Wave Tails in Minkowski and de Sitter space-times

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Preface

- **Embedding:** 4D Schwarzschild can be viewed as a curved surface in 6D Minkowski.
- **Question:** Can the Green's function of the graviton wave operator in a Schwarzschild geometry be computed from its 6D flat cousin?

$$\Box_x G[x, x'] = \Box_{x'} G[x, x'] = \frac{\delta^{(d)}[x - x']}{\sqrt[4]{|g[x]g[x']|}}$$

Preface

- **Causal structure:** In a curved spacetime, massless particles such as photons and gravitons do not travel solely on the light cone. They also propagate inside the light cone of their sources.
- **Nomenclature:** This inside-the-light-cone portion of the physical signal is called the *tail*.

Retarded Green's Function

- Causal structure of signals produced by physical sources is encoded in the **Green's function** of the appropriate wave operator.
- **Interpretation of G:** Field at observer location *x*, produced by space-time point source at *x'*.

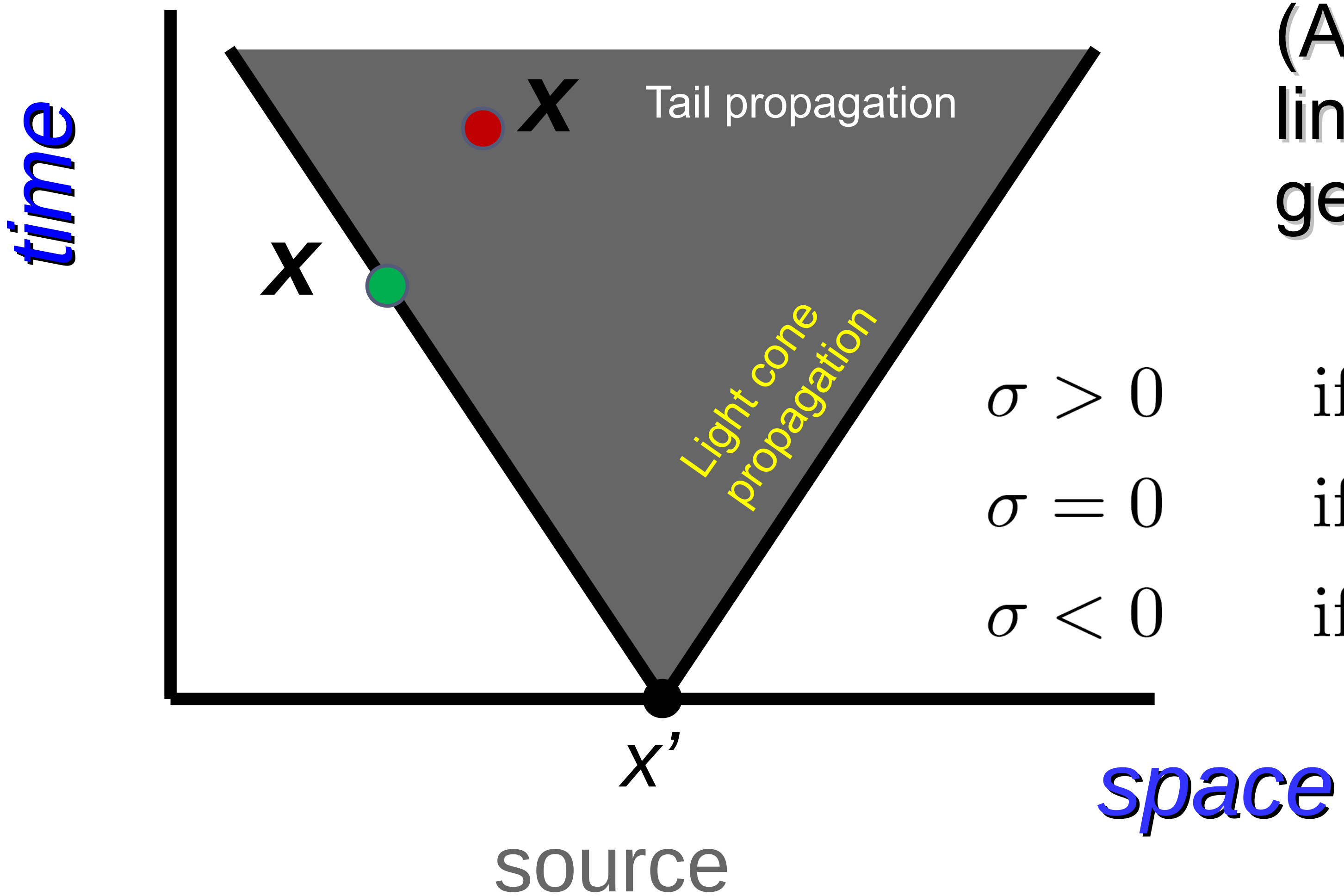
$$\Box_x \varphi[x] \equiv g^{\mu\nu}\nabla_\mu \nabla_\nu \varphi[x] = J[x]$$

$$\Box_x G[x,x'] = \Box_{x'} G[x,x'] = \frac{\delta^{(d)}[x-x']}{\sqrt[4]{|g[x]g[x']|}}$$

$$\varphi[x] = \int \mathrm{d}^d x' \sqrt{|g[x']|} G[x,x'] J[x']$$

(3+1)D Causal Structure of Retarded G

$$G[x,x'] = \frac{\Theta[t-t']}{4\pi}\left(\sqrt{\Delta_{x,x'}}\delta[\sigma_{x,x'}] + \Theta[\sigma_{x,x'}]V_{x,x'}\right)$$



(Assuming x, x' linked by unique geodesic.)

$$\sigma > 0 \qquad \text{if } x \leftrightarrow x' \text{ time-like}$$
$$\sigma = 0 \qquad \text{if } x \leftrightarrow x' \text{ null}$$
$$\sigma < 0 \qquad \text{if } x \leftrightarrow x' \text{ space-like}$$

Hadamard, (1923)

