Lorenz gauge solution in the frequency domain: Constrained EHS method, low-order and static modes

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In collaboration with Erik Forseth, Charles Evans, and Seth Hopper

Eccentric orbits on Schwarzchild: Previous work (partial list)

- Akcay 2011
  Circular orbits, frequency domain
- Warburton, Akcay, Barack, Gair & Sago 2012
  Eccentric orbits, FD, EHS, application to inspiral evolution
- Capra 15 talks: Warburton; Evans, Osburn & Forseth
- Capra 16 talks: Warburton; Hopper; Forseth & Osburn (update)
- Hopper & Evans 2013, 2010
  Eccentric orbits, FD, RWZ gauge to Lorenz gauge
- Barack & Sago 2010
  Eccentric orbits in LG, TD radiative modes, FD low order modes
- Sago, Barack & Detweiler 2009
  Circular orbits, comparison between Lorenz and RW gauges
- Detweiler & Poisson 2003
  Circular orbits, low order modes in Lorenz gauge
- Zerilli 1970

Outline

- Constrained equations for radiative modes
  $(l \geq 2,\ \omega \neq 0)$
- Homogeneous solutions of constrained equations
- Particular solution of constrained equations
  (Extended homogeneous solutions)
- Static modes: constrained solution
  $(m = 0,\ n = 0 \Rightarrow \omega = 0)$
- Low-order modes: constrained solution
  $(l < 0, 1)$
- Calculation of the dissipative self-force and results

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Lorenz gauge overview

- Lorenz gauge perturbation equation:

$$\Box\bar{p}_{\mu\nu} + 2R_{\mu\alpha\nu\beta}\bar{p}^{\alpha\beta} = -16\pi T_{\mu\nu}$$

Lorenz gauge condition:

$$\nabla^{\beta}\bar{p}_{\alpha\beta} = 0$$

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- Spherical harmonic decomposition (Martel & Poisson 2005 notation)

$$p_{\alpha\beta} = \sum_{l,m} \left(\begin{array}{c|c} h^{lm}_{ab} Y^{lm} & j^{lm}_{a} Y^{lm}_{B} + h^{lm}_{a} X^{lm}_{B} \\ \hline * & r^2 \left(K^{lm} \Omega_{AB} Y^{lm} + G^{lm} Y^{lm}_{AB} \right) + h^{lm}_{2} X^{lm}_{AB} \end{array} \right)$$

	Odd Parity	Even Parity
Harmonics:	X^{lm}_{A}, X^{lm}_{AB}	$Y^{lm}, Y^{lm}_{A}, Y^{lm}_{AB}, \Omega_{AB} Y^{lm}$
Amplitudes:	$h_t,\ h_r,\ h_2$	$h_{tt},\ h_{tr},\ h_{rr},\ j_t,\ j_r,\ K,\ G$

Constrained odd-parity frequency-domain equations

- Three unconstrained odd-parity field equations and one Lorenz gauge condition ($l \geq 2$)

$$0 = f(l+2)(l-1)\tilde{h}_2 - 4f\,(r-M)\,\tilde{h}_r - 2fr^2\frac{d\tilde{h}_r}{dr_*} - 2i\omega r^2\tilde{h}_t,$$

$$f^2\tilde{P}^t = \frac{d^2\tilde{h}_t}{dr_*^2} - \frac{2M}{r^2}\frac{d\tilde{h}_t}{dr_*} + \left[\omega^2 - \frac{f}{r^2}\left(l(l+1) - \frac{4M}{r}\right)\right]\tilde{h}_t - \frac{2ifM\omega}{r^2}\tilde{h}_r,$$

$$-\tilde{P}^r = \frac{d^2\tilde{h}_r}{dr_*^2} + \frac{2M}{r^2}\frac{d\tilde{h}_r}{dr_*} + \left[\omega^2 - \frac{f}{r^2}\left(l(l+1)+4f\right)\right]\tilde{h}_r - \frac{2iM\omega}{fr^2}\tilde{h}_t + \frac{f(l+2)(l-1)}{r^3}\tilde{h}_2,$$

$$-2f\tilde{P} = \frac{d^2\tilde{h}_2}{dr_*^2} - \frac{2f}{r}\frac{d\tilde{h}_2}{dr_*} + \left[\omega^2 - \frac{f}{r^2}\left(l(l+1)-4f\right)\right]\tilde{h}_2 + \frac{4f^2}{r}\tilde{h}_r.$$

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$$-\tilde{P}^r = \frac{d^2\tilde{h}_r}{dr_*^2} + \frac{2M}{r^2}\frac{d\tilde{h}_r}{dr_*} + \left[\omega^2 - \frac{f}{r^2}\left(l(l+1) + 4f\right)\right]\tilde{h}_r - \frac{2iM\omega}{fr^2}\tilde{h}_t + \frac{f(l+2)(l-1)}{r^3}\tilde{h}_2,$$

$$-2f\tilde{P} = \frac{d^2\tilde{h}_2}{dr_*^2} - \frac{2f}{r}\frac{d\tilde{h}_2}{dr_*} + \left[\omega^2 - \frac{f}{r^2}\left(l(l+1) - 4f\right)\right]\tilde{h}_2 + \frac{4f^2}{r}\tilde{h}_r.$$

- Solve the Lorenz gauge condition algebraically for $\tilde{h}_2$

$$\tilde{h}_2 = \frac{1}{(l+2)(l-1)}\left[4(r-M)\tilde{h}_r + 2r^2\frac{d\tilde{h}_r}{dr_*} + \frac{2i\omega r^2}{f}\tilde{h}_t\right],$$

Constrained odd-parity frequency-domain equations

- Three unconstrained odd-parity field equations and one Lorenz gauge condition ($l \geq 2$)

$$0 = f(l+2)(l-1)\tilde{h}_2 - 4f\,(r-M)\,\tilde{h}_r - 2fr^2\frac{d\tilde{h}_r}{dr_*} - 2i\omega r^2\tilde{h}_t,$$

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$$-2f\tilde{P} = \frac{d^2\tilde{h}_2}{dr_*^2} - \frac{2f}{r}\frac{d\tilde{h}_2}{dr_*} + \left[\omega^2 - \frac{f}{r^2}\left(l(l+1)-4f\right)\right]\tilde{h}_2 + \frac{4f^2}{r}\tilde{h}_r.$$

- Solve the Lorenz gauge condition algebraically for $\tilde{h}_2$

$$\tilde{h}_2 = \frac{1}{(l+2)(l-1)}\left[4(r-M)\tilde{h}_r + 2r^2\frac{d\tilde{h}_r}{dr_*} + \frac{2i\omega r^2}{f}\tilde{h}_t\right],$$

- Decouple $\tilde{h}_2$ from the field equations, which reduces the system to fourth order

$$f^2\tilde{P}^t = \frac{d^2\tilde{h}_t}{dr_*^2} - \frac{2M}{r^2}\frac{d\tilde{h}_t}{dr_*} + \left[\omega^2 - \frac{f}{r^2}\left(l(l+1) - \frac{4M}{r}\right)\right]\tilde{h}_t - \frac{2ifM\omega}{r^2}\tilde{h}_r,$$

$$-\tilde{P}^r = \frac{d^2\tilde{h}_r}{dr_*^2} + \frac{2(r-M)}{r^2}\frac{d\tilde{h}_r}{dr_*} + \left[\omega^2 - \frac{f}{r^2}\left(l(l+1) - \frac{4M}{r}\right)\right]\tilde{h}_r + \frac{2i\omega(r-3M)}{fr^2}\tilde{h}_t.$$

Even-parity frequency-domain equations

$$\tilde{j}_t = \frac{1}{l(l+1)}\left[\frac{i\omega f r^2}{2}\tilde{h}_{rr} + 2(r-M)\tilde{h}_{tr} + \frac{i\omega r^2}{2f}\tilde{h}_{tt} + i\omega r^2\tilde{K} + r^2\frac{d\tilde{h}_{tr}}{dr_*}\right],$$

$$\tilde{j}_r = \frac{1}{l(l+1)}\left[2(r-M)\tilde{h}_{rr} + \frac{i\omega r^2}{f}\tilde{h}_{tr} - 2r\tilde{K} + \frac{r^2}{2}\frac{d\tilde{h}_{rr}}{dr_*} + \frac{r^2}{2f^2}\frac{d\tilde{h}_{tt}}{dr_*} - \frac{r^2}{f}\frac{d\tilde{K}}{dr_*}\right],$$

$$\tilde{G} = \frac{1}{(l+2)(l-1)}\left[\frac{1}{f}\tilde{h}_{tt} - f\tilde{h}_{rr} + \frac{2i\omega}{f}\tilde{j}_t + \frac{4(r-M)}{r^2}\tilde{j}_r + 2\frac{d\tilde{j}_r}{dr_*}\right],$$

$$-f\tilde{Q}^{rr} - f^2\tilde{Q}^\flat - f^3\tilde{Q}^{tt} = \frac{d^2\tilde{h}_{tt}}{dr_*^2} + \frac{2(r-4M)}{r^2}\frac{d\tilde{h}_{tt}}{dr_*} + \left[\omega^2 + \frac{2M^2}{r^4} - \frac{f}{r^2}l(l+1)\right]\tilde{h}_{tt} + \frac{2Mf^2(3M-2r)}{r^4}\tilde{h}_{rr} - \frac{4iM\omega f}{r^2}\tilde{h}_{tr} + \frac{4Mf^2}{r^3}\tilde{K},$$

$$2f\tilde{Q}^{tr} = \frac{d^2\tilde{h}_{tr}}{dr_*^2} + \frac{4f}{r}\frac{d\tilde{h}_{tr}}{dr_*} + \left[\omega^2 + \frac{2(2M^2-r^2)}{r^4} - \frac{f}{r^2}\left(l(l+1)-4\right)\right]\tilde{h}_{tr} + \frac{i\omega(r-4M)}{fr^2}\tilde{h}_{tt} + \frac{i\omega f(r-4M)}{r^2}\tilde{h}_{rr} + \frac{2i\omega f}{r}\tilde{K},$$

$$-\frac{1}{f}\tilde{Q}^{rr} + \tilde{Q}^\flat - f\tilde{Q}^{tt} = \frac{d^2\tilde{h}_{rr}}{dr_*^2} + \frac{4(r-M)}{r^2}\frac{d\tilde{h}_{rr}}{dr_*} + \frac{2}{fr}\frac{d\tilde{h}_{tt}}{dr_*} - \frac{4}{r}\frac{d\tilde{K}}{dr_*} + \left[\omega^2 + \frac{2M^2}{r^4} - \frac{f}{r^2}\left(l(l+1)-4\right)\right]\tilde{h}_{rr} + \frac{2M(3M-2r)}{f^2r^4}\tilde{h}_{tt} + \frac{4i\omega(r-3M)}{fr^2}\tilde{h}_{tr} - \frac{4(r-M)}{r^3}\tilde{K},$$

$$-f^2\tilde{Q}^{tt} + \tilde{Q}^{rr} = \frac{d^2\tilde{K}}{dr_*^2} + \frac{4f}{r}\frac{d\tilde{K}}{dr_*} - \frac{1}{r}\frac{d\tilde{h}_{tt}}{dr_*} - \frac{f^2}{r}\frac{d\tilde{h}_{rr}}{dr_*} + \left[\omega^2 - \frac{f}{r^2}\left(l(l+1)-2\right)\right]\tilde{K} + \frac{2M}{r^3}\tilde{h}_{tt} - \frac{2i\omega f}{r}\tilde{h}_{tr} - \frac{2f^2(r+M)}{r^3}\tilde{h}_{rr},$$

