Effective source self-force calculations in the frequency domain

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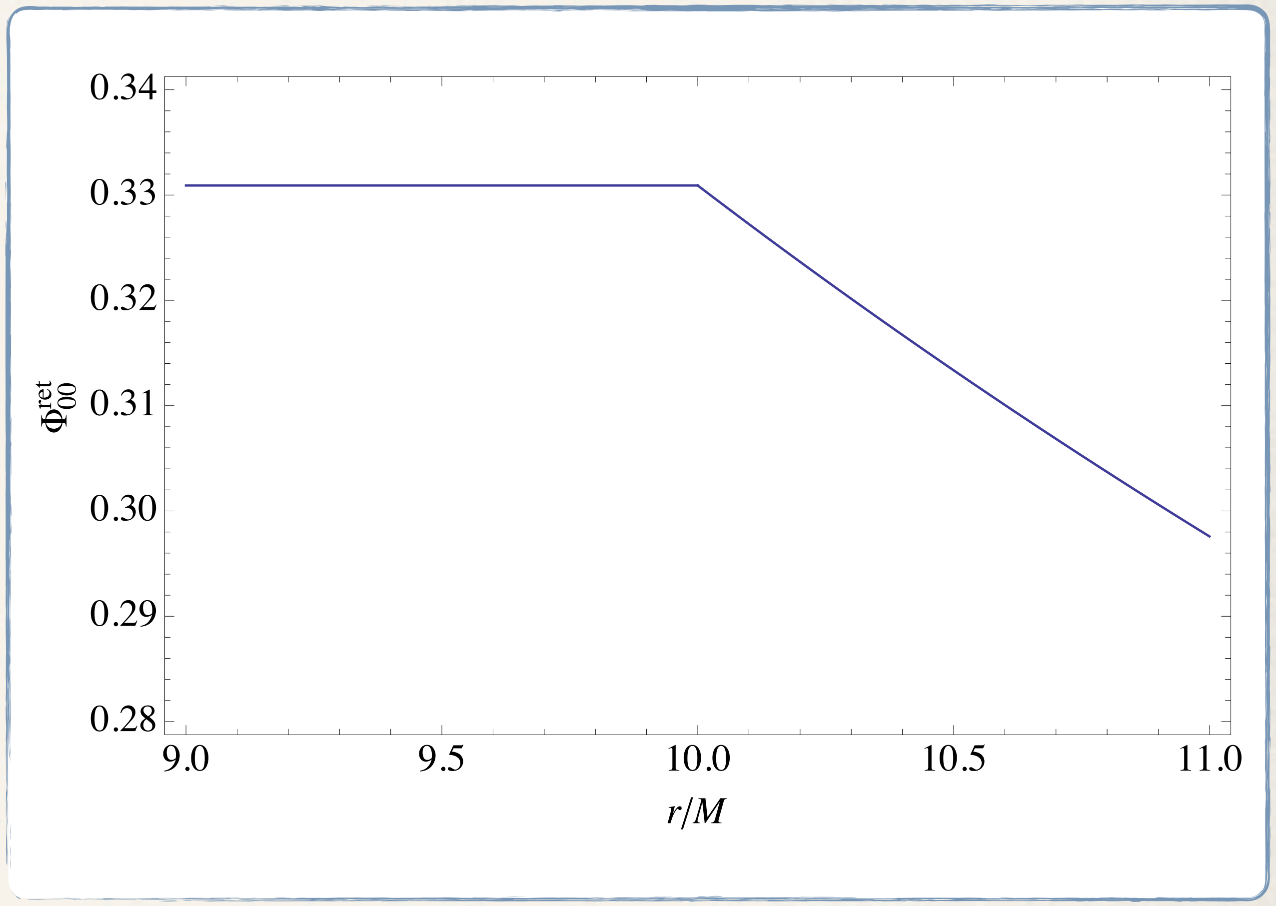
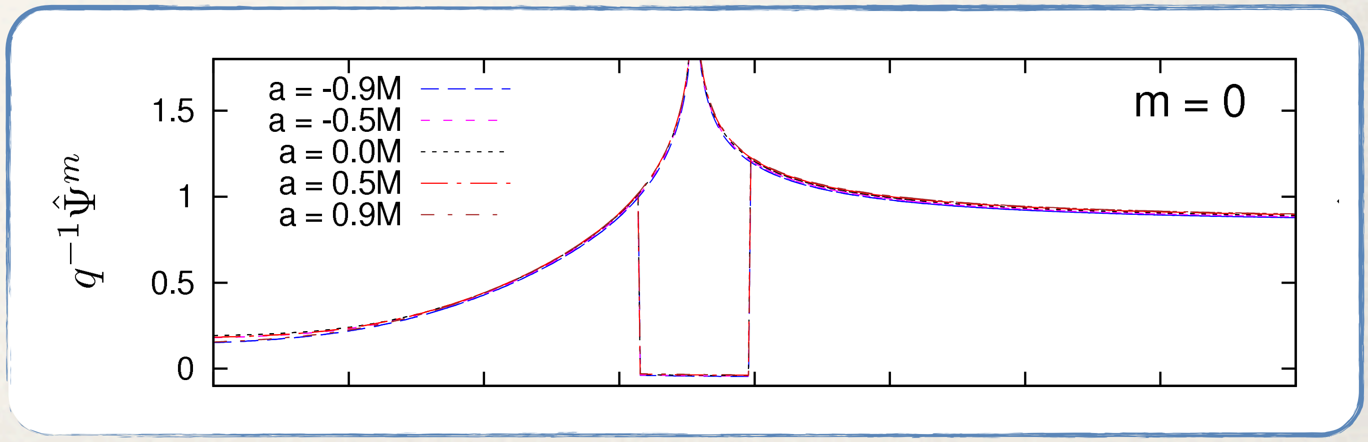
Motivation: Second order