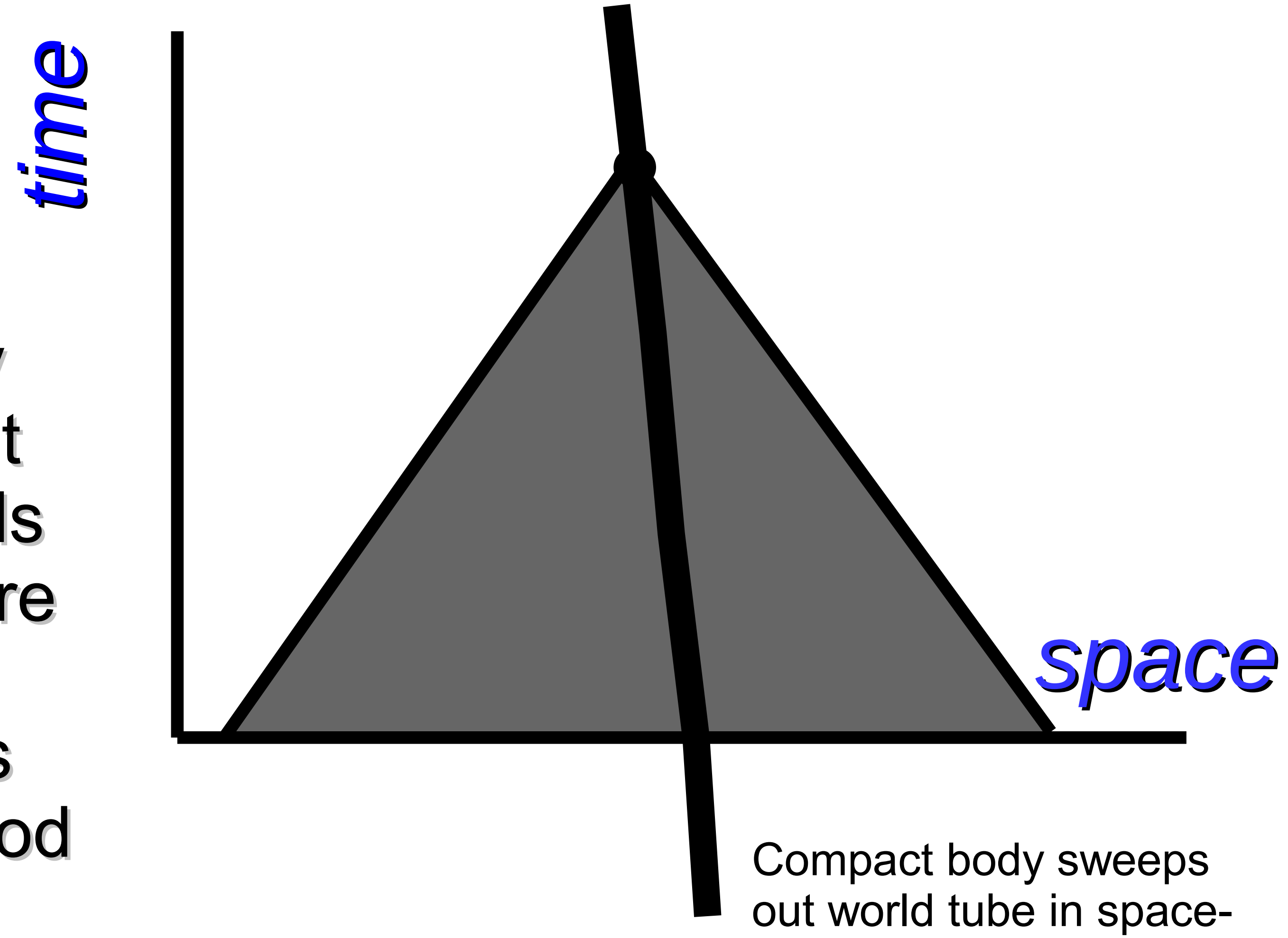
Motivation

- **Extreme-Mass-Ratio-Inspirals** (EMRIs): Solar mass compact objects orbit and subsequently plunge into super massive (M_{BH}>10^6 M_{sun}) Kerr black holes at the center of many galaxies, producing gravitational waves.
- **Testing GR** in the strong field regime: These GWs give detailed information regarding the geometry of Kerr BH.
- The dynamics of compact object of mass $m \ll M_{BH}$ in a BH geometry needs to be computed accurately to model GWs from EMRIs. In particular, there is a **tail induced self-force problem.**

See, for e.g., E. Poisson, arXiv: gr-qc/0410127

Motivation: EMRIs Self-Force Problem

- **Tail induced self-force:**
- In curved spacetime, motion of body at some instant of time depends on its own entire past history.
- Tail of G needs to be understood properly.

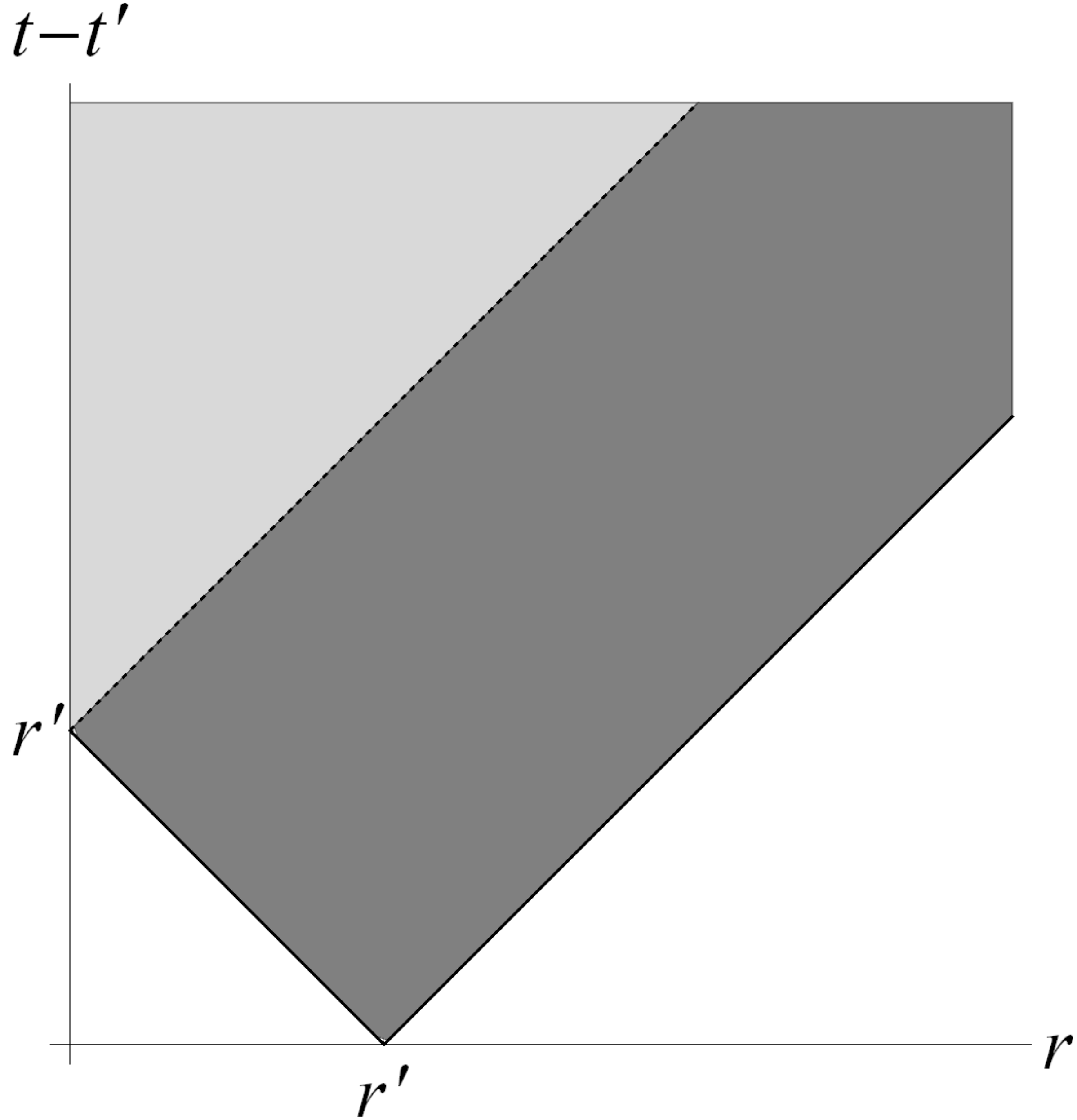


Compact body sweeps out world tube in space-time

B.S.DeWitt, R.W.Brehme, Annals Phys. 9, 220 (1960).

Motivation: Tails can be non-local

- Scalar, vector and tensor G can be computed in the **weak field limit of the Kerr geometry**, i.e., by replacing the BH with a spinning point mass.
- Up to first order in mass and spin, the tail portion of the graviton G breaks up into three distinct groups of terms, visualized on the right.