Even-parity frequency-domain equations

- Seven even-parity unconstrained equations, three LG conditions ($l \geq 2$)
-
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$$-f\tilde{Q}^{rr} - f^2\tilde{Q}^\flat - f^3\tilde{Q}^{tt} = \frac{d^2\tilde{h}_{tt}}{dr_*^2} + \frac{2(r-4M)}{r^2}\frac{d\tilde{h}_{tt}}{dr_*} + \left[\omega^2 + \frac{2M^2}{r^4} - \frac{f}{r^2}l(l+1)\right]\tilde{h}_{tt} + \frac{2Mf^2(3M-2r)}{r^4}\tilde{h}_{rr} - \frac{4iM\omega f}{r^2}\tilde{h}_{tr} + \frac{4Mf^2}{r^3}\tilde{K},$$

$$2f\tilde{Q}^{tr} = \frac{d^2\tilde{h}_{tr}}{dr_*^2} + \frac{4f}{r}\frac{d\tilde{h}_{tr}}{dr_*} + \left[\omega^2 + \frac{2(2M^2-r^2)}{r^4} - \frac{f}{r^2}\left(l(l+1)-4\right)\right]\tilde{h}_{tr} + \frac{i\omega(r-4M)}{fr^2}\tilde{h}_{tt} + \frac{i\omega f(r-4M)}{r^2}\tilde{h}_{rr} + \frac{2i\omega f}{r}\tilde{K},$$

$$-\frac{1}{f}\tilde{Q}^{rr} + \tilde{Q}^\flat - f\tilde{Q}^{tt} = \frac{d^2\tilde{h}_{rr}}{dr_*^2} + \frac{4(r-M)}{r^2}\frac{d\tilde{h}_{rr}}{dr_*} + \frac{2}{fr}\frac{d\tilde{h}_{tt}}{dr_*} - \frac{4}{r}\frac{d\tilde{K}}{dr_*} + \left[\omega^2 + \frac{2M^2}{r^4} - \frac{f}{r^2}\left(l(l+1)-4\right)\right]\tilde{h}_{rr} + \frac{2M(3M-2r)}{f^2r^4}\tilde{h}_{tt} + \frac{4i\omega(r-3M)}{fr^2}\tilde{h}_{tr} - \frac{4(r-M)}{r^3}\tilde{K},$$

$$-f^2\tilde{Q}^{tt} + \tilde{Q}^{rr} = \frac{d^2\tilde{K}}{dr_*^2} + \frac{4f}{r}\frac{d\tilde{K}}{dr_*} - \frac{1}{r}\frac{d\tilde{h}_{tt}}{dr_*} - \frac{f^2}{r}\frac{d\tilde{h}_{rr}}{dr_*} + \left[\omega^2 - \frac{f}{r^2}\left(l(l+1)-2\right)\right]\tilde{K} + \frac{2M}{r^3}\tilde{h}_{tt} - \frac{2i\omega f}{r}\tilde{h}_{tr} - \frac{2f^2(r+M)}{r^3}\tilde{h}_{rr},$$

Even-parity frequency-domain equations

- Seven even-parity unconstrained equations, three LG conditions ($l \geq 2$)
- Use gauge conditions to reduce the order of the system
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$$-f\tilde{Q}^{rr} - f^2\tilde{Q}^{\flat} - f^3\tilde{Q}^{tt} = \frac{d^2\tilde{h}_{tt}}{dr_*^2} + \frac{2(r-4M)}{r^2}\frac{d\tilde{h}_{tt}}{dr_*} + \left[\omega^2 + \frac{2M^2}{r^4} - \frac{f}{r^2}l(l+1)\right]\tilde{h}_{tt} + \frac{2Mf^2(3M-2r)}{r^4}\tilde{h}_{rr} - \frac{4iM\omega f}{r^2}\tilde{h}_{tr} + \frac{4Mf^2}{r^3}\tilde{K},$$

$$2f\tilde{Q}^{tr} = \frac{d^2\tilde{h}_{tr}}{dr_*^2} + \frac{4f}{r}\frac{d\tilde{h}_{tr}}{dr_*} + \left[\omega^2 + \frac{2(2M^2-r^2)}{r^4} - \frac{f}{r^2}\left(l(l+1)-4\right)\right]\tilde{h}_{tr} + \frac{i\omega(r-4M)}{fr^2}\tilde{h}_{tt} + \frac{i\omega f(r-4M)}{r^2}\tilde{h}_{rr} + \frac{2i\omega f}{r}\tilde{K},$$

$$-\frac{1}{f}\tilde{Q}^{rr} + \tilde{Q}^{\flat} - f\tilde{Q}^{tt} = \frac{d^2\tilde{h}_{rr}}{dr_*^2} + \frac{4(r-M)}{r^2}\frac{d\tilde{h}_{rr}}{dr_*} + \frac{2}{fr}\frac{d\tilde{h}_{tt}}{dr_*} - \frac{4}{r}\frac{d\tilde{K}}{dr_*} + \left[\omega^2 + \frac{2M^2}{r^4} - \frac{f}{r^2}\left(l(l+1)-4\right)\right]\tilde{h}_{rr} + \frac{2M(3M-2r)}{f^2r^4}\tilde{h}_{tt} + \frac{4i\omega(r-3M)}{fr^2}\tilde{h}_{tr} - \frac{4(r-M)}{r^3}\tilde{K},$$

$$-f^2\tilde{Q}^{tt} + \tilde{Q}^{rr} = \frac{d^2\tilde{K}}{dr_*^2} + \frac{4f}{r}\frac{d\tilde{K}}{dr_*} - \frac{1}{r}\frac{d\tilde{h}_{tt}}{dr_*} - \frac{f^2}{r}\frac{d\tilde{h}_{rr}}{dr_*} + \left[\omega^2 - \frac{f}{r^2}\left(l(l+1)-2\right)\right]\tilde{K} + \frac{2M}{r^3}\tilde{h}_{tt} - \frac{2i\omega f}{r}\tilde{h}_{tr} - \frac{2f^2(r+M)}{r^3}\tilde{h}_{rr},$$

Even-parity frequency-domain equations

- Seven even-parity unconstrained equations, three LG conditions ($l \geq 2$)
- Use gauge conditions to reduce the order of the system
- Four constrained second-order equations

$$-f\tilde{Q}^{rr} - f^2\tilde{Q}^\flat - f^3\tilde{Q}^{tt} = \frac{d^2\tilde{h}_{tt}}{dr_*^2} + \frac{2(r-4M)}{r^2}\frac{d\tilde{h}_{tt}}{dr_*} + \left[\omega^2 + \frac{2M^2}{r^4} - \frac{f}{r^2}l(l+1)\right]\tilde{h}_{tt} + \frac{2Mf^2(3M-2r)}{r^4}\tilde{h}_{rr} - \frac{4iM\omega f}{r^2}\tilde{h}_{tr} + \frac{4Mf^2}{r^3}\tilde{K},$$

$$2f\tilde{Q}^{tr} = \frac{d^2\tilde{h}_{tr}}{dr_*^2} + \frac{4f}{r}\frac{d\tilde{h}_{tr}}{dr_*} + \left[\omega^2 + \frac{2(2M^2-r^2)}{r^4} - \frac{f}{r^2}\left(l(l+1)-4\right)\right]\tilde{h}_{tr} + \frac{i\omega(r-4M)}{fr^2}\tilde{h}_{tt} + \frac{i\omega f(r-4M)}{r^2}\tilde{h}_{rr} + \frac{2i\omega f}{r}\tilde{K},$$

$$-\frac{1}{f}\tilde{Q}^{rr} + \tilde{Q}^\flat - f\tilde{Q}^{tt} = \frac{d^2\tilde{h}_{rr}}{dr_*^2} + \frac{4(r-M)}{r^2}\frac{d\tilde{h}_{rr}}{dr_*} + \frac{2}{fr}\frac{d\tilde{h}_{tt}}{dr_*} - \frac{4}{r}\frac{d\tilde{K}}{dr_*} + \left[\omega^2 + \frac{2M^2}{r^4} - \frac{f}{r^2}\left(l(l+1)-4\right)\right]\tilde{h}_{rr} + \frac{2M(3M-2r)}{f^2r^4}\tilde{h}_{tt} + \frac{4i\omega(r-3M)}{fr^2}\tilde{h}_{tr} - \frac{4(r-M)}{r^3}\tilde{K},$$

$$-f^2\tilde{Q}^{tt} + \tilde{Q}^{rr} = \frac{d^2\tilde{K}}{dr_*^2} + \frac{4f}{r}\frac{d\tilde{K}}{dr_*} - \frac{1}{r}\frac{d\tilde{h}_{tt}}{dr_*} - \frac{f^2}{r}\frac{d\tilde{h}_{rr}}{dr_*} + \left[\omega^2 - \frac{f}{r^2}\left(l(l+1)-2\right)\right]\tilde{K} + \frac{2M}{r^3}\tilde{h}_{tt} - \frac{2i\omega f}{r}\tilde{h}_{tr} - \frac{2f^2(r+M)}{r^3}\tilde{h}_{rr},$$

Even-parity frequency-domain equations

$$\tilde{j}_t = \frac{1}{l(l+1)}\left[\frac{i\omega f r^2}{2}\tilde{h}_{rr} + 2(r-M)\tilde{h}_{tr} + \frac{i\omega r^2}{2f}\tilde{h}_{tt} + i\omega r^2\tilde{K} + r^2\frac{d\tilde{h}_{tr}}{dr_*}\right],$$

$$\tilde{j}_r = \frac{1}{l(l+1)}\left[2(r-M)\tilde{h}_{rr} + \frac{i\omega r^2}{f}\tilde{h}_{tr} - 2r\tilde{K} + \frac{r^2}{2}\frac{d\tilde{h}_{rr}}{dr_*} + \frac{r^2}{2f^2}\frac{d\tilde{h}_{tt}}{dr_*} - \frac{r^2}{f}\frac{d\tilde{K}}{dr_*}\right],$$

$$\tilde{G} = \frac{1}{(l+2)(l-1)}\left[\frac{1}{f}\tilde{h}_{tt} - f\tilde{h}_{rr} + \frac{2i\omega}{f}\tilde{j}_t + \frac{4(r-M)}{r^2}\tilde{j}_r + 2\frac{d\tilde{j}_r}{dr_*}\right],$$

