Structure of the retarded scalar Green function on Schwarzschild spacetime.

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Capra 16, July 2013

Joint work with Marc Casals, UCD $\rightarrow$ CBPF

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Self-force

- GW astronomy: need for accurate results describing the 2-body motion of a small black hole (m) in the field of a large black hole (M).
- Calculate the self-force of the small black hole, and treat the motion as the deviation from a geodesic of the background gravitational field of the large black hole *or* as a geodesic of the perturbed spacetime.
- Work with the scalar field toy model.
- Equation of motion (scalar version of MiSaTaQuWa):

$$ma^{\alpha} = q(g^{\alpha\beta} + u^{\alpha} u^{\beta}) \nabla_{\beta} \Phi_{\text{rad}}.$$

- The term required is

$$\nabla_\alpha \Phi_{\rm rad} = \text{ local stuff } + q \lim_{\epsilon \to 0^+} \int_{-\infty}^{\tau-\epsilon} \nabla_\alpha G_{\rm ret}(z(\tau), z(\tau'))d\tau',$$

where $G_{\rm ret}(x, x')$ is the retarded Green's function, satisfying

$$\Box G_{\rm ret}(x, x') = -4\pi\delta_4(x, x'), \qquad G_{\rm ret}(x, x') = 0 \text{ if } x \notin J^+(x').$$

- NB $G_{\rm ret}$ is required *globally*.

Local representation: Hadamard form of $G_{\rm ret}$

- Within a convex normal neighbourhood $\mathcal{N}$ of x',

  $$G_{\rm ret}(x,x') = [U(x,x')\delta(\sigma(x,x')) + V(x,x')\theta(-\sigma(x,x'))]\theta(t-t'),$$

  where $\sigma(x,x')$ is Synge's world function, δ, θ are the usual distributions.
- Given σ, there is an algorithm for generating U, V (involves solving transport equations for the *Hadamard coefficients* V_k of V).
- But this form is not valid once light-crossings occur ($\Delta t = 27.62M$ for circular geodesic at $r = 6M$).

G_{ret} on Schwarzschild

- Useful simplification:

$$G_{\mathrm{ret}}(x, x') = \frac{1}{r \cdot r'} \hat{G}_{\mathrm{ret}}(x, x'),$$

where $\hat{G}_{\mathrm{ret}}$ is the retarded Green function for the conformally invariant wave equation on the *conformal Schwarzschild* spacetime with line element

$$d\hat{s}^2 = -\frac{f(r)}{r^2}(dt^2 - dr_*^2) + d\Omega^2, \tag{1}$$

where $f(r) = 1 - 2M/r$ and r_* is the usual tortoise coordinate.

Null separations

- $\hat{\sigma}_4 = \sigma(x^A, x^{A'}) + \frac{1}{2}\gamma^2$, where $\sigma(x^A, x^{A'})$ is the 2-dim the world function, γ is geodesic distance on the unit 2-sphere.
- Furthermore, we can write (globally) $\sigma = -\frac{1}{2}\eta^2$ where η is geodesic distance along a causal geodesic in M_2.
- Then a null geodesic connects (x, x') in Schwarzschild iff ditto in conformal Schwarzschild iff $\hat{\sigma}_k^{\mathrm{even/odd}} = 0$ where

$$\hat{\sigma}_k^{\mathrm{even/odd}} = -\frac{1}{2}\eta^2 + \frac{1}{2}(\gamma \pm 2k\pi)^2,$$

and even/odd refers to the number of light-crossings that the geodesic has passed through.

Mode sum decomposition

- Separation of variables:

$$\hat{G}_{\mathrm{ret}}(x,x') = \frac{1}{4\pi}\sum_{\ell=0}^{\infty}(2\ell+1)\mathcal{G}_\ell(x^A,x^{A'})P_\ell(\cos\gamma),$$

where P_ℓ are Legendre polynomials and $\mathcal{G}_\ell$ satisfies the PDE for the Green function on the 1+1 dimensional spacetime with line element

$$ds^2 = -\frac{f(r)}{r^2}(dt^2 - dr_*^2).$$

- The relevant 1+1 dim wave equation is

$$P\phi - \lambda^2\phi = \Box\phi - (\lambda^2 + \frac{1}{4}(1-\frac{8M}{r}))\phi = 0,$$

where $\lambda = \ell + \frac{1}{2}$.