C.M.DeWitt, B.S.DeWitt,, Physics 1, 3 (1964).
M.J.Pfenning, E.Poisson, Phys. Rev. D 65, 084001 (2002) [gr-qc/0012057]
Y.Z.C., G.D.Starkman, Phys. Rev. D 84, 124020 (2011), [arXiv:1108.1825 [astro-ph.CO]].

Causal structure from embedding
Two Examples

- In **odd dimensional flat spacetime**, massless particles do not propagate strictly on the light cone.

  Embed d-Minkowski in (d+1)-Minkowski.

  No massless tails in even d > 2.

- Study causal structure of waves in **de Sitter** by embedding it in one higher dimensional Minkowski.

G from Embedding: Minkowski

- Line "charge" in D=*d*+1 sources G in D=*d*.
- 2 different charge densities from solutions to 2nd order eigenvector ODE: $\left(\mathcal{D}_y + m^2\right)\langle y|m^2\rangle = m^2\langle y|m^2\rangle$
- Yet yield the same G_d.

$$\overline{G}_d\left[X - X'\right] = \int_{-\infty}^{+\infty} \mathrm{d}X'^d \frac{X'^d}{X^d}\overline{G}_{d+1}\left[X - X'\right]$$

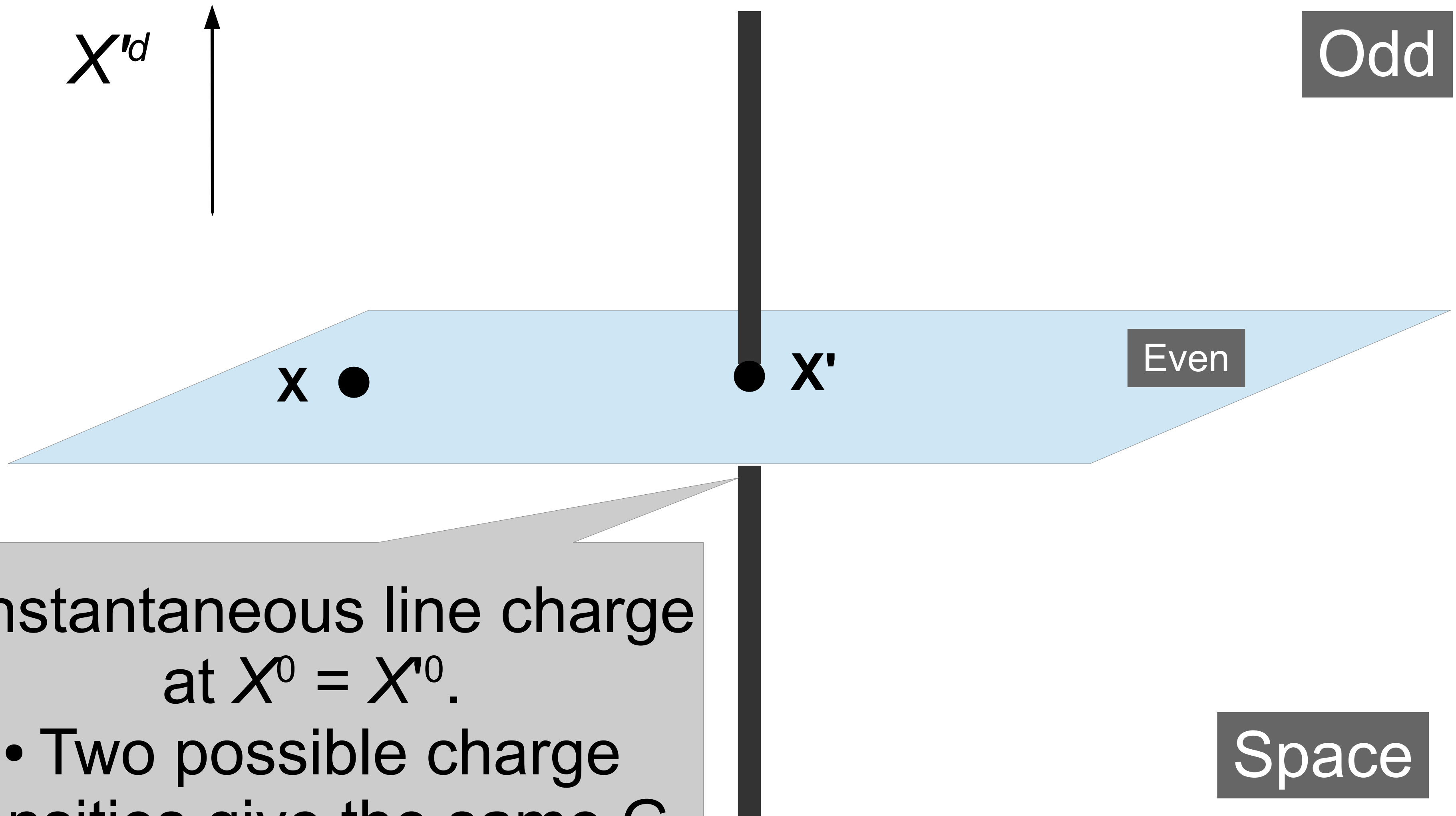
$$\overline{G}_d\left[X - X'\right] = \int_{-\infty}^{+\infty} \mathrm{d}X'^d\,\overline{G}_{d+1}\left[X - X'\right]$$

d-components (d+1)-components

H. Soodak and M. S. Tiersten, Am. J. Phys. 61 (5), May 1993
Y.-Z.C., arXiv:1305.6933, 1310.2939

G from Embedding: Minkowski

- Line "charge" in even D sources G in odd D.