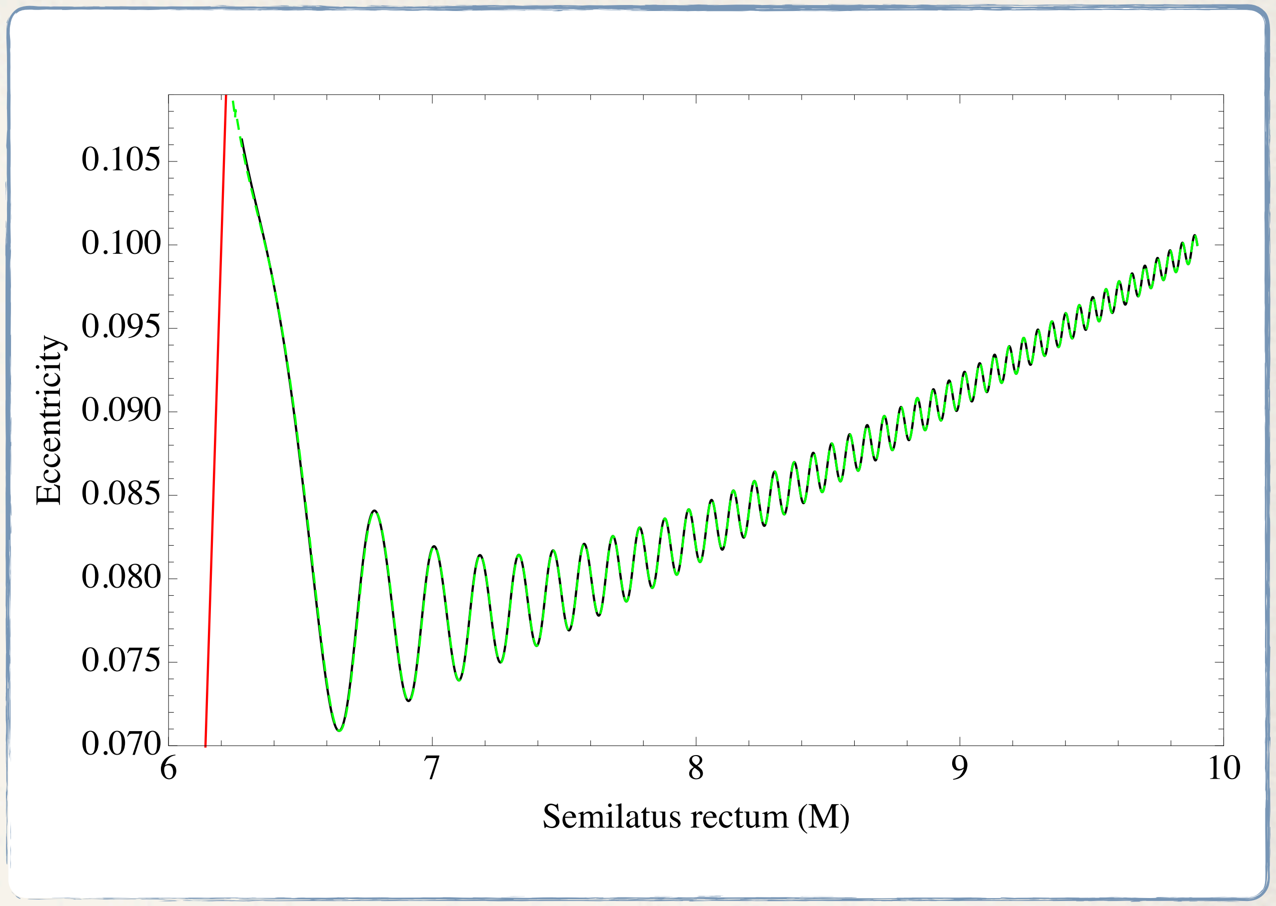
- Second order self-force will require high accuracy $\Rightarrow$ Frequency domain.
- Spherical harmonic modes at first order finite on world line $\Rightarrow$ mode-sum regularization.
- Second order, modes diverge logarithmically.
- At second order need derivatives of first order R field.
- Avoid computing retarded field on world line $\Rightarrow$ effective source (Adam's talk).





Motivation: First order

- Assessing the “geodesic” self-force approximation in orbital evolution appears to require higher accuracy than feasible with current 3+1D codes. (Niels’ talk)
- One option: evolve 1+1D time domain system.
- Main complication is deriving analytic expression for effective source in 1+1D - identical calculation as for frequency domain effective source.