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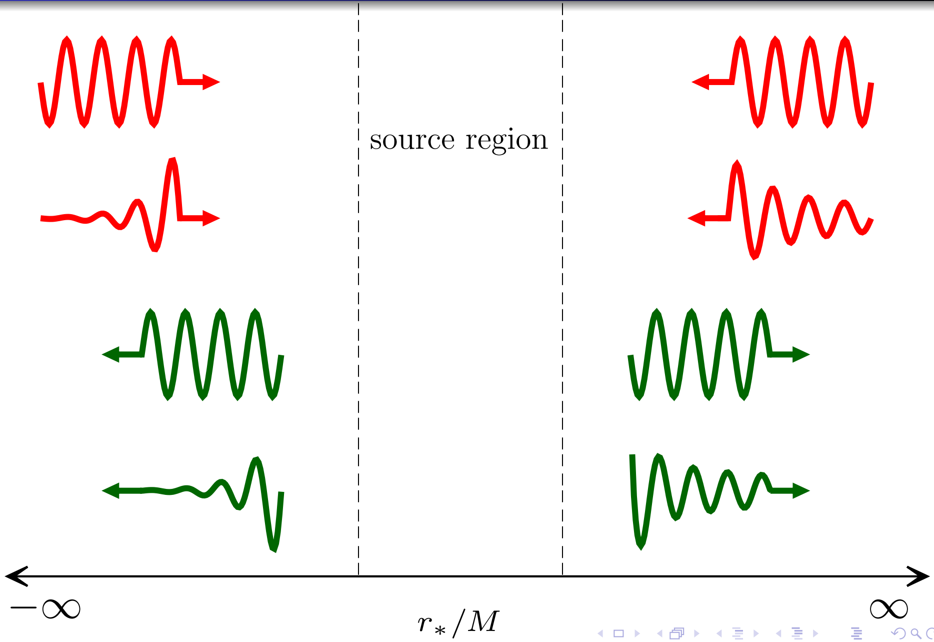
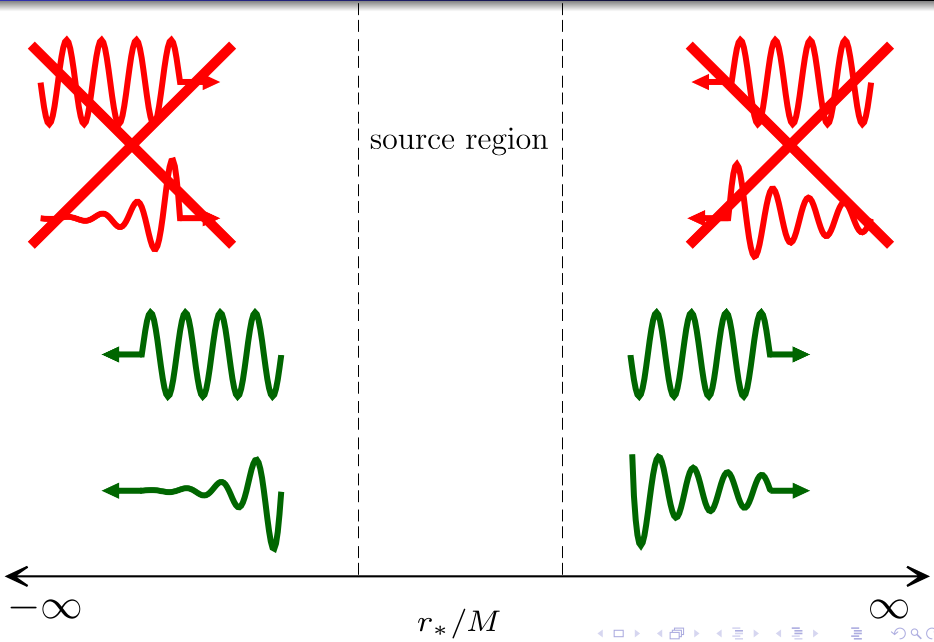
Causality of odd-parity constrained solutions



Causality of odd-parity constrained solutions



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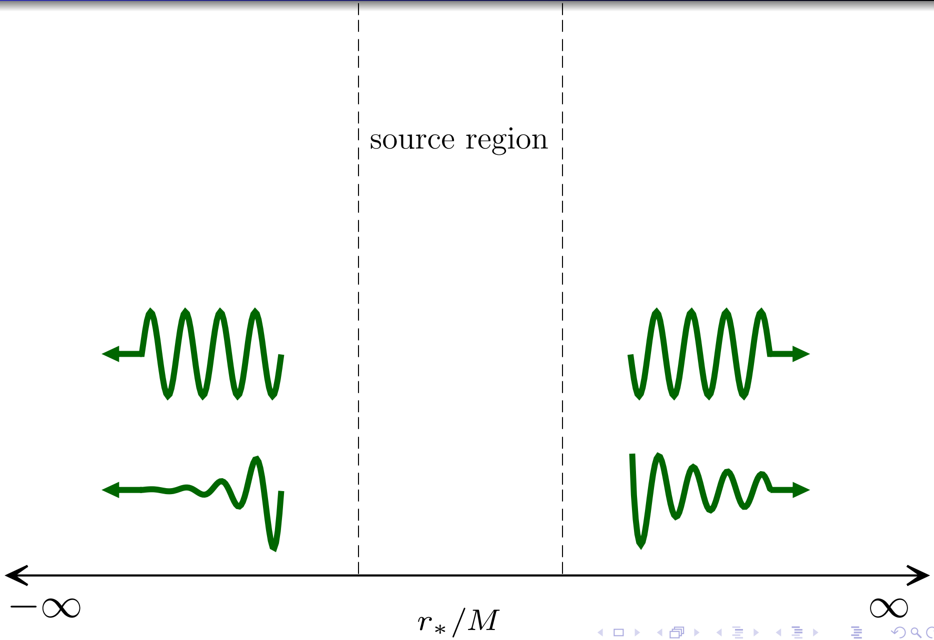

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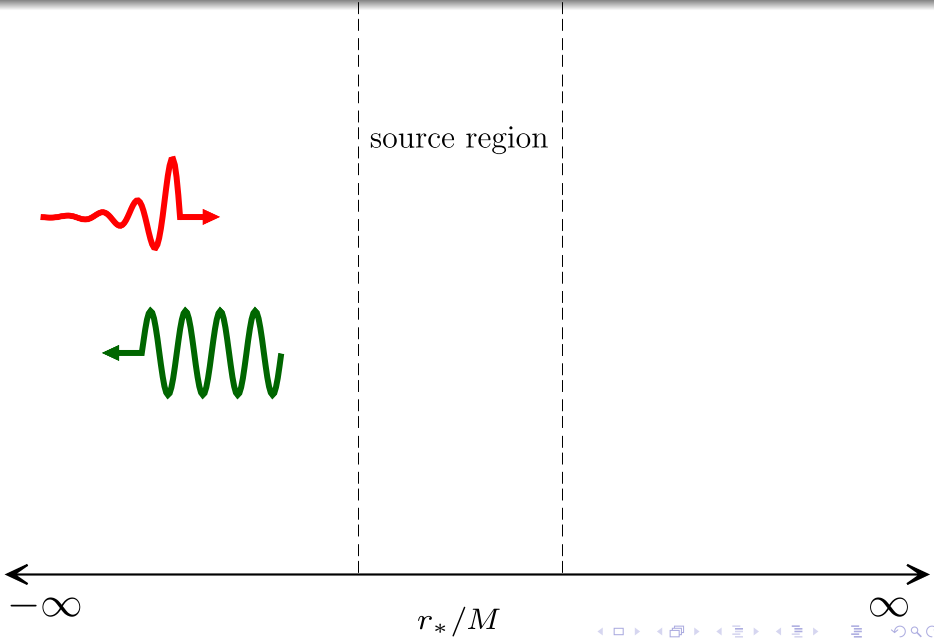
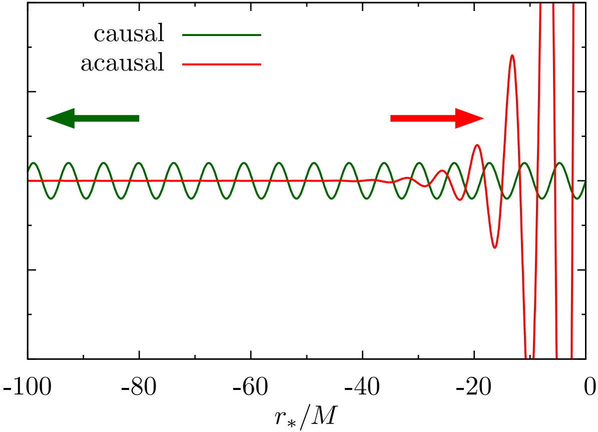


Near-horizon acausal growth

- One of the acausal homogeneous solutions we wish to avoid grows exponentially in the direction of integration.
- Roundoff error excites this unwanted solution.



downgoing

$$\begin{pmatrix} \tilde{h}_t \\ \tilde{h}_r \end{pmatrix}_0^- \sim \begin{pmatrix} 1 \\ 1/f \end{pmatrix} e^{-i\omega r_*}$$

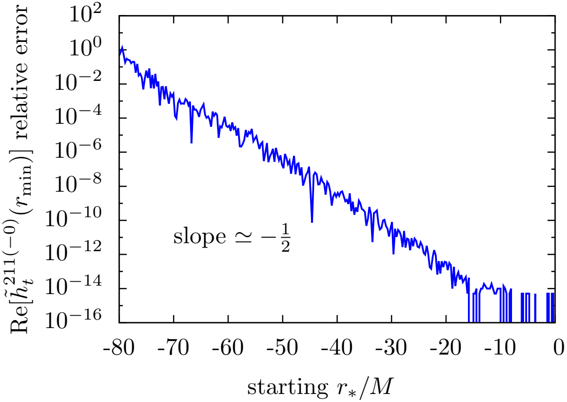
upgoing

$$\begin{pmatrix} \tilde{h}_t \\ \tilde{h}_r \end{pmatrix}^{(\mathrm{up})} \sim \begin{pmatrix} 1 \\ 1/f \end{pmatrix} e^{+i\omega r_*} e^{r_*/(2M)}$$

Exponential growth

Solution of near-horizon acausal growth problem

- Give initial conditions well away from horizon to avoid exponential growth
- The causal solution can still be accurately calculated in this region with Taylor series



$e = 0.764124$

$p = 8.75455$

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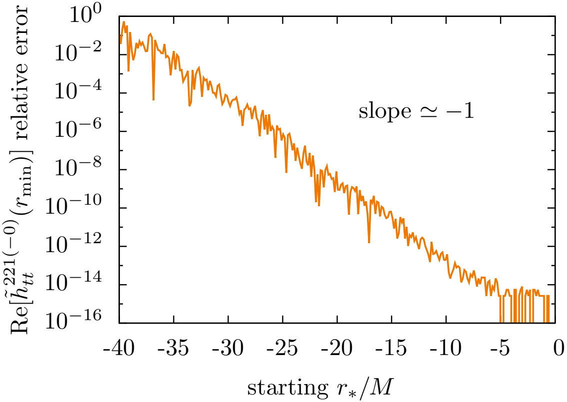
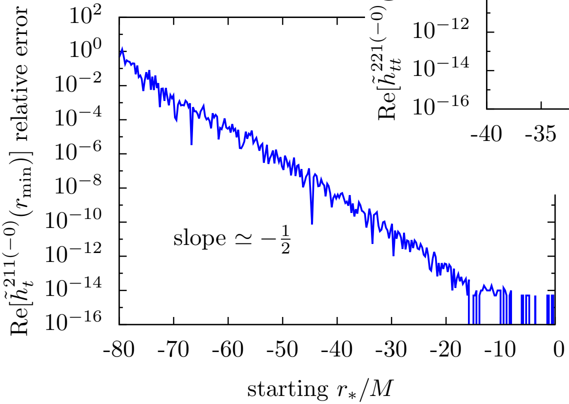
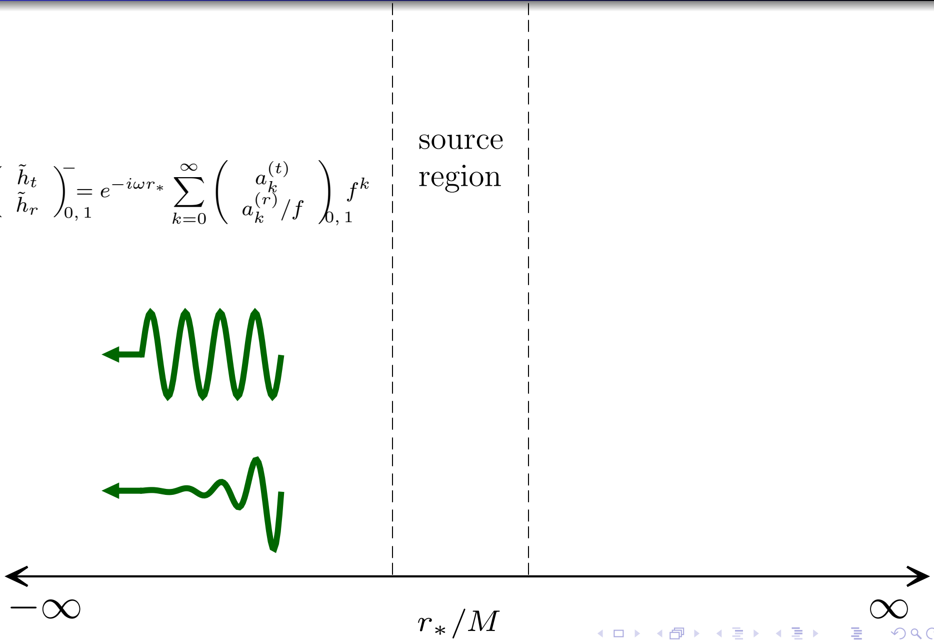