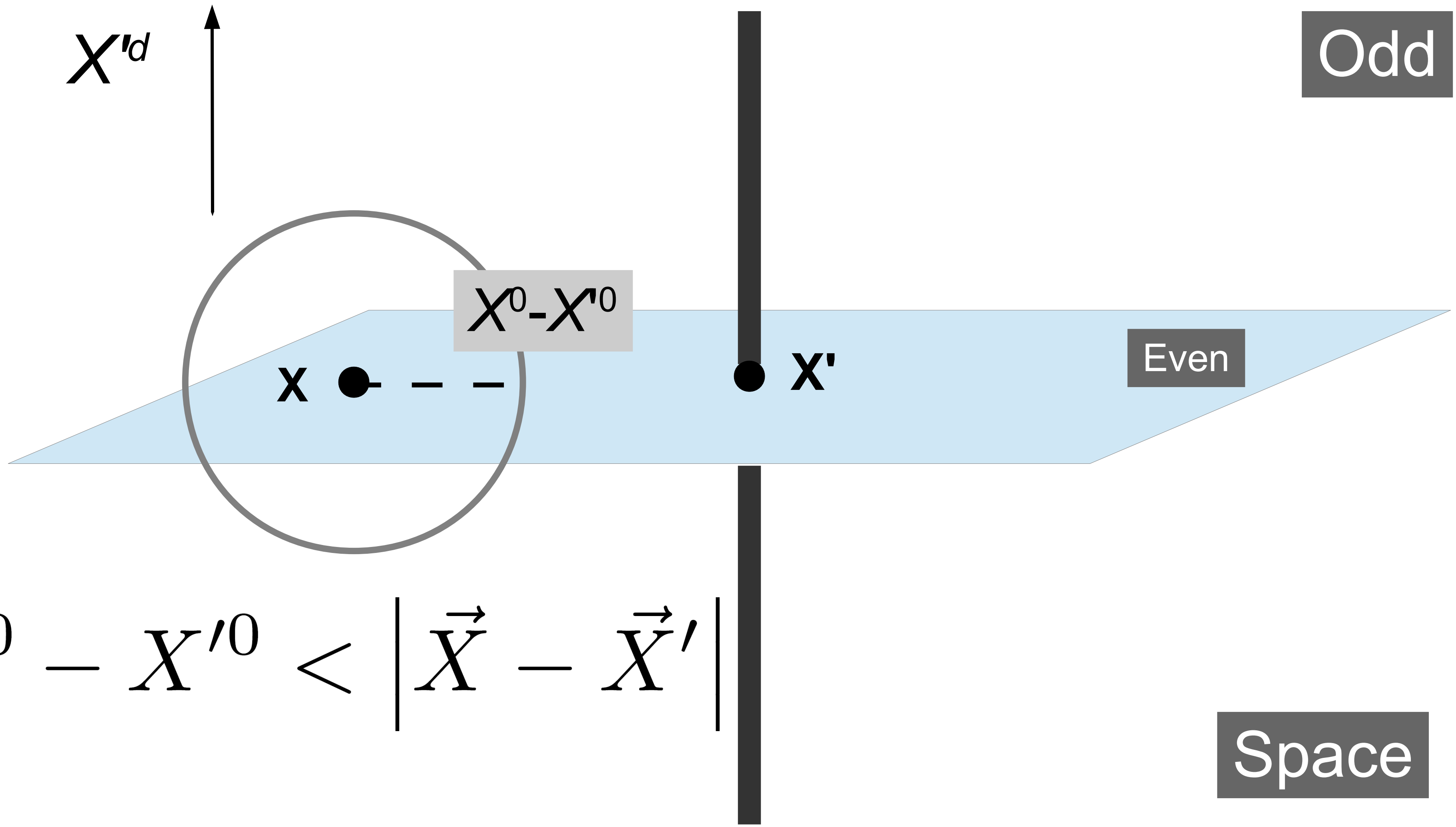


- Instantaneous line charge at $X^0 = X'^0$.
- Two possible charge densities give the same G.

H. Soodak and M. S. Tiersten, Am. J. Phys. 61 (5), May 1993
Y.-Z.C., arXiv:1305.6933, 1310.2939

Massless tails in odd D Minkowski

- No tails in even D(=4,6,8,...) Minkowski.



$$X^0 - X'^0 < \left|\vec{X} - \vec{X}'\right|$$

H. Soodak and M. S. Tiersten, Am. J. Phys. 61 (5), May 1993
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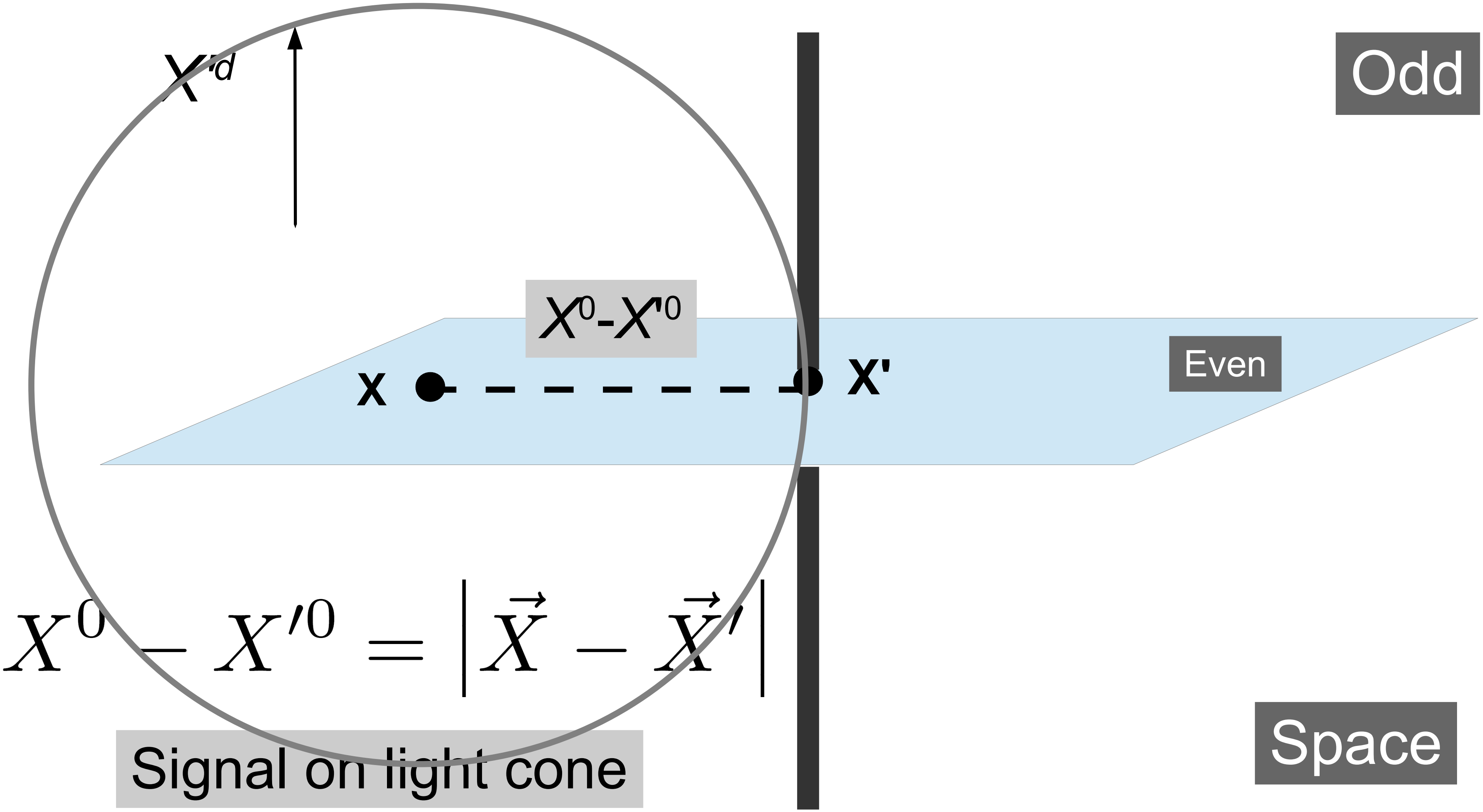