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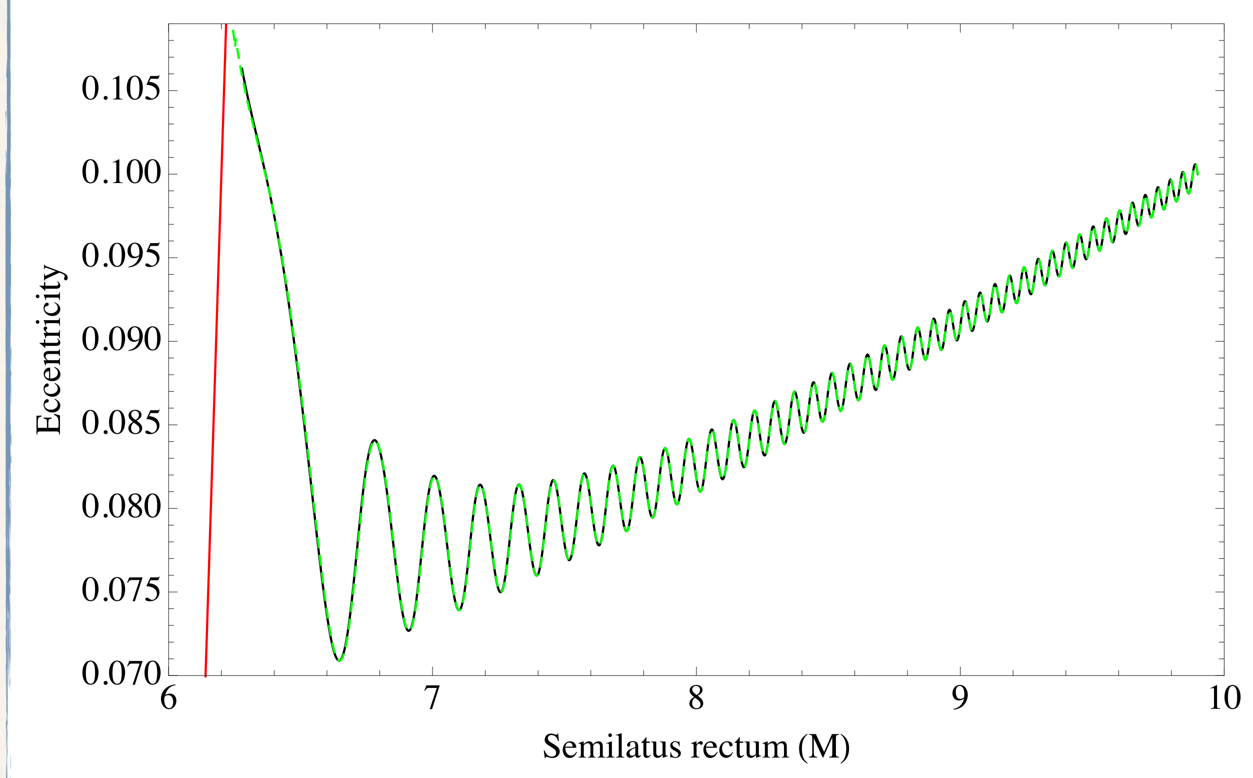
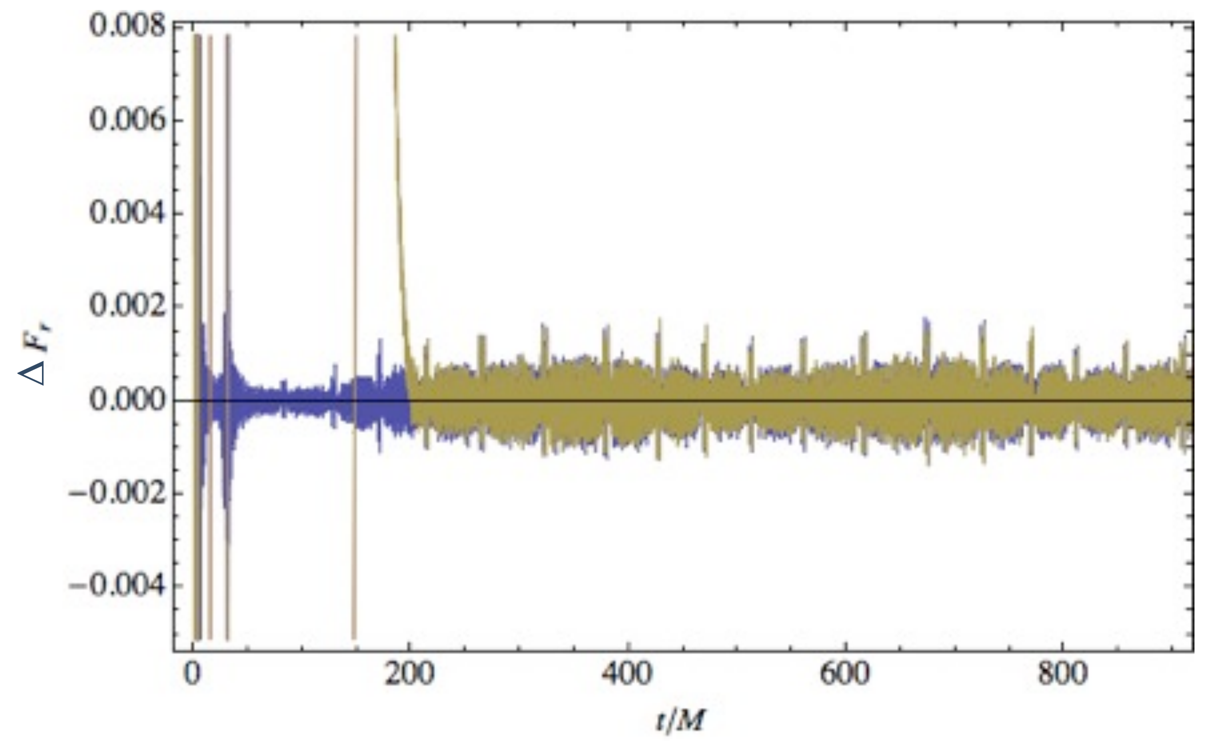
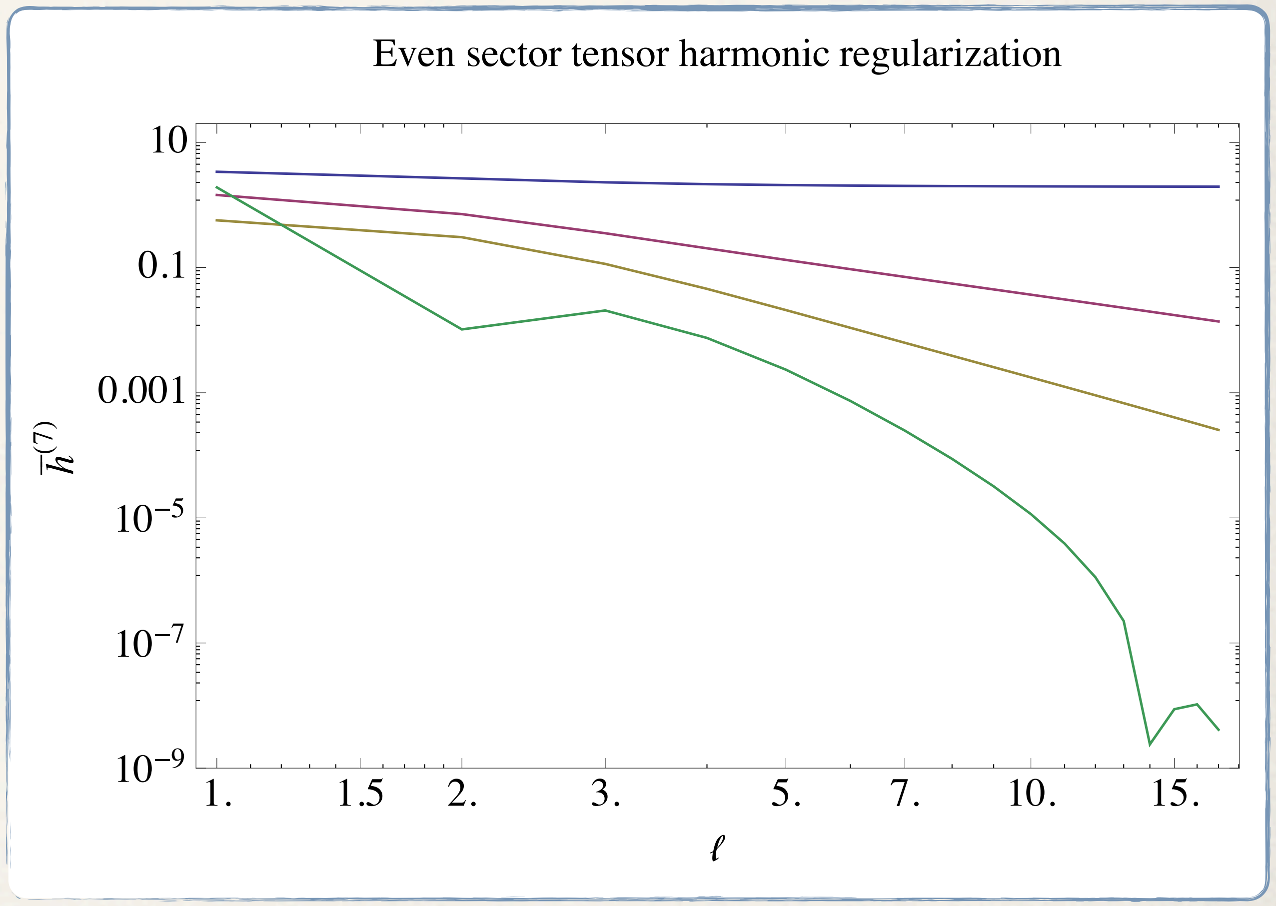