- A large body of work on this equation then moves to the frequency domain: write $\phi(t, r_*) = \sum_\omega \bar{\phi}(r_*; \omega) e^{i\omega t}$, which yields the Regge-Wheeler equation for $\bar{\phi}$. Proceed by analysing the spectrum: QNM, branch cut, large frequency arc (Casals - previous talk).
- Our aim is to apply PDE theory to the 1+1 dimensional problem, and then resum to obtain $G_{\rm ret}$.
- Principal technique: large-ℓ expansion for $\mathcal{G}_\ell$.

The spacetime M_2.

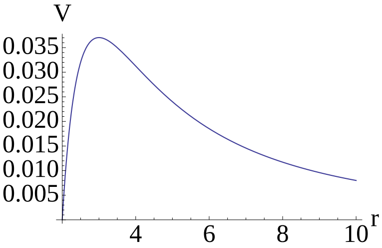
- A theoretical advantage is present: the 1+1 dimensional spacetime M_2 is (almost certainly) a causal domain (geodesically convex with a certain causality condition).
- **Theorem:** If Ω is a causal domain, then the results of Friedlander's book apply on Ω.
- In particular, results that are typically valid only locally in 3+1 are globally valid for the 1+1 problem.

M_2 is (almost certainly) a causal domain.

- Geodesic convexity: there is a unique geodesic connecting every pair (t, r_*) and (t', r'_*).
- Only timelike separations cause any difficulty:

$$\left(\frac{dr_*}{dt}\right)^2 = 1 - \frac{\alpha(r_*)}{E^2}, \qquad \alpha(r_*) = \frac{1}{r^2}\left(1 - \frac{2m}{r}\right).$$

- Most uniqueness problems are resolved simply by comparing slopes.
- Not so straightforward for particles with sub-critical energies $E < E_0 = 1/(3\sqrt{3}m)$ which reflect off the potential barrier at $r_+ = r_+(E)$.



- A geodesic from r_0 and sub-critical energy E arrives at the potential barrier after time

$$\Delta t = \int_{r_+(E)}^{r_0} \frac{f^{-1}}{\sqrt{1 - \frac{f}{E^2 r^2}}} dr.$$

- **Lemma 1:** If $E_1 < E_2 < E_0$, then $r_+(E_1) > r_+(E_2) > 3m$ (that is, $\frac{dr_+}{dE} < 0$).

- **Lemma 2:** If $\frac{d(\Delta t)}{dE} > 0$, then geodesics are unique.

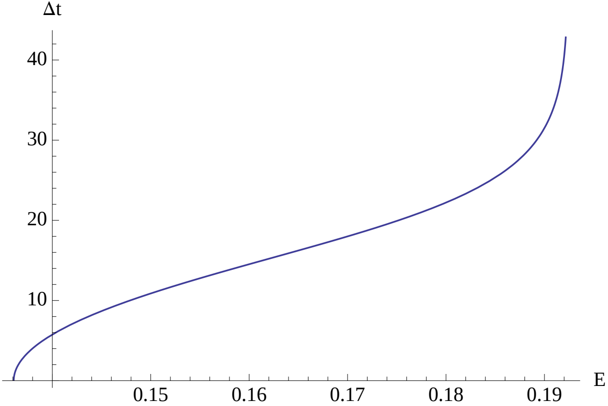


Figure: Arrival time for geodesics from $r = 6M$.