$$X^0 - X'^0 = \left| \vec{X} - \vec{X}' \right|$$

H. Soodak and M. S. Tiersten, Am. J. Phys. 61 (5), May 1993
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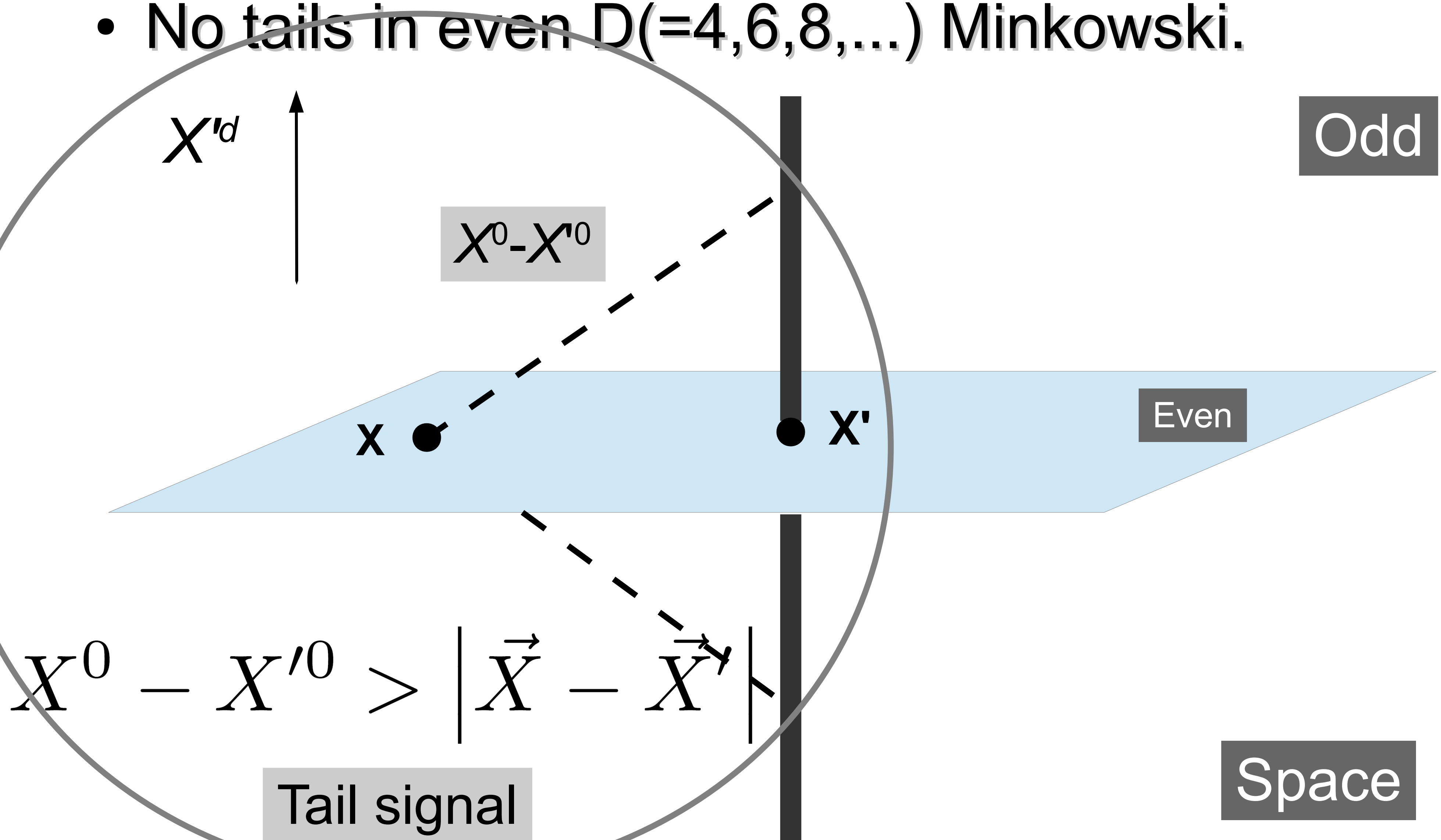


$$X^0 - X'^0 > \left|\vec{X} - \vec{X}'\right|$$

H. Soodak and M. S. Tiersten, Am. J. Phys. 61 (5), May 1993
Y.-Z.C., arXiv: 1305.6933, 1310.2939

G from Embedding: Minkowski

- Recursion relation follows from spatial translation symmetry:

$$\overline{G}_d\left[X - X'\right] = \int_{-\infty}^{+\infty} \mathrm{d}X'^d \, \overline{G}_{d+1}\left[X - X'\right]$$

$$\overline{G}_d\left[X - X'\right] = \int_{-\infty}^{+\infty} \mathrm{d}X'^{d-1} \int_{-\infty}^{+\infty} \mathrm{d}X'^d \, \overline{G}_{d+2}\left[X - X'\right]$$

$$\overline{G}_{d+2}\left[X - X'\right] = -\frac{1}{2\pi R} \frac{\partial \overline{G}_d\left[X - X'\right]}{\partial R}, \qquad R \equiv \left|\vec{X} - \vec{X}'\right|$$

H. Soodak and M. S. Tiersten, Am. J. Phys. 61 (5), May 1993
Y.-Z.C., arXiv:1305.6933, 1310.2939

G from Embedding: Minkowski

- G in Minkowski can be generated from d=2,3:

$$\overline{G}_d^{\pm}[X - X'] = \int_{\pm} \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{e^{-ik\cdot(X-X')}}{-k^2 + m^2}$$

$$\overline{G}_{d+2}[X - X'] = -\frac{1}{2\pi R} \frac{\partial \overline{G}_d [X - X']}{\partial R}$$

H. Soodak and M. S. Tiersten, Am. J. Phys. 61 (5), May 1993
Y.-Z.C., arXiv:1305.6933, 1310.2939

G from Embedding: Minkowski

- No massless tails for d=4,6,8,...; massless tails in all odd dimensions:

$$\overline{G}_d^{\pm}[X - X'] = \Theta[\pm(X^0 - X'^0)]\overline{\mathcal{G}}_d[\bar{\sigma}]$$

$$\overline{\mathcal{G}}_{d\ \mathrm{even}}[\bar{\sigma}] = \frac{1}{2(2\pi)^{\frac{d-2}{2}}}\left(\frac{\partial}{\partial\bar{\sigma}}\right)^{\frac{d-2}{2}}\Theta[\bar{\sigma}],$$

$$\overline{\mathcal{G}}_{d\ \mathrm{odd}}[\bar{\sigma}] = \frac{1}{(2\pi)^{\frac{d-1}{2}}}\left(\frac{\partial}{\partial\bar{\sigma}}\right)^{\frac{d-3}{2}}\left(\frac{\Theta[\bar{\sigma}]}{\sqrt{2\bar{\sigma}}}\right),$$

$$\bar{\sigma} \equiv \frac{1}{2}(X - X')^2 \equiv \frac{1}{2}\eta_{\mathfrak{A}\mathfrak{B}}(X - X')^{\mathfrak{A}}(X - X')^{\mathfrak{B}}.$$

H. Soodak and M. S. Tiersten, Am. J. Phys. 61 (5), May 1993
Y.-Z.C., arXiv:1305.6933, 1310.2939

G from Embedding: de Sitter

- De Sitter (dS) spacetime in 4D cosmology is associated with an exponentially expanding universe, with expansion rate *H*.
- From embedding perspective *d*-dim. dS with Hubble parameter *H* is defined is a hyperboloid in (*d*+1)-dim. Minkowski.

$$-\eta_{\mathfrak{A}\mathfrak{B}} X^{\mathfrak{A}} X^{\mathfrak{B}} \equiv -X^2 = \frac{1}{H^2}$$

G from Embedding: de Sitter

- Foliate ambient (d+1)-Minkowski (outside l.c. of 0^A) with d-dimensional de Sitter. hyperboloids.

$$X^{\mathfrak{A}}[\rho, \tau, \vec{\theta}] = \rho\left(\sinh[\tau], \cosh[\tau]\widehat{n}[\vec{\theta}]\right)$$

- Massive G_d in dS is sourced by line charge in (d+1)-Minkowski:

$$G_d[x, x'] = \int_0^\infty \mathrm{d}\rho' \, (H\rho')^{\frac{d-3}{2}} \frac{I_{\nu_\pm}[m\rho']}{I_{\nu_\pm}[m/H]} \overline{G}_{d+1}\left[X[\rho = H^{-1}, x] - X'[\rho', x']\right]$$