Other benefits

- Equivalence of world tube and window function approaches to effective source.
- Recover standard mode-sum scheme in the limit of a zero-width effective source.
- Regularization parameters for tensor harmonic modes - avoid tensor/scalar re-expansion



Effective source approach

- Basic idea: use approximation to Detweiler-Whiting singular field, Φ^{S}, to derive an evolution equation for approximation to the regular field, Φ^{R}.
  [Barack and Golbourn (2007), Detweiler and Vega (2008)]

$$\Box\Phi^{\mathrm{ret}} = -4\pi q \int \frac{\delta^4(x - z(\tau))}{\sqrt{-g}} d\tau = \Box\tilde{\Phi}^{\mathrm{S}} + \mathcal{O}(\epsilon^n)$$

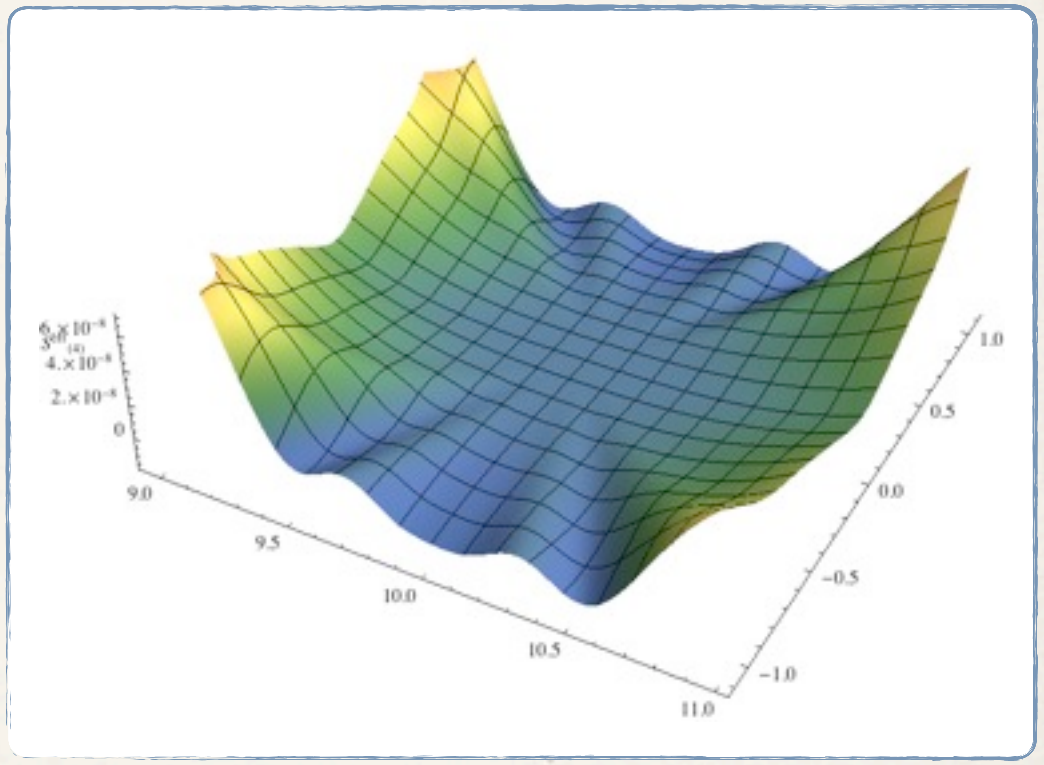
$$\Downarrow$$

$$\Box\tilde{\Phi}^{\mathrm{R}} = S_{\mathrm{eff}}$$

- No distributional sources and no singular fields. Motion purely determined by Φ^{R}.

Effective source approach

- S_{eff} is typically finite, but of limited differentiability on the world line.
- Typically solve for Φ^{R} in time domain using 3+1D or 2+1D m-mode.
- Potentially more accuracy using 1+1D or 1D frequency domain because:
  - Extra mode decomposition smoothens out the source.
  - Lower dimensionality is generally more accurate to numerically solve.
- Both 1+1D and frequency domain require *l*,*m* modes of singular field.
- Desirable to have this mode decomposition analytically.



Worked example - scalar, circular

- Decompose Detweiler-Whiting singular field into spherical harmonic and Fourier modes (circular orbit ⇒ Fourier decomposition trivial since $\omega = m\,\Omega$)

$$\tilde{\Phi}^{\mathrm{S}} = \sum_{\ell m \omega} \Phi^{\mathrm{S}}_{\ell m \omega}(r) Y_{\ell m}(\theta, \phi) e^{-i\omega t}$$

- Decomposition analytic with methods from mode-sum regularization.
- In a coordinate system where the world line is on the north pole

$$\Phi^{\mathrm{S}}_{l,m'=0} = -\frac{(2l+1)|\Delta r|}{2r_0(r_0-2M)}\sqrt{1-\frac{3M}{r_0}}\left[1-\frac{(r_0-M)\Delta r}{r_0(r_0-2M)}\right] + \frac{1}{\pi r_0}\sqrt{\frac{r_0-3M}{(r_0-2M)}}\left[2\mathcal{K}+\frac{(\mathcal{E}-2\mathcal{K})}{r_0}\Delta r+\frac{(2l+1)^2\mathcal{E}}{4r_0(r_0-2M)}\Delta r^2\right]$$

Worked example - scalar, circular

- Spherical harmonic modes in unrotated coordinate system (where particle is on an equatorial orbit) obtained by rotating using Wigner-D symbol

$$\Phi^{\mathrm{S}}_{lm} = \sum_{m'=-\ell}^{\ell} \Phi^{\mathrm{S}}_{lm'} D^{\ell}_{mm'}(0, \pi/2, \Omega t)$$

- Effective source obtained by applying wave operator to singular field.
- Additional complexity relative to standard Barack-Ori mode sum scheme:
  - Need decomposition for $\Delta r \neq 0$.
  - Need to be careful to take account of time dependence of rotation.
  - Second t-derivatives in wave operator mean we need $m' \leq 2$ modes.