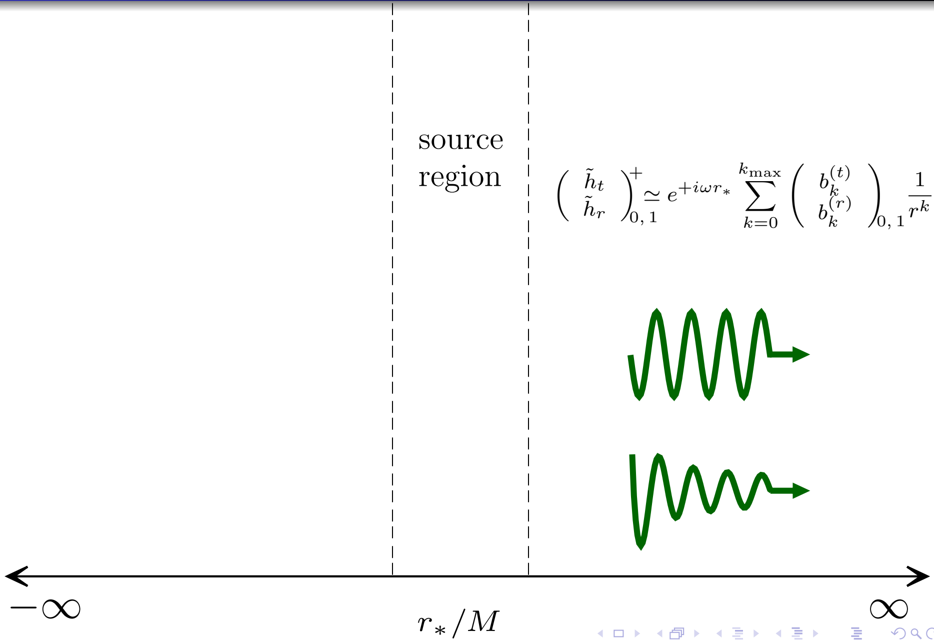
$$e = 0.764124$$
$$p = 8.75455$$

Causality of odd-parity constrained solutions

$$\begin{pmatrix} \tilde{h}_t \\ \tilde{h}_r \end{pmatrix}^{-}_{0,\,1} = e^{-i\omega r_*} \sum_{k=0}^{\infty} \begin{pmatrix} a_k^{(t)} \\ a_k^{(r)}/f \end{pmatrix}_{0,\,1} f^k$$



Causality of odd-parity constrained solutions



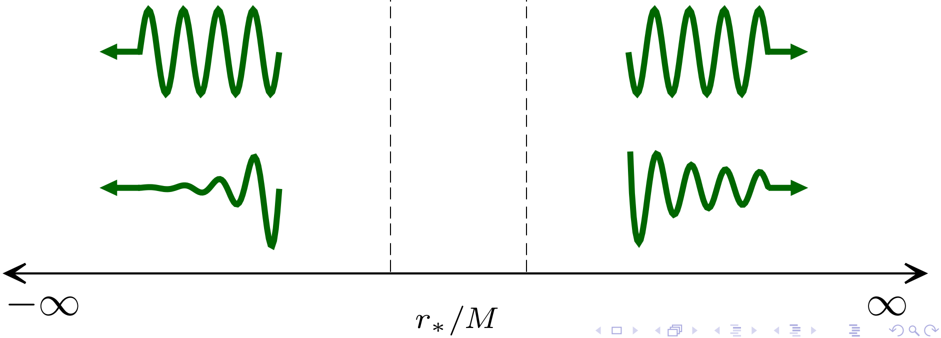
$$\begin{pmatrix} \tilde{h}_t \\ \tilde{h}_r \end{pmatrix}^{+}_{0,\,1} \simeq e^{+i\omega r_*} \sum_{k=0}^{k_{\max}} \begin{pmatrix} b_k^{(t)} \\ b_k^{(r)} \end{pmatrix}_{0,\,1} \frac{1}{r^k}$$

Causality of odd-parity constrained solutions

$$\begin{pmatrix}\tilde{h}_t \\ \tilde{h}_r\end{pmatrix}^{-}_{0,\,1} = e^{-i\omega r_*}\sum_{k=0}^{\infty}\begin{pmatrix} a_k^{(t)} \\ a_k^{(r)}/f\end{pmatrix}_{0,\,1} f^k$$

source region

$$\begin{pmatrix}\tilde{h}_t \\ \tilde{h}_r\end{pmatrix}^{+}_{0,\,1} \simeq e^{+i\omega r_*}\sum_{k=0}^{k_{\max}}\begin{pmatrix} b_k^{(t)} \\ b_k^{(r)}\end{pmatrix}_{0,\,1} \frac{1}{r^k}$$



Outline

- Constrained equations for radiative modes
  ($l \geq 2$, $\omega \neq 0$)
- Homogeneous solutions of constrained equations
- Particular solution of constrained equations
  (Extended homogeneous solutions)
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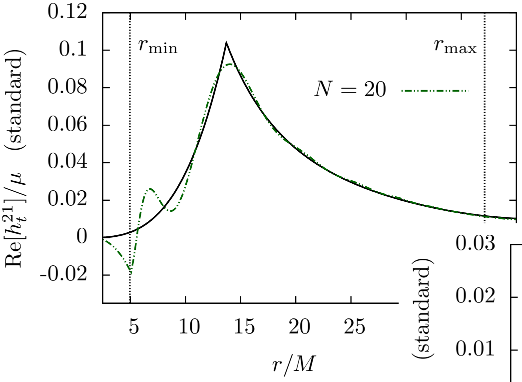
Constrained system solution via variation of parameters



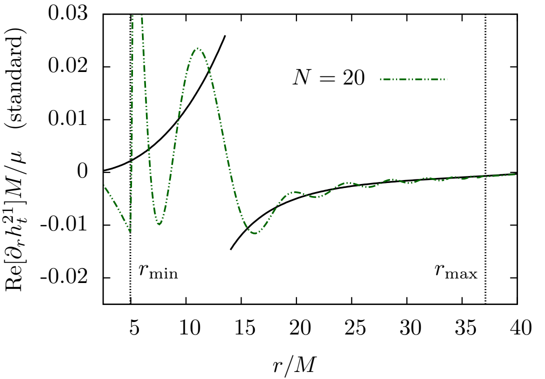
$$\omega_{mn} = n\Omega_r + m\Omega_\phi,$$

$$h_t^{lm}(t,r) = \lim_{N\to\infty} \sum_{n=-N}^{N} \tilde{h}_t^{lmn}(r) e^{-i\omega_{mn} t}$$

$e = 0.764124$

$p = 8.75455$

$t = 80M$



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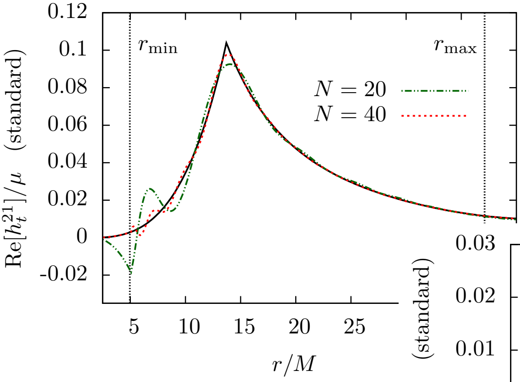

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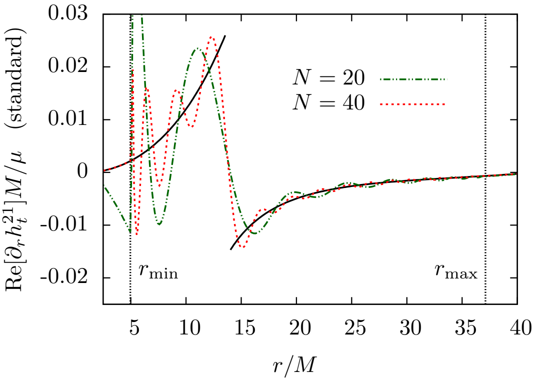

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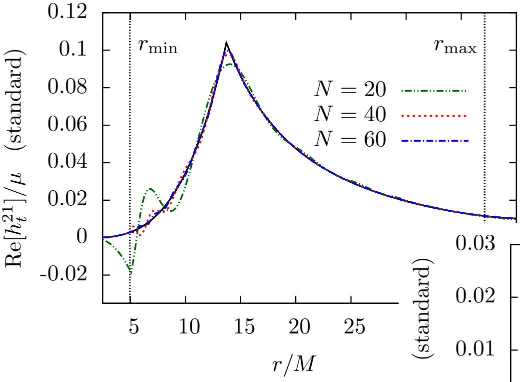

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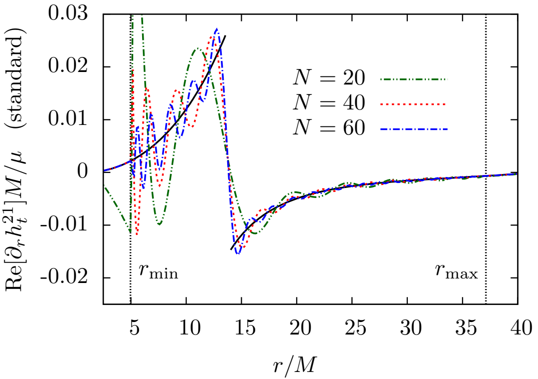
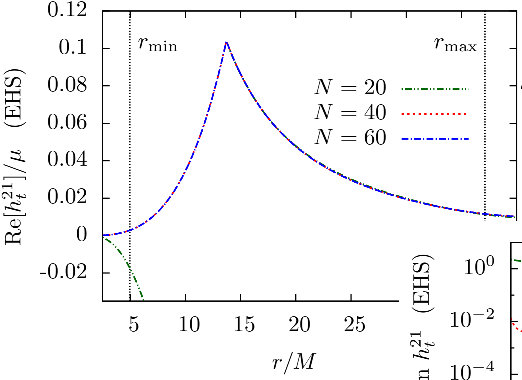


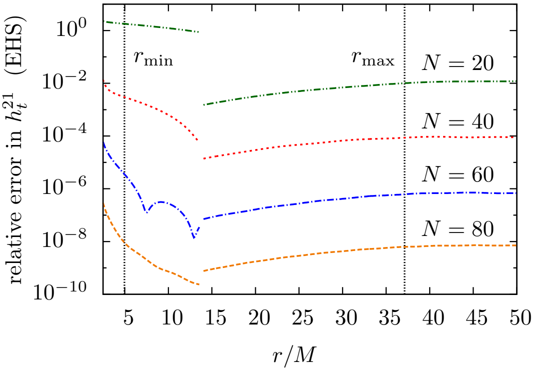
Extended homogeneous solutions for a system



$$z \equiv r - r_p(t),$$

$$h_t(t,r) = h_t^+(t,r)\theta(z) + h_t^-(t,r)\theta(-z)$$

$$h_t^{\pm}(t,r) = \lim_{N\to\infty} \sum_{n=-N}^{N} \tilde{h}_t^{n\pm}(r) e^{-i\omega_n t}$$