- Two possible charge densities from 2nd order eigenvector ODE gives the same G_d:

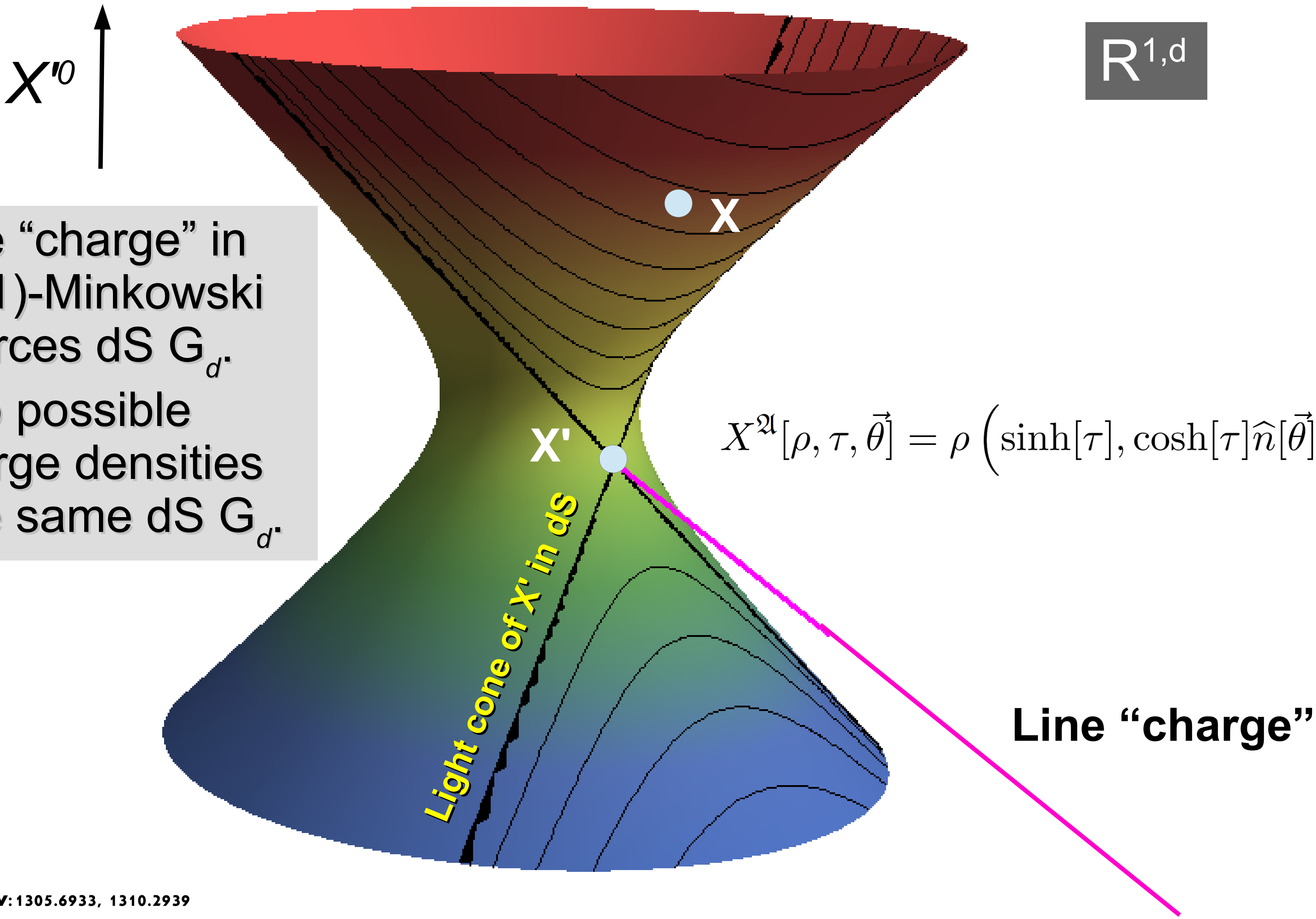
$$\nu_\pm \equiv \pm\sqrt{\left(\frac{d-1}{2}\right)^2 - \left(\frac{m}{H}\right)^2},$$

$$\left(\mathcal{D}_\rho + (\rho m)^2\right)\langle y|(m/H)^2\rangle = (m/H)^2\langle y|(m/H)^2\rangle$$

Y.-Z.C., arXiv: 1305.6933, 1310.2939

G from Embedding: de Sitter

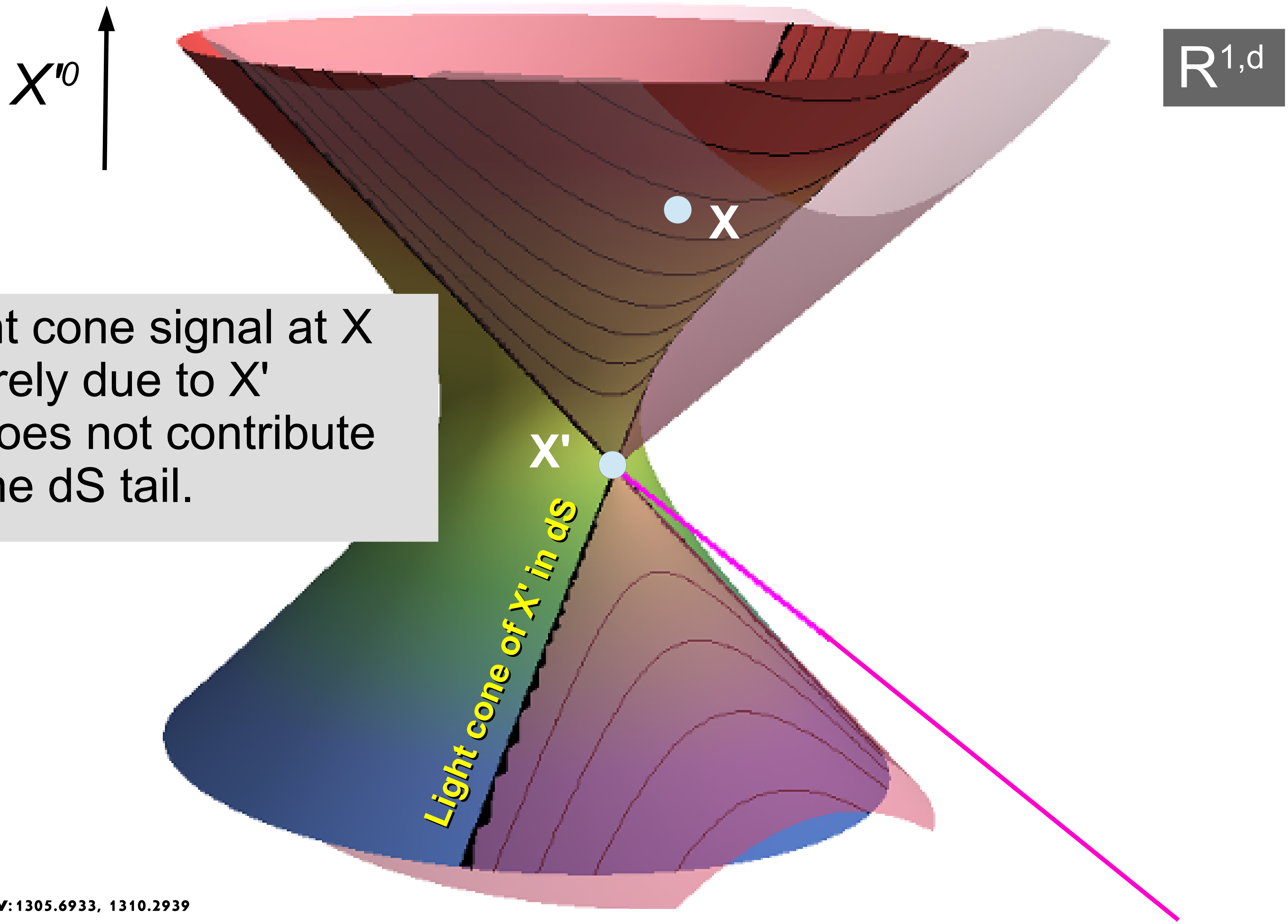
- Line "charge" in (d+1)-Minkowski sources dS G_d.
- Two possible charge densities give same dS G_d.