Worked example - scalar, circular

- Standard mode-sum frequency domain approach:

$$\left[\frac{d^2}{dr^2} + \frac{2(r-M)}{fr^2}\frac{d}{dr} + \frac{1}{f}\left(\frac{\omega^2}{f} - \frac{\ell(\ell+1)}{r^2}\right)\right]\Phi^{\rm ret}_{\ell m} = \alpha_{\ell m}\delta(r-r_0)$$

  - Find solutions to homogeneous equation which satisfy outgoing boundary conditions on horizon and at infinity, respectively.

$$\left[\frac{d^2}{dr^2} + \frac{2(r-M)}{fr^2}\frac{d}{dr} + \frac{1}{f}\left(\frac{\omega^2}{f} - \frac{\ell(\ell+1)}{r^2}\right)\right]\tilde{\Phi}^{\rm ret\pm}_{\ell m} = 0$$

  - Construct inhomogeneous solutions by matching on the world line

$$\Phi^{\rm ret\pm}_{\ell m} = c^{\pm}_{\ell m}\tilde{\Phi}^{\rm ret\pm}_{\ell m} \qquad c^{\pm}_{\ell m} = \alpha_{\ell m}\frac{\tilde{\Phi}^{\mp}_{\ell m}}{W}$$

where W is the Wronskian of the homogeneous solutions.

Worked example - scalar, circular

- Effective source in frequency domain:

$$\left[\frac{d^2}{dr^2} + \frac{2(r-M)}{fr^2}\frac{d}{dr} + \frac{1}{f}\left(\frac{\omega^2}{f} - \frac{\ell(\ell+1)}{r^2}\right)\right]\Phi^{\text{ret}}_{\ell m} = S^{\text{eff}}_{\ell m}$$

  - Find solutions to homogeneous equation which satisfy outgoing boundary conditions on horizon and at infinity, respectively.

$$\left[\frac{d^2}{dr^2} + \frac{2(r-M)}{fr^2}\frac{d}{dr} + \frac{1}{f}\left(\frac{\omega^2}{f} - \frac{\ell(\ell+1)}{r^2}\right)\right]\tilde{\Phi}^{\text{ret}\pm}_{\ell m} = 0$$

  - Construct inhomogeneous solutions using variation of parameters

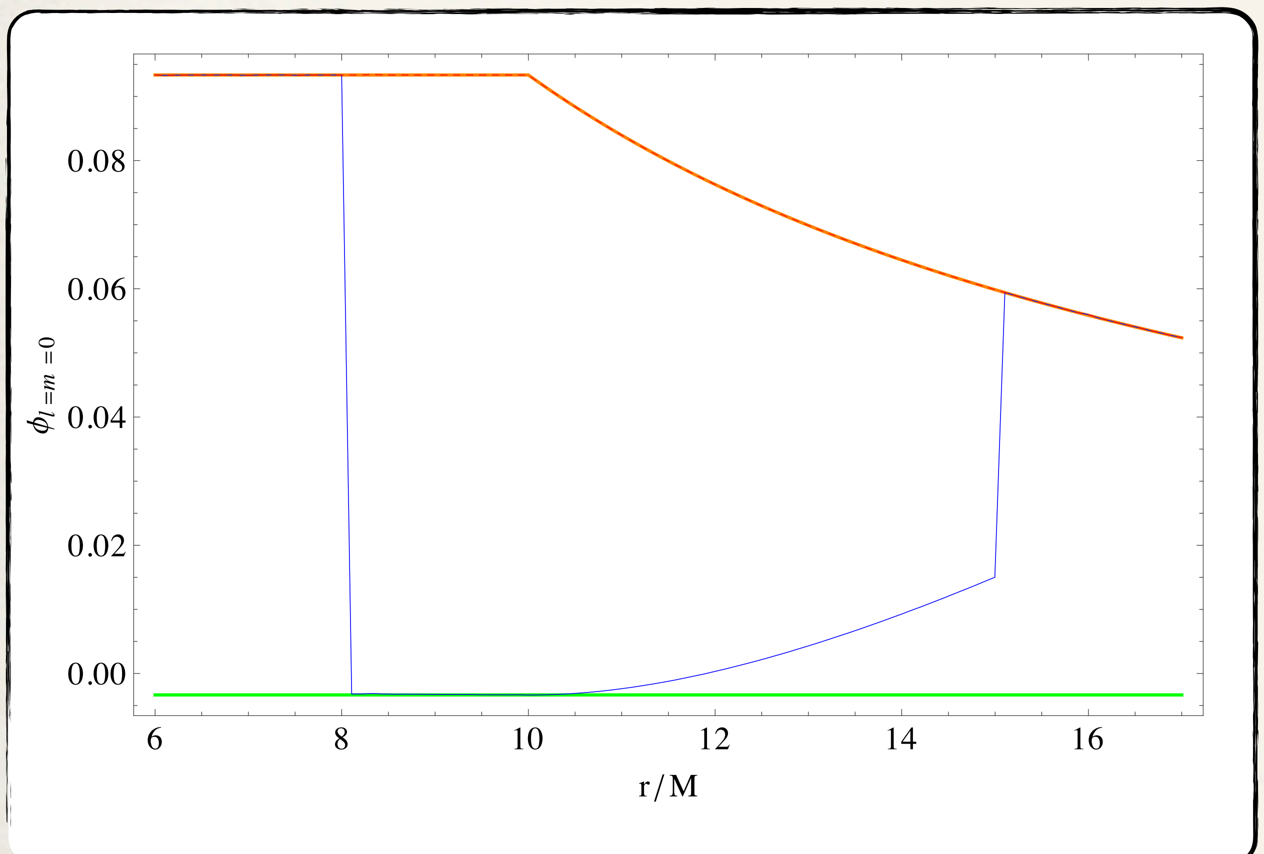
$$\Phi^{\text{ret}}_{\ell m} = c^{+}_{\ell m}(r)\tilde{\Phi}^{\text{ret}+}_{\ell m} + c^{-}_{\ell m}(r)\tilde{\Phi}^{\text{ret}-}_{\ell m}$$

