scale.

Gauge invariant red-shift

Detweiler showed that for circular orbits in Schwarszchild there are two gauge invariant quantities that carry out non-trivial information about the conservative SF dynamics: Ω and $u^t \equiv U$. In practical calculations we compute:

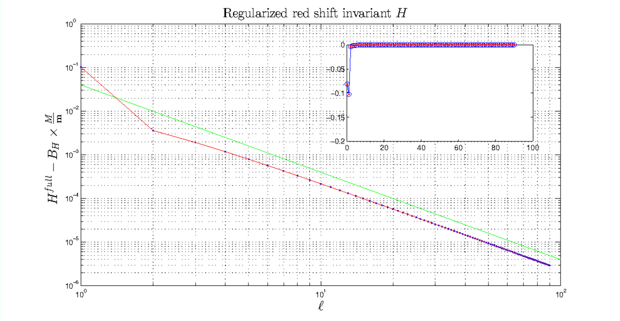
$$H \equiv \frac{1}{2} h^R_{\alpha\beta} u^\alpha u^\beta, \quad \frac{d\tau}{d\tilde{\tau}} = 1 + H,$$

where $\tilde{\tau}$ is the proper time along the geodesic of the effective metric $\tilde{g} = g + h^R$ and τ along the projection on g.

$$H^{\rm LLR} = \sum_{\ell=0}^{\infty} \left[H^{{\rm Rad}\,\ell} - (B^H - \delta B^H) - (C^H - \delta C^H)/L \right] - (D^H - \delta D^H),$$

with $\delta B^H = \delta C^H = \delta D^H = 0$, for circular orbits.

H in ℓ-modes



ℓ-modes of H after regularization.

Preliminary values

r_0/M	$F^{r\,\mathrm{IRG}}(r_0^+) \times \frac{M^2}{\mu^2}$	$F^{r\,\mathrm{ORG}}(r_0) \times \frac{M^2}{\mu^2}$	$F^r(r_0^+) \times \frac{M^2}{\mu^2}$
10	1.49E-02 (1)	1.3580536E-02	1.969800E-02 (1)
12	1.09E-02 (1)	1.0019806E-02	1.4776563E-02 (3)
20	4.37E-03 (5)	4.0997900E-03	6.147348E-03 (1)
25	2.88E-03 (3)	2.7292140E-03	4.100090E-03 (1)
50	7.60E-04 (9)	7.3864055E-04	1.110554E-03 (1)

r_0/M	$\Delta U \times \frac{M}{\mu}$	$\Delta U_{\mathrm{SD}} \times \frac{M}{\mu}$
10	-0.12912222 (1)	-0.1291222
12	-0.10193561 (1)	-0.1019355
20	-0.0558278 (1)	-0.05582771
25	-0.0435999 (1)	-0.04359984
50	-0.020844686 (3)	-0.02084465
100	-0.010205291 (2)	-0.01020528

Summary and future work

- We have obtained the gauge transformation from the radiation gauge to a locally Lorenz radiation gauge. This transformation naturally has a string singularity, but it is possible to construct a regular solution in each half spacetime. The regular halves can be combined into a string-free solution at the cost of introducing a discontinuity across the sphere intersecting the particle
- The new mode-sum formula to obtain the GSF in a new locally Lorenz radiation gauge (Schwarzschild and Kerr).
- We have calculated numerically ℓ-modes contributions to SF and showed that the results from our implementation are consistent with all the regularization parameters given by the mode-sum formula.
- Extend the numerical implementation to obtain the SF for non-circular orbits.
- Compute numerically the gravitational SF and the gauge invariant quantity H for the Kerr case.