- The causality condition required is the following: for all pairs of points $p, q \in M_2$, the set

$$J^+(p) \cap J^-(q)$$

is either compact or empty.
- $J^\pm(p) = \overline{D^\pm(p)}$, the closure of the chronological future (past) of $p \in M_2$.
- Thanks to global conformal flatness of M_2, the sets in question are either empty or are closed rectangles with sides at $\pm 45^\circ$.

Hadarmard-Bessel series

- Back to the main theme: large$-\ell$ asymptotics of

$$\frac{1}{r^2 f}(-\partial_t^2\phi + \partial_{r_*}^2\phi) - (\lambda^2 + \frac{1}{4}(1 - \frac{8M}{r}))\phi = 0.$$

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- Lewis, Keller, Bleistein, others (NYU, 1960's): $\sum_k a_k(x)e^{i\lambda s}/(i\lambda^k)$.

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- Lewis, Keller, Bleistein, others (NYU, 1960's): $\sum_k a_k(x)e^{i\lambda s}/(i\lambda^k)$.
- The following result is due to Zauderer; cited in Friedlander.

$$\mathcal{G}_\ell(x^A, x^{A'}) = \frac{1}{2}\sum_{k=0}^{\infty} U_k \left(\frac{2\eta}{\lambda}\right)^k J_k(\lambda\eta)\theta(-\sigma)\theta(\Delta t).$$

- J_k are Bessel functions;
- $\sigma = -\eta^2/2$ where η is the 2-dim geodesic distance;
- U_k are the Hadamard coefficients for the retarded Green function of the operator P - that is, for the equation above with $\lambda = 0$.
- The U_k satisfy certain recurrence relations in the form of transport equations along the geodesic from x^A to $x^{A'}$.
- These coefficients and the series for $\mathcal{G}_\ell$ are defined globally on M_2.
- The result is **not** perturbative: holds for all $\lambda \in \mathbb{C}\backslash\{0\}$.

Large$-\ell$ expansion: singularity structure of $G_{\rm ret}$

- We apply a large-argument asymptotic expansion for the Bessel functions (large$-\lambda$ - i.e. large$-\ell$).
- Collecting inverse powers of λ and resumming allows us to identify the singular and the non-singular (continuous) parts of $G_{\rm ret}$:

$$
\begin{aligned}
G_{\rm ret}^{\rm sing} \quad = \quad & \frac{2}{r \cdot r'} \frac{U_0}{\eta^{1/2}\sqrt{\sin\gamma}} \times \sum_{k=0}^{\infty} (-1)^k \Big\{ \\
& \left[\delta(\eta - (\gamma + 2k\pi)) + \mu_0^-(\eta,\gamma)\theta(\eta - (\gamma + 2k\pi))\right] \\
& + \frac{1}{\pi}\left[\mathrm{PV}\left(\frac{1}{\eta - (2k\pi - \gamma)}\right) + \mu_0^+(\eta,\gamma)\ln|\eta - (2k\pi - \gamma)|\right]\Big\}
\end{aligned}
$$

$$
\mu_0^{\pm} = \frac{1}{8}(\cot\gamma \pm \frac{1}{\eta} \pm 16\eta\frac{U_1}{U_0}).
$$

Comments

- Ori's observation: spherical symmetry induces a 4-fold recursion in the singularity structure of the retarded Green's function as successive caustics are met:

$$\delta(\sigma) \to PV\left(\frac{1}{\sigma}\right) \to -\delta(\sigma) \to -PV\left(\frac{1}{\sigma}\right) \to \delta(\sigma) \to \cdots$$

- Established in general spacetimes via Penrose limits by Harte & Drivas; see also previous work in Schwarzschild by Dolan & Ottewill and in $\mathbb{M}_2 \times \mathbb{S}_2$ by Casals & Nolan.
- Four-fold recursion demonstrated for the "tail" term: $\theta \to \log \to \cdots$.
- The result above identifies exactly the locations of the singularities at $\hat{\sigma}_k^{\mathrm{even/odd}} = 0$.