$$c^{+}_{\ell m}(r) = \int_{2M}^{r} \frac{\tilde{\phi}^{-}(r')}{W(r')} S^{\text{eff}}_{\ell m}\, dr', \quad c^{-}_{\ell m}(r) = \int_{r}^{\infty} \frac{\tilde{\phi}^{+}(r')}{W(r')} S^{\text{eff}}_{\ell m}\, dr'$$

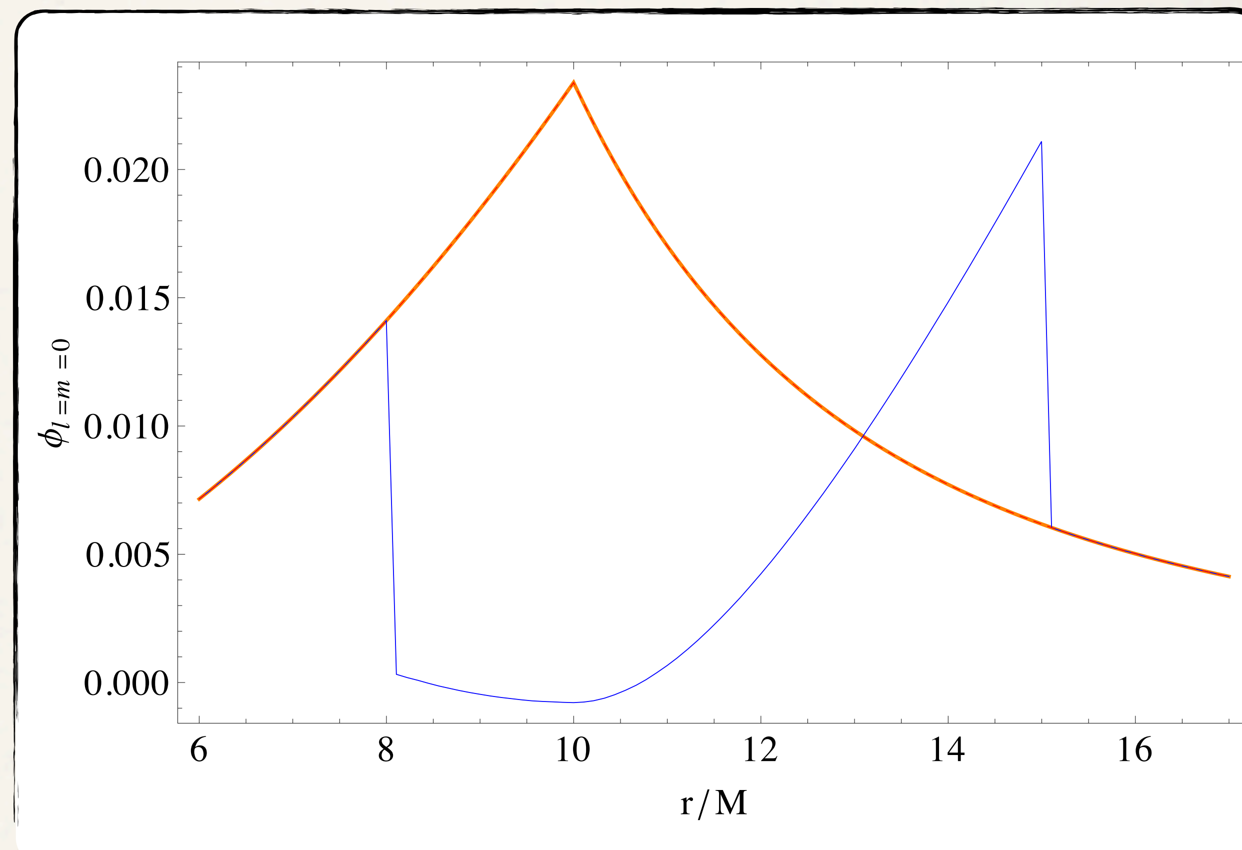
where W is the Wronskian of the homogeneous solutions.

Worked example - scalar, circular

l=0, *m*=0



l=2, *m*=0



Complete calculation done for circular orbits, all *l*, *m* modes using both window function and world tube. Recover correct values for regular field and self-force on world line.

Window function - world tube equivalence

- Detweiler-Whiting singular field defined through a Hadamard form Green function which is not defined globally.
- Need to introduce a method for restricting the singular field to a region near the particle. Two common approaches:

Window function

- Multiply the singular field by a function which is 1 at the particle and goes to 0 far away