$$X^{\mathfrak{A}}[\rho, \tau, \vec{\theta}] = \rho \left(\sinh[\tau], \cosh[\tau] \widehat{n}[\vec{\theta}] \right)$$

Y.-Z.C., arXiv: 1305.6933, 1310.2939

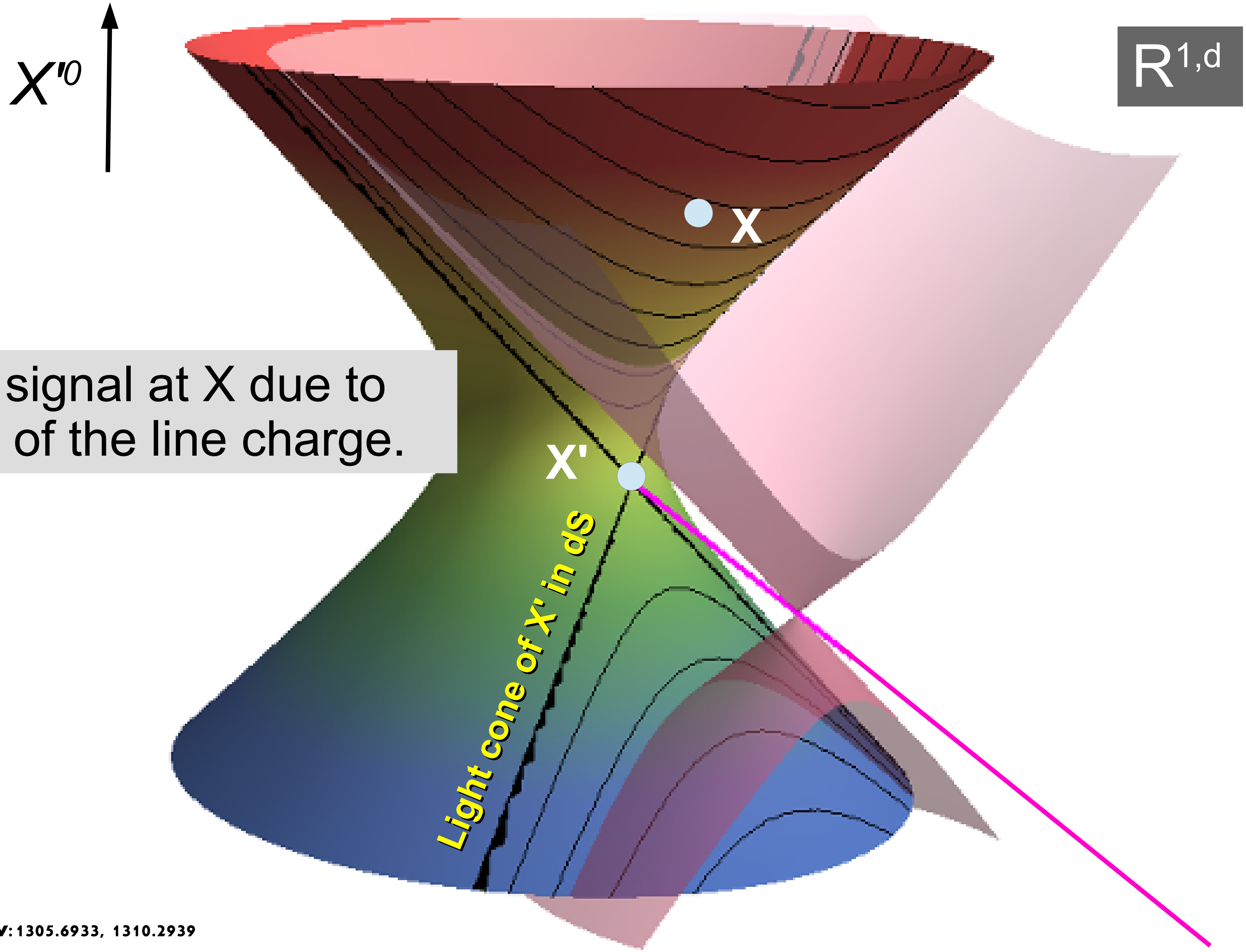
Causal structure: de Sitter



- Light cone signal at X entirely due to X'
- X' does not contribute to the dS tail.