Calculations

- Ultimate aim is to calculate $G_{\mathrm{ret}}(t, r, \theta, \phi; t', r', \theta', \phi')$ for pairs of points on the orbit of the small black hole.
- Consider geodesic motion: $G_{\mathrm{ret}}(\Delta t, r, r', \gamma)$.
- Given inputs $\Delta t, r, r'$, we must first determine the timelike geodesic of M_2 that connects (t, r) and (t', r'), and calculate the total proper time η_* along this geodesic segment.
- Solve transport equations $(\eta \frac{dX}{d\eta} = f(X, \eta))$ for N variables along this geodesic (cf. Ottewill and Wardell) to determine $U_0(N = 6)$ and $U_1(N = 96)$.
- This yields one data point on the graph of $G_{\mathrm{ret}}(\Delta t)$.

Transport equations

- In 2-d, U_0 is the square root of the van Vleck determinant:

$$U_0 = \triangle^{1/2} \qquad \Leftrightarrow \qquad \sigma^A \nabla_A U_0 = (1 - \Box\sigma) U_0, \quad [U_0] = 1.$$

- The transport equations are

$$2\sigma^A \nabla_A U_k + (\Box\sigma + 2(k-1)) U_k = \frac{1}{2} P U_{k-1}, \quad k \geq 1.$$

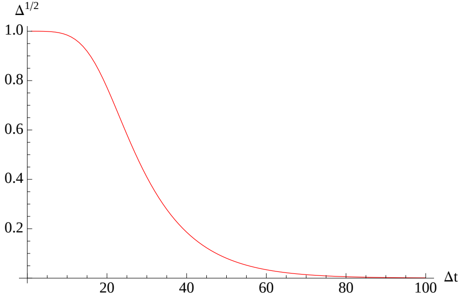

Figure: Decay of $U_0 = \triangle^{1/2}$ for $r = r' = 6M$.

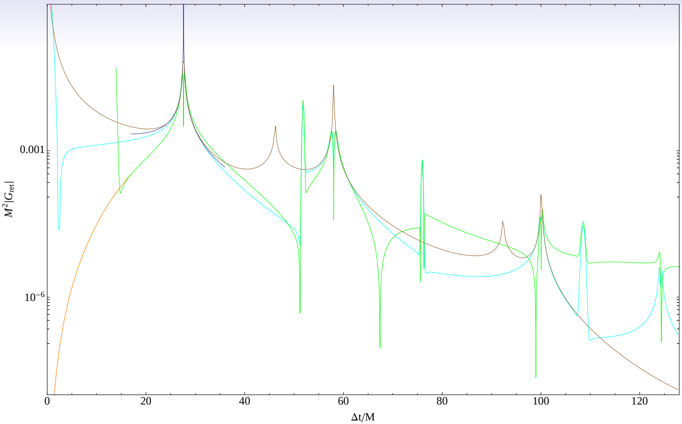

Figure: Log-plot of approximations to $G_{\rm ret}$ as functions of Δt for points on a timelike circular geodesic at $r = 6M$. Cyan: the Bessel expansion including just the $k = 0$ term, summed up to $\ell = 100$. Brown: leading order in the large-ℓ expansion including only the $k = 0$ term. Green: QNM sum. Blue: large-ℓ asymptotics in the QNM sum. First two due to Casals; last two due to Casals, Dolan, Ottewill and Wardell.

$G_{\rm ret}$ as a sum over Hadamard forms

- Begin with $G_{\rm ret} = \sum_\ell \mathcal{G}_\ell P_\ell$,

$$\mathcal{G}_\ell(x^A, x^{A'}) = \frac{1}{2}\sum_{k=0}^{\infty} U_k \left(\frac{2\eta}{\lambda}\right)^k J_k(\lambda\eta)\theta(-\sigma)\theta(\Delta t).$$

- Expand $P_\ell(\cos\gamma)$ in Bessel functions:

$$P_\ell(\cos\gamma) = \sum_{j=0}^{\infty} \alpha_j(\gamma)\frac{J_j(\lambda\gamma)}{\lambda^j}.$$