$$\Box\Phi^{\mathrm{R}} = -\Box(W\Phi^{\mathrm{S}})$$

World tube

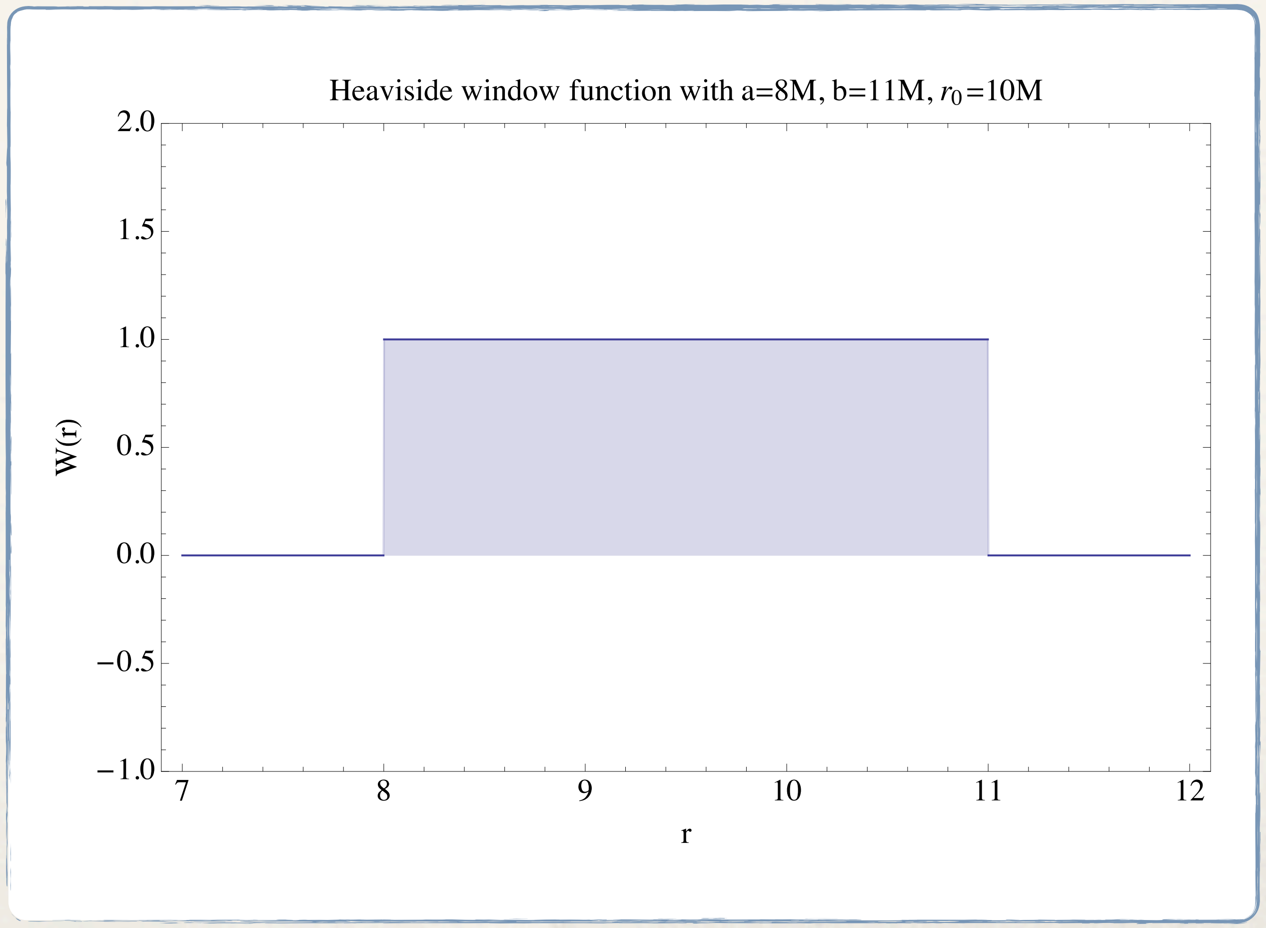
- World tube around the particle.
- Inside solve for the R field, outside solve for retarded field.
- On the world tube boundary, apply the boundary condition

$$\Phi^{\mathrm{ret}} = \Phi^{\mathrm{S}} + \Phi^{\mathrm{R}}$$

Window function - world tube equivalence

- Both approaches can be shown to be equivalent in frequency domain by choosing a Heaviside distribution as the window function

$$W(r) = \Pi\left(\frac{\Delta r - (b + a - 2r_0)/2}{b - a}\right) = \begin{cases} 1 & \left|\frac{\Delta r - (b+a-2r_0)/2}{b-a}\right| < 1/2 \\ 0 & \left|\frac{\Delta r - (b+a-2r_0)/2}{b-a}\right| > 1/2 \end{cases}$$



Window function - world tube equivalence

- Effective source splits into two terms, one coming from the interior of the puncture region and the other from the boundary of the puncture

$$S^{\text{eff}}_{lm} = -\Box_{lm}(\mathcal{W}\Phi^P_{lm}) \equiv S^I_{lm}\Pi(x) + S^B_{lm}$$

where

$$S^I_{lm} = -\frac{d^2\Phi^P_{lm}}{r^2} - \frac{2(r-M)}{fr^2}\frac{d\Phi^P_{lm}}{dr} + \frac{1}{f}\left(\frac{2}{f} + \frac{l(l+1)}{r^2}\right)\Phi^P_{lm}$$

$$S^B_{lm} = -\left[\frac{\delta'(x_a) + \delta'(-x_b)}{(b-a)^2} + \frac{2(r-M)(\delta(x_a) - \delta(x_b))}{fr^2(b-a)}\right]\Phi^P_{lm} - \frac{2(\delta(x_a) - \delta(x_b))}{b-a}\frac{d\Phi^P_{lm}}{dr}$$

$$x_a = \frac{a-r}{a-b}, \quad x_b = \frac{b-r}{a-b}$$

Window function - world tube equivalence

- Integrating the δ-function terms analytically, we find that the scaling coefficients are equivalent to world tube jumps

$$c^+(r) = \begin{cases} 0 & r < a \\ L_B(\phi^-/W) & a \le r < b \\ L_B(\phi^-/W) + R_B(\phi^-/W) & r \ge b \end{cases} + \Pi(x(r)) \int_a^r \frac{\tilde{\phi}^-}{W} S^I_{\text{eff}}\, dr$$

$$c^-(r) = \begin{cases} 0 & r > b \\ R_B(\phi^+/W) & b \ge r > a \\ L_B(\phi^+/W) + R_B(\phi^+/W) & r \le a \end{cases} + \Pi(x(r)) \int_r^b \frac{\tilde{\phi}^+}{W} S^I_{\text{eff}}\, dr$$

$$L_B[f(r)] = \int_{a^-}^{a^+} f(r) S^B_{\text{eff}}\, dr = \alpha(a) f(a) + \beta(a) f'(a)$$

$$R_B[f(r)] = \int_{b^-}^{b^+} f(r) S^B_{\text{eff}}\, dr = -\alpha(b) f(b) - \beta(b) f'(b)$$

$$\alpha(x) = -\frac{2(x-M)}{x(x-2M)} \Phi^P_{lm}(x) - \frac{d\Phi^P_{\ell m}}{dr}(x)$$

$$\beta(x) = \Phi^P_{lm\omega}(x)$$

Relation to mode-sum scheme

- Taking the limit of the world tube width to a point, i.e. $a \to r_0$, $b \to r_0$, we recover the familiar Barack-Ori mode sum regularization method.
- Effective source turns into jumps on the world line

$$c_0^{+R} \equiv L_B \left[\frac{\tilde{\phi}^-}{W}\right]_{a=r_0^-}, \qquad c_0^{-R} \equiv R_B \left[\frac{\tilde{\phi}^+}{W}\right]_{b=r_0^+}$$

$$c_0^{+S} \equiv R_B \left[\frac{\tilde{\phi}^-}{W}\right]_{a=r_0^\pm}, \qquad c_0^{-S} \equiv L_B \left[\frac{\tilde{\phi}^+}{W}\right]_{b=r_0^\pm}$$