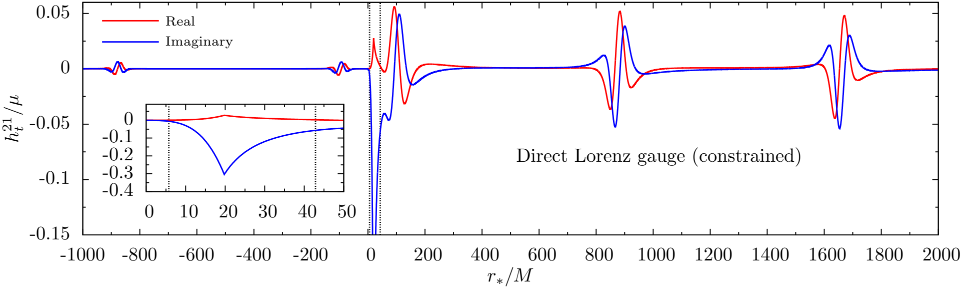


$e = 0.764124$

$p = 8.75455$

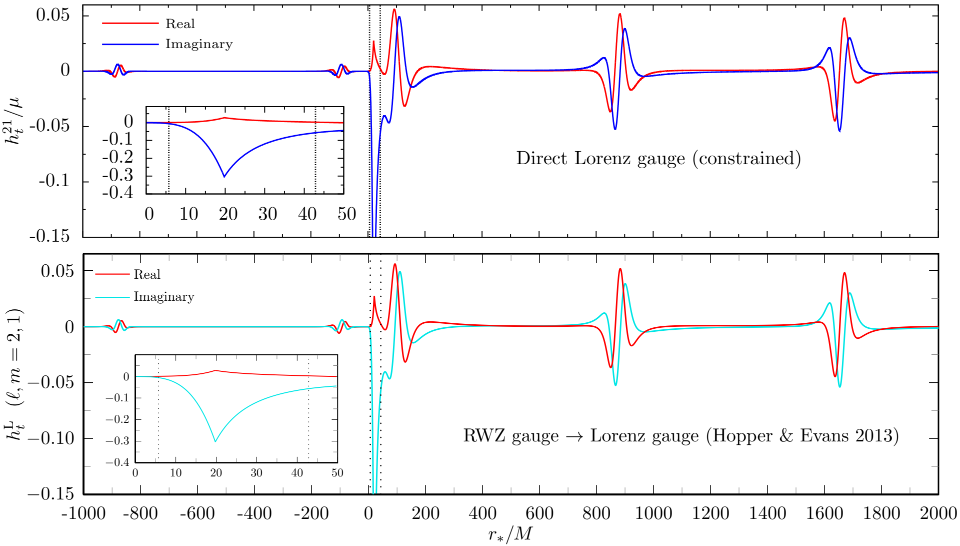
$t = 80M$

Time domain solution, comparison of methods



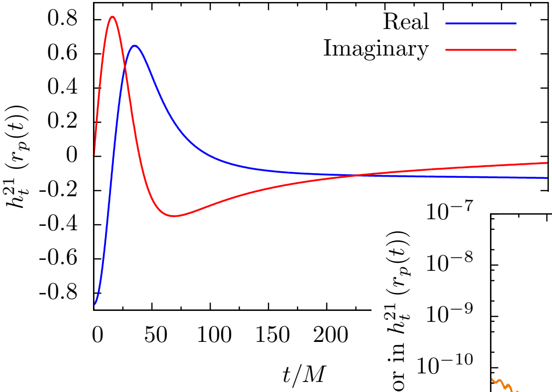
$e = 0.764124,$ $p = 8.75455,$ $t = 93.58M$

Time domain solution, comparison of methods

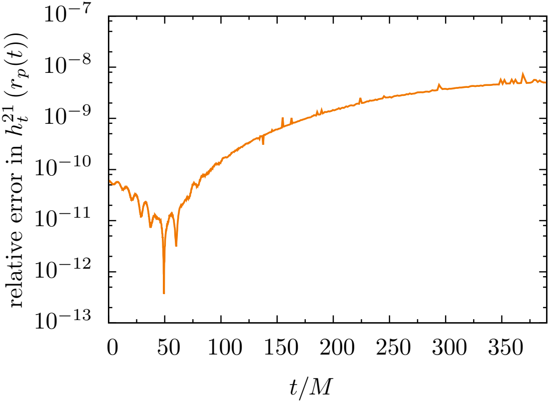


$e = 0.764124, \qquad p = 8.75455, \qquad t = 93.58M$

Comparison of methods: Relative error



We can compare a gauge transformation from RWG to Lorenz gauge with the constrained direct Lorenz gauge result at $r_p(t)$



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Odd-parity static modes ($m = 0$, $n = 0 \implies \omega = 0$)

Zero-frequency form of odd-parity constrained equations

$$\tilde{h}_2 = \frac{1}{(l+2)(l-1)}\left[4(r-M)\tilde{h}_r + 2r^2\frac{d\tilde{h}_r}{dr_*}\right],$$

$$-\tilde{P}^r = 0 = \frac{d^2\tilde{h}_r}{dr_*^2} + \frac{2(r-M)}{r^2}\frac{d\tilde{h}_r}{dr_*} - \frac{f}{r^2}\left(l(l+1) - \frac{4M}{r}\right)\tilde{h}_r,$$

$$f^2\tilde{P}^t = \frac{d^2\tilde{h}_t}{dr_*^2} - \frac{2M}{r^2}\frac{d\tilde{h}_t}{dr_*} - \frac{f}{r^2}\left(l(l+1) - \frac{4M}{r}\right)\tilde{h}_t,$$

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$$\tilde{h}_t^{(+0)} \simeq \frac{1}{r^l} + \mathcal{O}(\frac{1}{r^{l+1}}),$$
$$\tilde{h}_t^{(+1)} \simeq r^{l+1} + \mathcal{O}(r^l),$$

- Causality no longer dictates choice of homogeneous solutions
-

$$\tilde{h}_t^{(-0)} \simeq f + \frac{l(l+1)}{2} f^2 + \mathcal{O}(f^3),$$
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Even-parity static modes ($m = 0$, $n = 0 \implies \omega = 0$)

Zero-frequency form of even-parity constrained equations

$$\tilde{j}_t = \frac{1}{l(l+1)}\left[2(r-M)\tilde{h}_{tr} + r^2\frac{d\tilde{h}_{tr}}{dr_*}\right],$$

$$\tilde{j}_r = \frac{1}{l(l+1)}\left[2(r-M)\tilde{h}_{rr} - 2r\tilde{K} + \frac{r^2}{2}\frac{d\tilde{h}_{rr}}{dr_*} + \frac{r^2}{2f^2}\frac{d\tilde{h}_{tt}}{dr_*} - \frac{r^2}{f}\frac{d\tilde{K}}{dr_*}\right],$$

$$\tilde{G} = \frac{1}{(l+2)(l-1)}\left[\frac{1}{f}\tilde{h}_{tt} - f\tilde{h}_{rr} + \frac{4(r-M)}{r^2}\tilde{j}_r + 2\frac{d\tilde{j}_r}{dr_*}\right],$$

$$2f\tilde{Q}^{tr} = 0 = \frac{d^2\tilde{h}_{tr}}{dr_*^2} + \frac{4f}{r}\frac{d\tilde{h}_{tr}}{dr_*} + \left(\frac{2(2M^2-r^2)}{r^4} - \frac{f}{r^2}\left(l(l+1)-4\right)\right)\tilde{h}_{tr},$$

$$f\tilde{Q}^{rr} + f^2\tilde{Q}^{\flat} + f^3\tilde{Q}^{tt} = \frac{d^2\tilde{h}_{tt}}{dr_*^2} + \frac{2(r-4M)}{r^2}\frac{d\tilde{h}_{tt}}{dr_*} + \left(\frac{2M^2}{r^4} - \frac{f}{r^2}l(l+1)\right)\tilde{h}_{tt} + \frac{2Mf^2(3M-2r)}{r^4}\tilde{h}_{rr} + \frac{4Mf^2}{r^3}\tilde{K},$$

$$\frac{1}{f}\tilde{Q}^{rr} - \tilde{Q}^{\flat} + f\tilde{Q}^{tt} = \frac{d^2\tilde{h}_{rr}}{dr_*^2} + \frac{4(r-M)}{r^2}\frac{d\tilde{h}_{rr}}{dr_*} + \frac{2}{fr}\frac{d\tilde{h}_{tt}}{dr_*} - \frac{4}{r}\frac{d\tilde{K}}{dr_*},$$

$$+\left(\frac{2M^2}{r^4} - \frac{f}{r^2}\left(l(l+1)-4\right)\right)\tilde{h}_{rr} + \frac{2M(3M-2r)}{f^2r^4}\tilde{h}_{tt}, - \frac{4(r-M)}{r^3}\tilde{K}$$

$$f^2\tilde{Q}^{tt} - \tilde{Q}^{rr} = \frac{d^2\tilde{K}}{dr_*^2} + \frac{4f}{r}\frac{d\tilde{K}}{dr_*} - \frac{1}{r}\frac{d\tilde{h}_{tt}}{dr_*} - \frac{f^2}{r}\frac{d\tilde{h}_{rr}}{dr_*} - \frac{f}{r^2}\left(l(l+1)-2\right)\tilde{K} + \frac{2M}{r^3}\tilde{h}_{tt} - \frac{2f^2(r+M)}{r^3}\tilde{h}_{rr},$$

Even-parity static modes ($m = 0$, $n = 0 \implies \omega = 0$)

Zero-frequency form of even-parity constrained equations

$$\tilde{j}_t = \frac{1}{l(l+1)}\left[2(r-M)\tilde{h}_{tr} + r^2\frac{d\tilde{h}_{tr}}{dr_*}\right],$$

$$\tilde{j}_r = \frac{1}{l(l+1)}\left[2(r-M)\tilde{h}_{rr} - 2r\tilde{K} + \frac{r^2}{2}\frac{d\tilde{h}_{rr}}{dr_*} + \frac{r^2}{2f^2}\frac{d\tilde{h}_{tt}}{dr_*} - \frac{r^2}{f}\frac{d\tilde{K}}{dr_*}\right],$$

$$\tilde{G} = \frac{1}{(l+2)(l-1)}\left[\frac{1}{f}\tilde{h}_{tt} - f\tilde{h}_{rr} + \frac{4(r-M)}{r^2}\tilde{j}_r + 2\frac{d\tilde{j}_r}{dr_*}\right],$$

$$2f\tilde{Q}^{tr} = 0 = \frac{d^2\tilde{h}_{tr}}{dr_*^2} + \frac{4f}{r}\frac{d\tilde{h}_{tr}}{dr_*} + \left(\frac{2(2M^2-r^2)}{r^4} - \frac{f}{r^2}\left(l(l+1)-4\right)\right)\tilde{h}_{tr},$$

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$$\frac{1}{f}\tilde{Q}^{rr} - \tilde{Q}^{\flat} + f\tilde{Q}^{tt} = \frac{d^2\tilde{h}_{rr}}{dr_*^2} + \frac{4(r-M)}{r^2}\frac{d\tilde{h}_{rr}}{dr_*} + \frac{2}{fr}\frac{d\tilde{h}_{tt}}{dr_*} - \frac{4}{r}\frac{d\tilde{K}}{dr_*},$$

$$+\left(\frac{2M^2}{r^4} - \frac{f}{r^2}\left(l(l+1)-4\right)\right)\tilde{h}_{rr} + \frac{2M(3M-2r)}{f^2r^4}\tilde{h}_{tt}, - \frac{4(r-M)}{r^3}\tilde{K}$$

$$f^2\tilde{Q}^{tt} - \tilde{Q}^{rr} = \frac{d^2\tilde{K}}{dr_*^2} + \frac{4f}{r}\frac{d\tilde{K}}{dr_*} - \frac{1}{r}\frac{d\tilde{h}_{tt}}{dr_*} - \frac{f^2}{r}\frac{d\tilde{h}_{rr}}{dr_*} - \frac{f}{r^2}\left(l(l+1)-2\right)\tilde{K} + \frac{2M}{r^3}\tilde{h}_{tt} - \frac{2f^2(r+M)}{r^3}\tilde{h}_{rr},$$