- Expand Bessel functions:

$$\begin{aligned} J_k(\lambda x) &= \frac{1}{\sqrt{2\pi\lambda x}}\sum_{m=0}^{\infty} E_m(\lambda x - \frac{\pi}{2}k - \frac{\pi}{4})\frac{a_{k,m}}{(2x)^m\lambda^m}, \\ E_k(x) &= \frac{e^{ik\pi/2}}{2}(e^{ix} + (-1)^k e^{-ix}). \end{aligned}$$

- Expand sums, collect powers of λ (Cauchy product formula).
- Re-expand in powers of ℓ and collect terms by phase.

- This results in

$$G_{\rm ret}^{\ell\geq 1} = \sum_{k=0}^{\infty}\left(\sum_{\ell=1}^{\infty}\frac{e^{\pm i\ell(\eta\pm\gamma)}}{\ell^k}\right)V_k^{(\pm,\pm)}(\eta,\gamma).$$

- Define

$$\mathcal{A}_k(x) = \sum_{\ell=1}^{\infty}\frac{e^{i\ell x}}{\ell^k}.$$

Then

$$\mathcal{A}_1(x) = \mathcal{D}(x) + i\mathcal{U}(x),$$

with

$$\begin{aligned}
\mathcal{D}(x) &= -\ln|x| - 2\sum_{n=0}^{\infty}\ln\left|1-\frac{x^2}{4n^2\pi^2}\right|, \\
\mathcal{U}(x) &= \frac{1}{2}(\pi - x) + \pi\sum_{n=1}^{\infty}[\theta(x-2n\pi)-\theta(-x-2n\pi)] - \pi\theta(-x)
\end{aligned}$$

.

- Notice that

$$\mathcal{A}_k(x) = \underbrace{\mathcal{A}_k(0)}_{=\zeta(k)} + i\int_0^x \mathcal{A}_{k-1}(y)dy, \quad k \geq 2,$$

and

$$\mathcal{A}_0(x) = -i\mathcal{A}_1'(x) = \sum \mathrm{PV} + \delta.$$

- Thus we have the regularity results

$$\mathcal{A}_k \in C^{k-2}(\mathbb{R}), \quad \mathcal{A}_k^{(k-1)} \in L^1_{\mathrm{loc}}(\mathbb{R}).$$

- The overall structure is

$$\begin{aligned} G_{\mathrm{ret}} \quad = \quad & \sum_{N=-\infty}^{\infty}\sum_{j=0}^{\infty}\{A_j(\eta)B_j(\gamma)\times \\ & \left[(\hat{\sigma}_N^{\mathrm{even}})^j \ln|\hat{\sigma}_N^{\mathrm{odd}}| + (\hat{\sigma}_N^{\mathrm{even}})^j\theta(\hat{\sigma}_N^{\mathrm{even}})\right] \\ & + \left[(\hat{\sigma}_N^{\mathrm{odd}})^j \mathrm{PV}\left(\frac{1}{\hat{\sigma}_N^{\mathrm{odd}}}\right) + (\hat{\sigma}_N^{\mathrm{odd}})^j \log(\hat{\sigma}_N^{\mathrm{odd}})\right]\Big\} \end{aligned}$$

Conclusions/To-Do List

- First identification of a “sum over Hadamard forms” for spacetimes with a 4-fold singularity structure (cf. Einstein static universe and Bertotti-Robinson: $G_{\mathrm{ret}} = \sum \delta + \theta$ is known; 2-fold singularity structure).
- Exact form for singular part of G_{ret} as data to support other approaches (quasi-local, spectral methods, matched expansions).
- Calculation of U_1: include this term in the ‘flat’ sum and the large$-\ell$ sum.
- Calculation of $\eta, U_0, U_k, k \geq 1$ using numerical PDE solvers in 1+1 dimensions.
- Calculate $\nabla_\alpha G_{\mathrm{ret}}$; carry out self-force calculations.