- Recover standard mode-sum matching condition

$$c_0^{\pm} = c_0^{\pm R} + c_0^{\pm S} \alpha_{lm} \frac{\tilde{\phi}_0^{\mp}}{W_0}$$

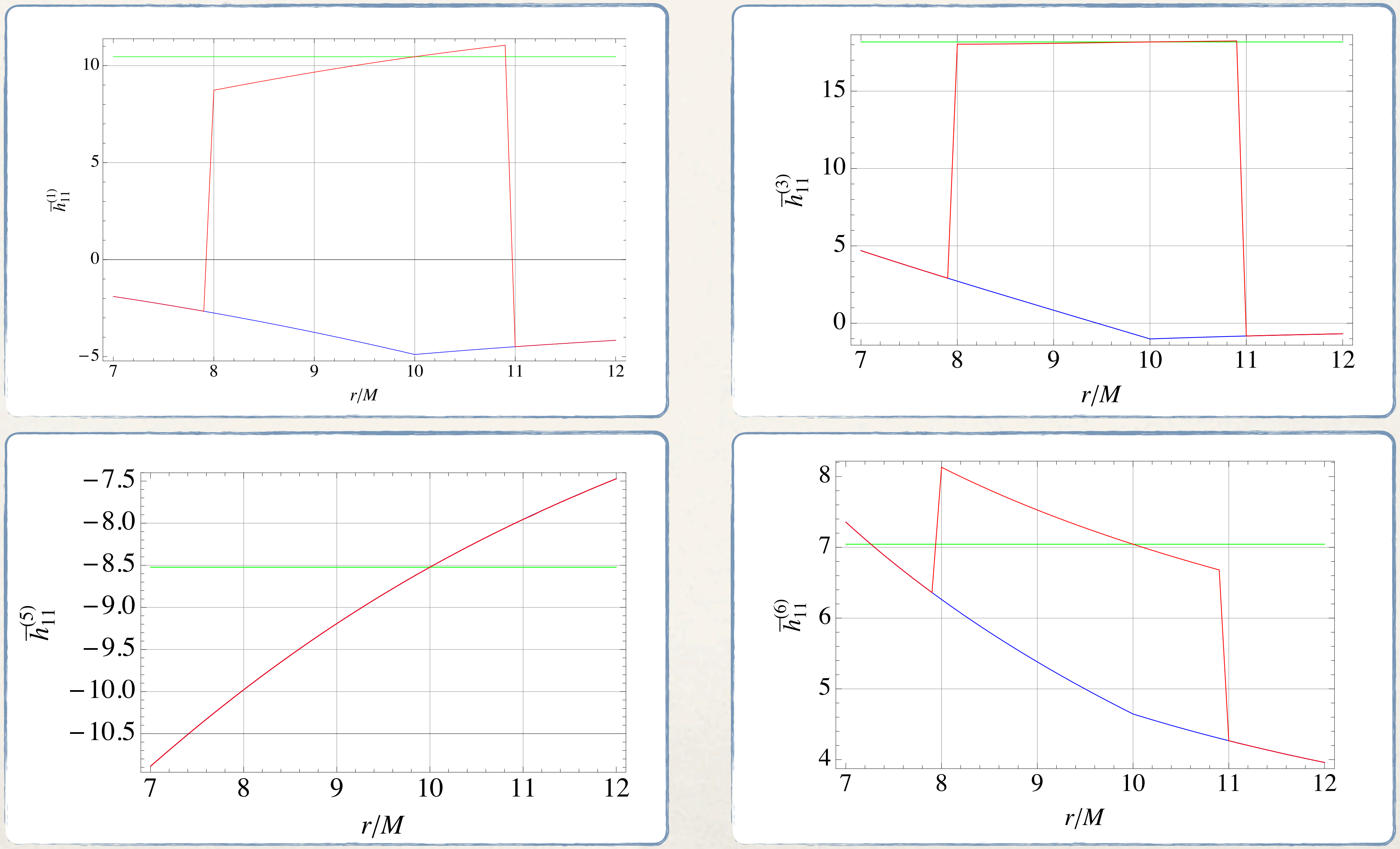
"Regularization parameters" and regularized field

$$\phi_0^S = c_0^{+S} \tilde{\phi}_0^+ + c_0^{-S}$$

$$\phi_0^R = c_0^{+R} \tilde{\phi}_0^+ + c_0^{-R} \tilde{\phi}_0^-$$

First order gravitational case

- Proceeds in exactly the same way apart from technical details (m ≤ 2, tensor harmonics, monopole, etc.)



First order gravitational case

- Work-in-progress (80% complete)
- Done so far:
  - All modes for "scalar" components: $h^{(1)}$, $h^{(3)}$, $h^{(6)}$
  - $h^{(4)}$ obtained from gauge conditions
  - $h^{(2)}$ zero, $h^{(9)}$, $h^{(5)}$ already regular (sourceless) in circular orbits case
  - Monopole with zero-width world tube
- To do:
  - "vector" and "tensor": $h^{(7)}$, $h^{(8)}$, $h^{(10)}$ - some factors missing
  - Check computed self-force, h.u.u
  - Monopole with extended world tube

Conclusions

- Applied effective source approach in frequency domain (and 1+1D).
- Obtained agreement with mode-sum calculations mode-by-mode to round-off (or better).
- Regularization of individual *l,m* modes including arbitrary number of derivatives.
- Complete calculation for scalar, “almost” done gravity.
- Tensor harmonic regularization parameters obtained by setting $\Delta r=0$.
- To do
  - Extension to second order - straightforward, but technical
  - Use in 1+1D evolution to assess “geodesic” approximation