Even-parity static modes ($m = 0$, $n = 0 \Longrightarrow \omega = 0$)

Zero-frequency form of even-parity constrained equations

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$$\tilde{j}_r = \frac{1}{l(l+1)}\left[2(r-M)\tilde{h}_{rr} - 2r\tilde{K} + \frac{r^2}{2}\frac{d\tilde{h}_{rr}}{dr_*} + \frac{r^2}{2f^2}\frac{d\tilde{h}_{tt}}{dr_*} - \frac{r^2}{f}\frac{d\tilde{K}}{dr_*}\right],$$

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$$\tilde{h}_{tr} = 0,$$

$$f\tilde{Q}^{rr} + f^2\tilde{Q}^\flat + f^3\tilde{Q}^{tt} = \frac{d^2\tilde{h}_{tt}}{dr_*^2} + \frac{2(r-4M)}{r^2}\frac{d\tilde{h}_{tt}}{dr_*} + \left(\frac{2M^2}{r^4} - \frac{f}{r^2}l(l+1)\right)\tilde{h}_{tt} + \frac{2Mf^2(3M-2r)}{r^4}\tilde{h}_{rr} + \frac{4Mf^2}{r^3}\tilde{K},$$

$$\frac{1}{f}\tilde{Q}^{rr} - \tilde{Q}^\flat + f\tilde{Q}^{tt} = \frac{d^2\tilde{h}_{rr}}{dr_*^2} + \frac{4(r-M)}{r^2}\frac{d\tilde{h}_{rr}}{dr_*} + \frac{2}{fr}\frac{d\tilde{h}_{tt}}{dr_*} - \frac{4}{r}\frac{d\tilde{K}}{dr_*}$$
$$+ \left(\frac{2M^2}{r^4} - \frac{f}{r^2}\left(l(l+1)-4\right)\right)\tilde{h}_{rr} + \frac{2M(3M-2r)}{f^2r^4}\tilde{h}_{tt} - \frac{4(r-M)}{r^3}\tilde{K},$$

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Even-parity static modes ($m = 0$, $n = 0 \Longrightarrow \omega = 0$)

Zero-frequency form of even-parity constrained equations

$$\tilde{j}_t = 0,$$

$$\tilde{j}_r = \frac{1}{l(l+1)}\left[2(r-M)\tilde{h}_{rr} - 2r\tilde{K} + \frac{r^2}{2}\frac{d\tilde{h}_{rr}}{dr_*} + \frac{r^2}{2f^2}\frac{d\tilde{h}_{tt}}{dr_*} - \frac{r^2}{f}\frac{d\tilde{K}}{dr_*}\right],$$

$$\tilde{G} = \frac{1}{(l+2)(l-1)}\left[\frac{1}{f}\tilde{h}_{tt} - f\tilde{h}_{rr} + \frac{4(r-M)}{r^2}\tilde{j}_r + 2\frac{d\tilde{j}_r}{dr_*}\right],$$

$$\tilde{h}_{tr} = 0,$$

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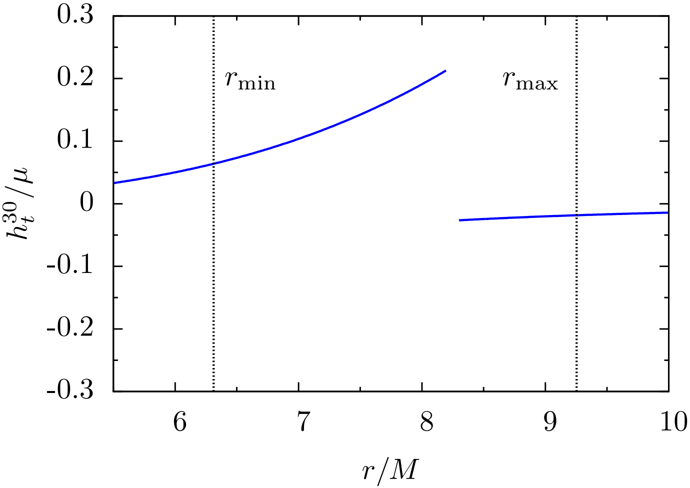
$$\begin{pmatrix}\tilde{h}_{tt}\\ \tilde{h}_{rr}\\ \tilde{K}\end{pmatrix} \simeq \frac{1}{r^{l+1}}\sum_{k=0}^{k_{\max}}\left[\frac{1}{r^k}\begin{pmatrix}a_k^{(tt)}\\ a_k^{(rr)}\\ a_k^{(K)}\end{pmatrix} + \frac{1}{r^{k+2}}\begin{pmatrix}b_k^{(tt)}\\ b_k^{(rr)}\\ b_k^{(K)}\end{pmatrix}\ln\left(\frac{r}{M}\right)\right].$$

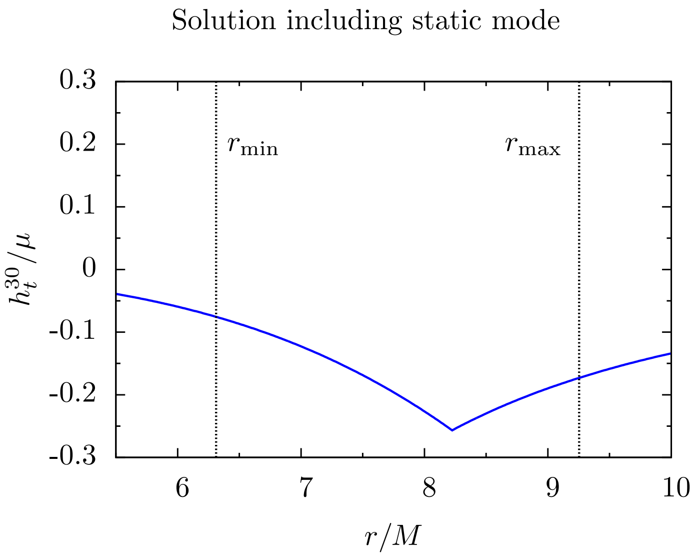
Solution including the contribution from every frequency except $\omega = 0$



$e = 0.188917$

$p = 7.50478$

$t = 89.5219M$



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Outline

- Constrained equations for radiative modes
  $(l \geq 2,\ \omega \neq 0)$
- Homogeneous solutions of constrained equations
- Particular solution of constrained equations
  (Extended homogeneous solutions)
- Static modes: constrained solution
  $(m = 0,\ n = 0 \Rightarrow \omega = 0)$
- Low-order modes: constrained solution
  $(l < 0, 1)$
- Calculation of the dissipative self-force and results

Low-order modes

- Eccentric orbits: Five cases to consider

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- $l = 1$, $m = \pm 1$: 6 unconstrained equations, 3 gauge conditions $\Rightarrow$ 4 ***partially*** constrained equations with 1 outstanding gauge condition

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Presently unclear if fully constrained (second order) equations can be found for $l = 1$, $m = \pm 1$

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Self-force overview

- Standard mode-sum regularization approach

$$F^{\mu}_{\text{full}}(x;x_{\text{p}}) = \mu\, k^{\mu\nu\gamma\delta}(x;x_p)\bar{p}_{\nu\gamma;\delta},$$

$$F^{\mu} = \sum_{l'=0}^{\infty}\left[F^{\mu l'}_{\text{full}\pm} - A^{\mu}_{\pm}(l'+\frac{1}{2}) - B^{\mu}\right] \equiv \sum_{l'=0}^{\infty} F^{\mu l'}_{\text{reg}},$$

- Scalar spherical harmoinc decomposition in l', m' modes for regularization

$$F^{\mu}_{\text{full}\pm} = \sum_{l'=0}^{\infty} F^{\mu l'}_{\text{full}\pm} = \sum_{l'=0}^{\infty}\sum_{m'=-l'}^{l'} Y_{l'm'}(\theta_p,\phi_p)\mathcal{A}^{\mu l'm'}_{\pm},$$

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- Tensor spherical harmoinc decomposition in l, m modes convenient except for regularization

$$F^{\mu}_{\text{full}\pm} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} F^{\mu\, lm}_{\text{full}\pm} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\begin{pmatrix} f^{a}_{lm}Y_{lm} \\ -------- \\ f^{lm}_{e}Y^{A}_{lm} + f^{lm}_{o}X^{A}_{lm}\end{pmatrix}_{\pm} = \sum_{l,m}\begin{pmatrix} f^{t}_{lm}Y_{lm} \\ f^{r}_{lm}Y_{lm} \\ f^{lm}_{e}Y^{\theta}_{lm} + f^{lm}_{o}X^{\theta}_{lm} \\ f^{lm}_{e}Y^{\phi}_{lm} + f^{lm}_{o}X^{\phi}_{lm}\end{pmatrix}_{\pm}$$

$$\mathcal{A}^{\mu l' m'}_{\pm} = \mathcal{A}^{\mu\, l'm'}_{\pm}(f^{t}_{lm\pm}, f^{r}_{lm\pm}, f^{lm}_{e\pm}, f^{lm}_{o\pm}) \leftarrow \text{mixes } l \text{ modes}$$

Dissipative self-force overview

- The dissipative self-force requires no regularization

$$p^{(\text{diss})}_{\mu\nu} = \frac{1}{2}\left(p^{\text{ret}}_{\mu\nu} - p^{\text{adv}}_{\mu\nu}\right),$$

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- Tensor and scalar spherical harmonic decompositions are equally valid

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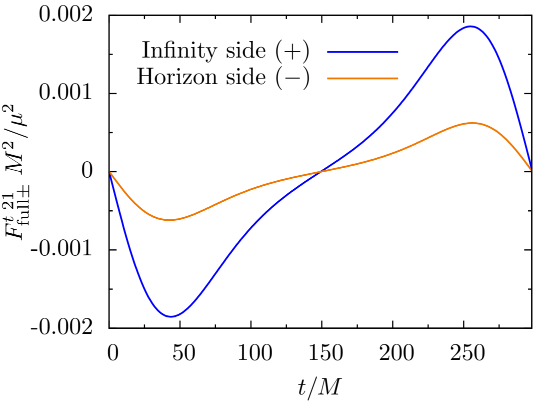
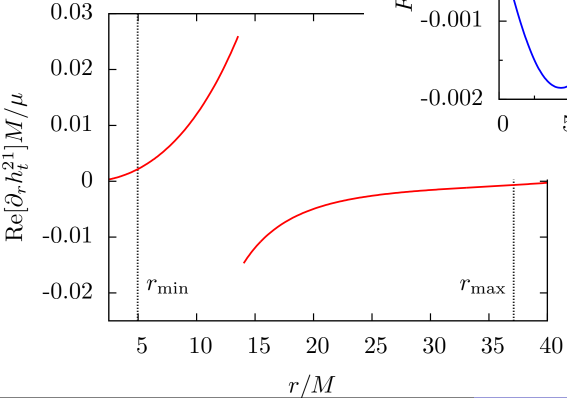
- How can we extract the effects of $F_{(\text{diss})}^{\mu\; lm}$ from $F_{\text{full}\pm}^{\mu\; lm}$?

$F^{\mu\ lm}_{\mathrm{full}\pm}$ is the unregularized l,m part of self-force evaluated on infinity (+) or horizon (−) side of r_p





← $e = 0.764124$

$p = 8.75455$

$t = 80M$

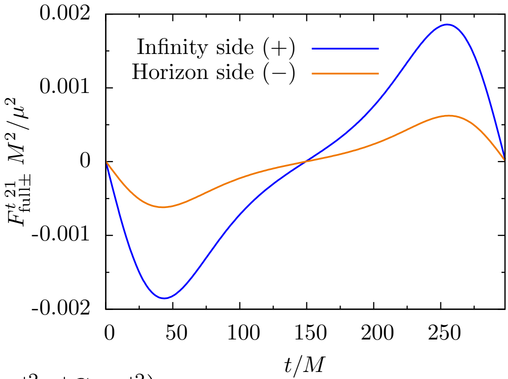
Energy flux and self-work (single l, m mode)

- Locally compute work done by self-force

$$E = \mu f_p u^t$$

$$\langle \dot{E}^{\text{work}}_{lm\pm} \rangle = \frac{1}{T_r} \int_0^{T_r} f_p \frac{d\tau}{dt} F^{t\;lm}_{\text{full}\pm} dt$$

- Compute energy flux at $r \simeq \infty$ and $r \simeq 2M$



$$\langle \dot{E}^{\text{rad}}_{lm} \rangle = \frac{1}{64\pi} \frac{(l+2)!}{(l-2)!} \sum_n \omega^2_{mn} \left(|C^+_{lmn}|^2 + |C^-_{lmn}|^2 \right)$$

	$\langle \dot{E}_{21} \rangle \; M^2/\mu^2$
Infinity side self-work	$7.61849062972 \times 10^{-7}$
Horizon side self-work	$7.61849062973 \times 10^{-7}$
Energy flux	$7.61849063003 \times 10^{-7}$

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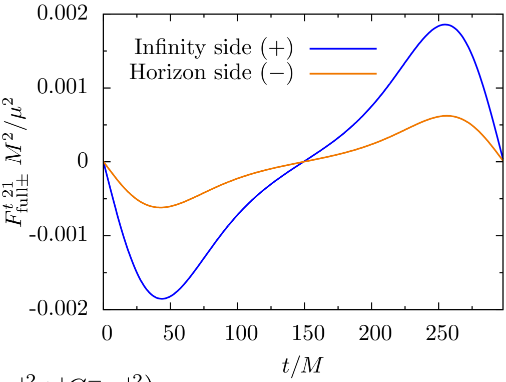

$$F^{t\, lm}_{(\text{diss})} = \frac{1}{2}\left(F^{t\, lm}_{\text{full}\pm}(t) + F^{t\, lm}_{\text{full}\pm}(-t) \right)$$

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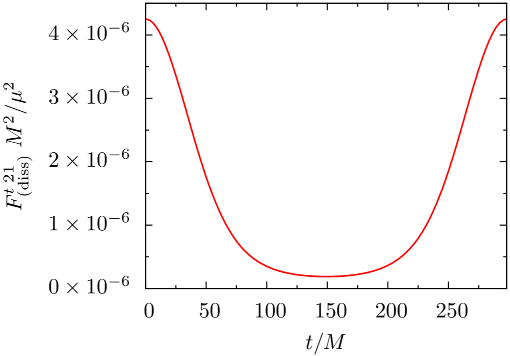

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Angular momentum flux and torque (single l, m mode)

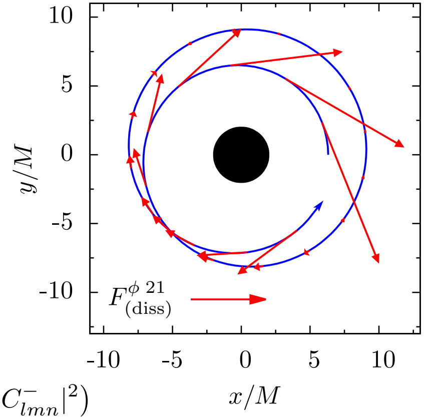
- Locally time-average torque from self-force

$$L = \mu r_p^2 u^\phi$$

$$\langle \dot{L}_{lm}^{\text{torq}} \rangle = \frac{1}{T_r}\int_0^{T_r} r_p^2 \frac{d\tau}{dt} F^{\phi\, lm}_{\text{full}\pm} dt$$

- Compute angular momentum flux at $r \sim \infty$ and $r \simeq 2M$

$$\langle \dot{L}_{lm}^{\text{rad}} \rangle = \frac{m}{64\pi}\frac{(l+2)!}{(l-2)!}\sum_n \omega_{mn}\left(|C^+_{lmn}|^2 + |C^-_{lmn}|^2\right)$$



$$F^{\phi\, lm}_{(\text{diss})} = \frac{1}{2}\left(F^{\phi\, lm}_{\text{full}\pm}(t) + F^{\phi\, lm}_{\text{full}\pm}(-t)\right)$$

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	$\langle \dot{L}_{21} \rangle\ M/\mu^2$
Infinity side torque	$1.39157808634 \times 10^{-5}$
Horizon side torque	$1.39157808635 \times 10^{-5}$
Ang. momentum flux	$1.39157808640 \times 10^{-5}$

Conclusions

- We use the Lorenz gauge conditions to reduce the system of ODEs in size from 10 to 6 (2 odd-parity, 4 even-parity).
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- Each low-order mode ($l = 0, 1$) is a special case handled separately. All except one can be solved by fully constraining the equations with the gauge conditions.
- We calculate dissipative effects of the self-force by time averaging over a period and compare locally determined work and torque with energy and angular momentum fluxes.

Acknowledgements

Movie

Constrained low order mode example: Monopole

$$0 = -i\omega r^2 \tilde{h}_{tt} - i\omega f^2 r^2 \tilde{h}_{rr} - 2i\omega f r^2 \tilde{K} - 2 f r^2 \frac{d\tilde{h}_{tr}}{dr_*} - 4f(r-M)\tilde{h}_{tr},$$

$$0 = 4f(r-M)\tilde{h}_{rr} - 2r^2 \frac{d\tilde{K}}{dr_*} - 4fr\tilde{K} + fr^2 \frac{d\tilde{h}_{rr}}{dr_*} + \frac{r^2}{f}\frac{d\tilde{h}_{tt}}{dr_*} + 2i\omega r^2 \tilde{h}_{tr},$$

$$-f\tilde{Q}^{rr} - f^2\tilde{Q}^{\flat} - f^3\tilde{Q}^{tt} = \frac{d^2\tilde{h}_{tt}}{dr_*^2} + \frac{2(r-4M)}{r^2}\frac{d\tilde{h}_{tt}}{dr_*} + \left(\omega^2 + \frac{2M^2}{r^4}\right)\tilde{h}_{tt} + \frac{2Mf^2(3M-2r)}{r^4}\tilde{h}_{rr} - \frac{4iM\omega f}{r^2}\tilde{h}_{tr} + \frac{4Mf^2}{r^3}\tilde{K},$$

$$-\frac{1}{f}\tilde{Q}^{rr} + \tilde{Q}^{\flat} - f\tilde{Q}^{tt} = \frac{d^2\tilde{h}_{rr}}{dr_*^2} + \frac{2}{r}\frac{d\tilde{h}_{rr}}{dr_*} + \left(\omega^2 - \frac{2(2r^2 - 8Mr + 7M^2)}{r^4}\right)\tilde{h}_{rr} + \frac{2M(3M-2r)}{f^2 r^4}\tilde{h}_{tt} - \frac{4iM\omega}{fr^2}\tilde{h}_{tr} + \frac{4(r-3M)}{r^3}\tilde{K},$$

$$2f\tilde{Q}^{tr} = \frac{d^2\tilde{h}_{tr}}{dr_*^2} + \frac{2f}{r}\frac{d\tilde{h}_{tr}}{dr_*} + \left(\omega^2 - \frac{2(r^2 - 2Mr + 2M^2)}{r^4}\right)\tilde{h}_{tr} - \frac{2iM\omega}{fr^2}\tilde{h}_{tt} - \frac{2ifM\omega}{r^2}\tilde{h}_{rr},$$

$$-f^2\tilde{Q}^{tt} + \tilde{Q}^{rr} = \frac{d^2\tilde{K}}{dr_*^2} + \frac{2f}{r}\frac{d\tilde{K}}{dr_*} + \left(\omega^2 - \frac{2f(r-4M)}{r^3}\right)\tilde{K} + \frac{2M}{r^3}\tilde{h}_{tt} - \frac{2f^2(3M-r)}{r^3}\tilde{h}_{rr},$$

Constrained low order mode example: Monopole

$$0 = -i\omega r^2 \tilde{h}_{tt} - i\omega f^2 r^2 \tilde{h}_{rr} - 2i\omega f r^2 \tilde{K} - 2fr^2 \frac{d\tilde{h}_{tr}}{dr_*} - 4f(r-M)\tilde{h}_{tr},$$

$$0 = 4f(r-M)\tilde{h}_{rr} - 2r^2 \frac{d\tilde{K}}{dr_*} - 4fr\tilde{K} + fr^2 \frac{d\tilde{h}_{rr}}{dr_*} + \frac{r^2}{f}\frac{d\tilde{h}_{tt}}{dr_*} + 2i\omega r^2 \tilde{h}_{tr},$$

$$-f\tilde{Q}^{rr} - f^2\tilde{Q}^{\flat} - f^3\tilde{Q}^{tt} = \frac{d^2\tilde{h}_{tt}}{dr_*^2} + \frac{2(r-4M)}{r^2}\frac{d\tilde{h}_{tt}}{dr_*} + \left(\omega^2 + \frac{2M^2}{r^4}\right)\tilde{h}_{tt} + \frac{2Mf^2(3M-2r)}{r^4}\tilde{h}_{rr} - \frac{4iM\omega f}{r^2}\tilde{h}_{tr} + \frac{4Mf^2}{r^3}\tilde{K},$$

$$-\frac{1}{f}\tilde{Q}^{rr} + \tilde{Q}^{\flat} - f\tilde{Q}^{tt} = \frac{d^2\tilde{h}_{rr}}{dr_*^2} + \frac{2}{r}\frac{d\tilde{h}_{rr}}{dr_*} + \left(\omega^2 - \frac{2(2r^2-8Mr+7M^2)}{r^4}\right)\tilde{h}_{rr} + \frac{2M(3M-2r)}{f^2r^4}\tilde{h}_{tt} - \frac{4iM\omega}{fr^2}\tilde{h}_{tr} + \frac{4(r-3M)}{r^3}\tilde{K},$$

$$2f\tilde{Q}^{tr} = \frac{d^2\tilde{h}_{tr}}{dr_*^2} + \frac{2f}{r}\frac{d\tilde{h}_{tr}}{dr_*} + \left(\omega^2 - \frac{2(r^2-2Mr+2M^2)}{r^4}\right)\tilde{h}_{tr} - \frac{2iM\omega}{fr^2}\tilde{h}_{tt} - \frac{2ifM\omega}{r^2}\tilde{h}_{rr},$$

$$-f^2\tilde{Q}^{tt} + \tilde{Q}^{rr} = \frac{d^2\tilde{K}}{dr_*^2} + \frac{2f}{r}\frac{d\tilde{K}}{dr_*} + \left(\omega^2 - \frac{2f(r-4M)}{r^3}\right)\tilde{K} + \frac{2M}{r^3}\tilde{h}_{tt} - \frac{2f^2(3M-r)}{r^3}\tilde{h}_{rr},$$

$$\tilde{h}_{tt} = \frac{i(r^4\omega^2 + 6r^2 - 12Mr + 4M^2)}{2r^3\omega}\tilde{h}_{tr} + \left(\frac{5M}{r} - 3\right)\tilde{K} + \frac{i(3r^2 - 10Mr + 8M^2)}{fr^2\omega}\frac{d\tilde{h}_{tr}}{dr_*} - r\frac{d\tilde{K}}{dr_*} + \frac{ir}{2\omega}\frac{d^2\tilde{h}_{tr}}{dr_*^2},$$

$$\tilde{h}_{rr} = -\frac{i(r^4\omega^2 - 2r^2 + 12Mr - 12M^2)}{2f^2r^3\omega}\tilde{h}_{tr} + \frac{r-M}{f^2r}\tilde{K} - \frac{i}{f^2\omega}\frac{d\tilde{h}_{tr}}{dr_*} + \frac{r}{f^2}\frac{d\tilde{K}}{dr_*} - \frac{ir}{2f^2\omega}\frac{d^2\tilde{h}_{tr}}{dr_*^2}.$$

Constrained low order mode example: Monopole

$$0 = -i\omega r^2 \tilde{h}_{tt} - i\omega f^2 r^2 \tilde{h}_{rr} - 2i\omega f r^2 \tilde{K} - 2fr^2 \frac{d\tilde{h}_{tr}}{dr_*} - 4f(r-M)\tilde{h}_{tr},$$

$$0 = 4f(r-M)\tilde{h}_{rr} - 2r^2 \frac{d\tilde{K}}{dr_*} - 4fr\tilde{K} + fr^2 \frac{d\tilde{h}_{rr}}{dr_*} + \frac{r^2}{f}\frac{d\tilde{h}_{tt}}{dr_*} + 2i\omega r^2 \tilde{h}_{tr},$$

$$-f\tilde{Q}^{rr} - f^2\tilde{Q}^{\flat} - f^3\tilde{Q}^{tt} = \frac{d^2\tilde{h}_{tt}}{dr_*^2} + \frac{2(r-4M)}{r^2}\frac{d\tilde{h}_{tt}}{dr_*} + \left(\omega^2 + \frac{2M^2}{r^4}\right)\tilde{h}_{tt} + \frac{2Mf^2(3M-2r)}{r^4}\tilde{h}_{rr} - \frac{4iM\omega f}{r^2}\tilde{h}_{tr} + \frac{4Mf^2}{r^3}\tilde{K},$$

$$-\frac{1}{f}\tilde{Q}^{rr} + \tilde{Q}^{\flat} - f\tilde{Q}^{tt} = \frac{d^2\tilde{h}_{rr}}{dr_*^2} + \frac{2}{r}\frac{d\tilde{h}_{rr}}{dr_*} + \left(\omega^2 - \frac{2(2r^2 - 8Mr + 7M^2)}{r^4}\right)\tilde{h}_{rr} + \frac{2M(3M-2r)}{f^2r^4}\tilde{h}_{tt} - \frac{4iM\omega}{fr^2}\tilde{h}_{tr} + \frac{4(r-3M)}{r^3}\tilde{K},$$

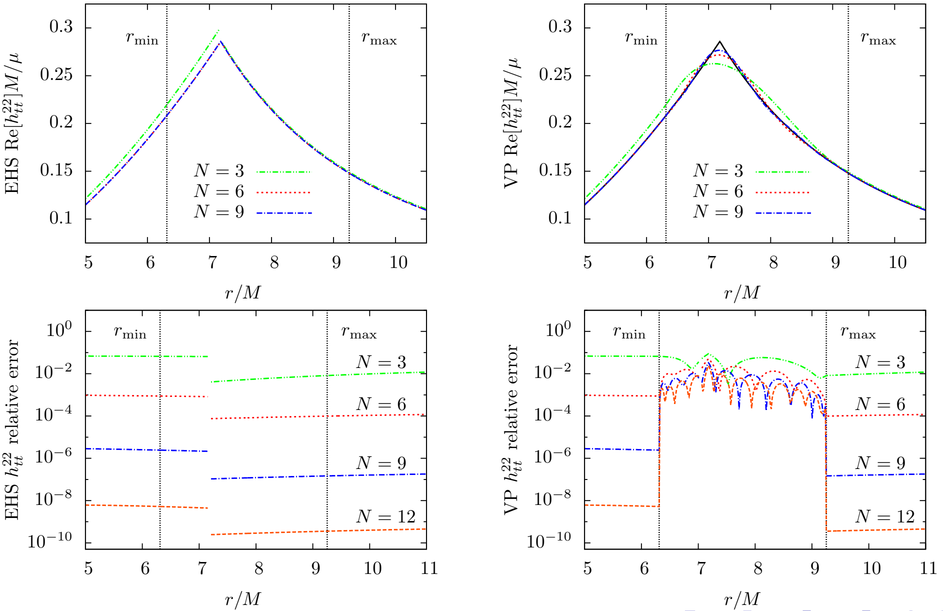
$$2f\tilde{Q}^{tr} = \frac{d^2\tilde{h}_{tr}}{dr_*^2} + \frac{2f}{r}\frac{d\tilde{h}_{tr}}{dr_*} + \left(\omega^2 - \frac{2(r^2 - 2Mr + 2M^2)}{r^4}\right)\tilde{h}_{tr} - \frac{2iM\omega}{fr^2}\tilde{h}_{tt} - \frac{2ifM\omega}{r^2}\tilde{h}_{rr},$$

$$-f^2\tilde{Q}^{tt} + \tilde{Q}^{rr} = \frac{d^2\tilde{K}}{dr_*^2} + \frac{2f}{r}\frac{d\tilde{K}}{dr_*} + \left(\omega^2 - \frac{2f(r-4M)}{r^3}\right)\tilde{K} + \frac{2M}{r^3}\tilde{h}_{tt} - \frac{2f^2(3M-r)}{r^3}\tilde{h}_{rr},$$

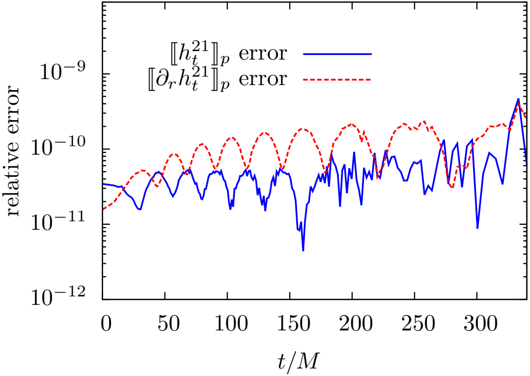
$$\tilde{h}_{tt} = \frac{i(r^4\omega^2 + 6r^2 - 12Mr + 4M^2)}{2r^3\omega}\tilde{h}_{tr} + \left(\frac{5M}{r} - 3\right)\tilde{K} + \frac{i(3r^2 - 10Mr + 8M^2)}{fr^2\omega}\frac{d\tilde{h}_{tr}}{dr_*} - r\frac{d\tilde{K}}{dr_*} + \frac{ir}{2\omega}\frac{d^2\tilde{h}_{tr}}{dr_*^2},$$

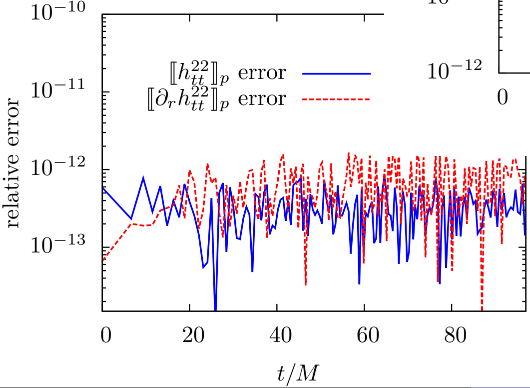
$$\tilde{h}_{rr} = -\frac{i(r^4\omega^2 - 2r^2 + 12Mr - 12M^2)}{2f^2r^3\omega}\tilde{h}_{tr} + \frac{r-M}{f^2r}\tilde{K} - \frac{i}{f^2\omega}\frac{d\tilde{h}_{tr}}{dr_*} + \frac{r}{f^2}\frac{d\tilde{K}}{dr_*} - \frac{ir}{2f^2\omega}\frac{d^2\tilde{h}_{tr}}{dr_*^2}.$$

Even-parity results



Test jump conditions





Odd-parity constrained, causal homogeneous solutions

$$\begin{pmatrix} \tilde{h}_t \\ \tilde{h}_r \end{pmatrix}_0^- \sim \begin{pmatrix} 1 \\ 1/f \end{pmatrix} e^{-i\omega r_*}, \qquad \begin{pmatrix} \tilde{h}_t \\ \tilde{h}_r \end{pmatrix}_0^+ \sim \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{+i\omega r_*},$$

$$\begin{pmatrix} \tilde{h}_t \\ \tilde{h}_r \end{pmatrix}_1^- \sim \begin{pmatrix} f \\ -1 \end{pmatrix} e^{-i\omega r_*}, \qquad \begin{pmatrix} \tilde{h}_t \\ \tilde{h}_r \end{pmatrix}_1^+ \sim \frac{1}{r} \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{+i\omega r_*}.$$

Even-parity constrained, causal homogeneous solutions

$$\begin{pmatrix}\tilde{h}_{tt}\\ \tilde{h}_{tr}\\ \tilde{h}_{rr}\\ \tilde{K}\end{pmatrix}_0^- \sim \begin{pmatrix}1\\ 1/f\\ 1/f^2\\ 0\end{pmatrix} e^{-i\omega r_*}, \qquad \begin{pmatrix}\tilde{h}_{tt}\\ \tilde{h}_{tr}\\ \tilde{h}_{rr}\\ \tilde{K}\end{pmatrix}_0^+ \sim \frac{1}{r}\begin{pmatrix}1\\ 0\\ -1\\ 0\end{pmatrix} e^{+i\omega r_*},$$

$$\begin{pmatrix}\tilde{h}_{tt}\\ \tilde{h}_{tr}\\ \tilde{h}_{rr}\\ \tilde{K}\end{pmatrix}_1^- \sim f \begin{pmatrix}1\\ -1/f\\ 0\\ 2/(4i\omega M-1)\end{pmatrix} e^{-i\omega r_*}, \qquad \begin{pmatrix}\tilde{h}_{tt}\\ \tilde{h}_{tr}\\ \tilde{h}_{rr}\\ \tilde{K}\end{pmatrix}_1^+ \sim \frac{1}{r}\begin{pmatrix}0\\ 1\\ -2\\ 0\end{pmatrix} e^{+i\omega r_*},$$

$$\begin{pmatrix}\tilde{h}_{tt}\\ \tilde{h}_{tr}\\ \tilde{h}_{rr}\\ \tilde{K}\end{pmatrix}_2^- \sim f^2 \begin{pmatrix}1\\ -1/f\\ 1/f^2\\ 0\end{pmatrix} e^{-i\omega r_*}, \qquad \begin{pmatrix}\tilde{h}_{tt}\\ \tilde{h}_{tr}\\ \tilde{h}_{rr}\\ \tilde{K}\end{pmatrix}_2^+ \sim \frac{1}{r^2}\begin{pmatrix}0\\ 1\\ -2\\ 1\end{pmatrix} e^{+i\omega r_*},$$

$$\begin{pmatrix}\tilde{h}_{tt}\\ \tilde{h}_{tr}\\ \tilde{h}_{rr}\\ \tilde{K}\end{pmatrix}_3^- \sim \begin{pmatrix}0\\ 0\\ 0\\ 1\end{pmatrix} e^{-i\omega r_*}, \qquad \begin{pmatrix}\tilde{h}_{tt}\\ \tilde{h}_{tr}\\ \tilde{h}_{rr}\\ \tilde{K}\end{pmatrix}_3^+ \sim \frac{1}{r^3}\begin{pmatrix}0\\ 0\\ -2\\ 1\end{pmatrix} e^{+i\omega r_*}.$$