Y.-Z.C., arXiv: 1305.6933, 1310.2939

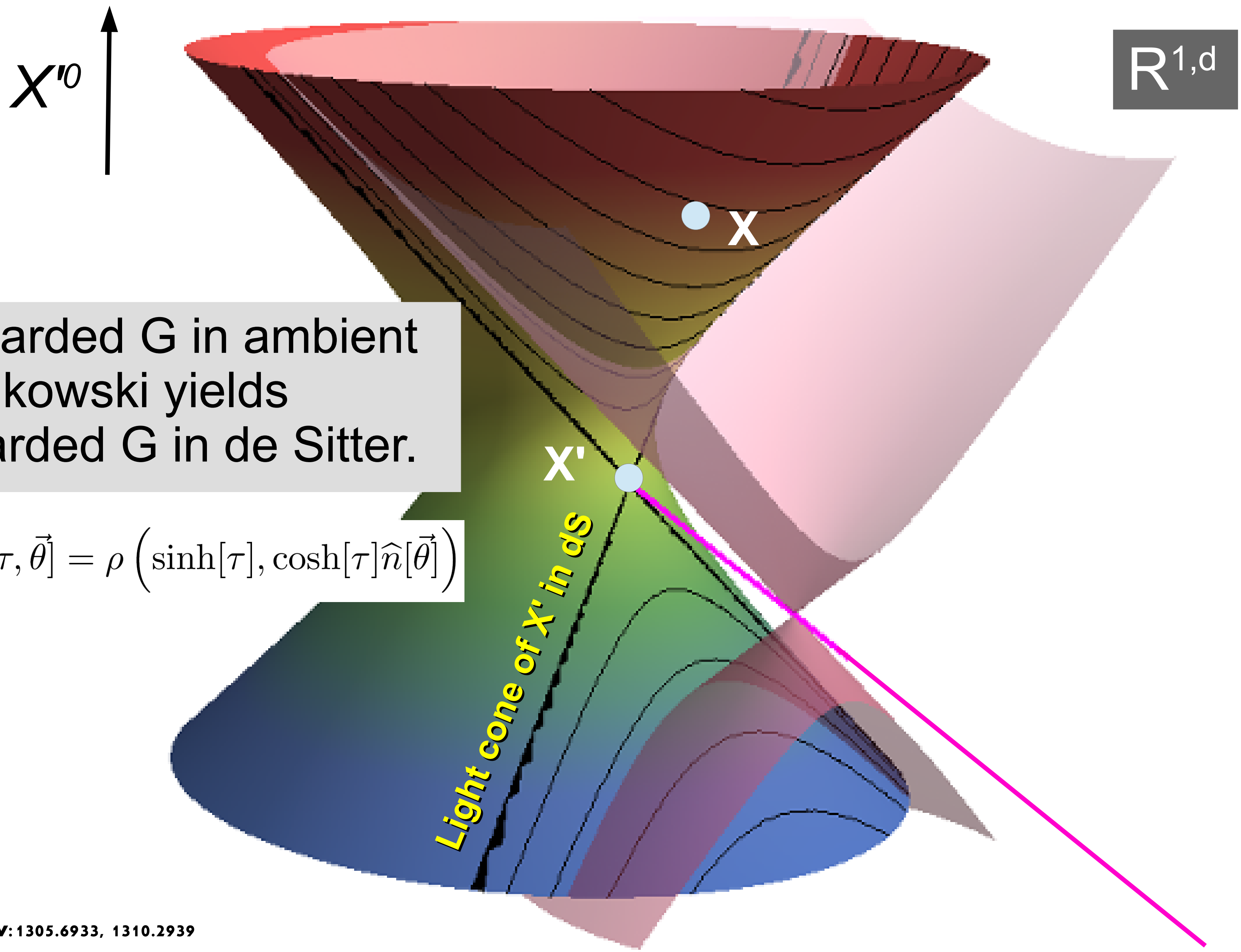
Causal structure: de Sitter



- Tail signal at X due to rest of the line charge.

Y.-Z.C., arXiv: 1305.6933, 1310.2939

Causal structure: de Sitter

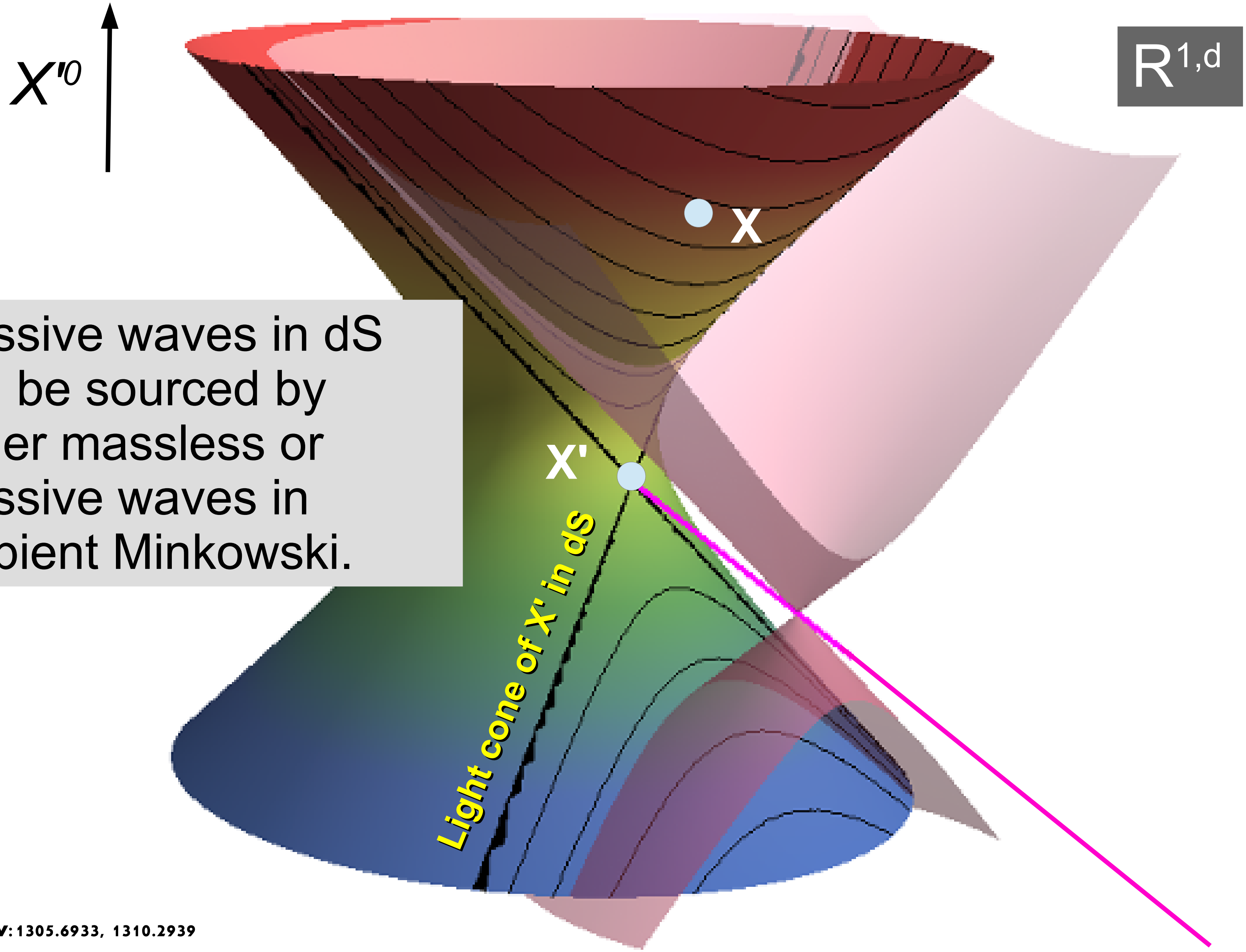


- Retarded G in ambient Minkowski yields retarded G in de Sitter.

$$X^{\mathfrak{A}}[\rho, \tau, \vec{\theta}] = \rho \left(\sinh[\tau], \cosh[\tau] \widehat{n}[\vec{\theta}] \right)$$

Y.-Z.C., arXiv: 1305.6933, 1310.2939

Causal structure: de Sitter



- Massive waves in dS can be sourced by either massless or massive waves in ambient Minkowski.

Y.-Z.C., arXiv: 1305.6933, 1310.2939

G from Embedding: de Sitter

- In "closed slicing" parametrization, massive G_d in d-dimensional dS is:

$$G_d^{(\text{Closed}|\pm)}[x,x'] = \Theta[\pm(\tau-\tau')]\mathcal{G}_d[x,x']$$

$$\mathcal{G}_{\text{even }d}[x,x'] = \frac{\pi H^{d-2}}{(2\pi)^{\frac{d}{2}}}\left(-\frac{\partial}{\partial Z}\right)^{\frac{d-2}{2}}\left(\Theta[-Z-1]P_{\nu-\frac{1}{2}}[-Z]\right)$$

$$\mathcal{G}_{\text{odd }d}[x,x'] = \frac{H^{d-2}}{(2\pi)^{\frac{d-1}{2}}}\left(-\frac{\partial}{\partial Z}\right)^{\frac{d-3}{2}}\left(\frac{\Theta[-Z-1]}{\sqrt{Z^2-1}}\cosh\left[\nu\ln\left[-Z+\sqrt{Z^2-1}\right]\right]\right)$$

$$\nu \equiv \sqrt{\left(\frac{d-1}{2}\right)^2-\left(\frac{m}{H}\right)^2} \qquad Z[x,x'] \equiv \left(H^2 X[\rho,x]\cdot X'[\rho',x']\right)\Big|_{\rho=\rho'=H^{-1}}$$

- dS light cone: $Z = -1$; inside light cone: $Z < -1$.
- Tail terms can be read off readily.

Y.-Z.C., arXiv: 1305.6933, 1310.2939

4D de Sitter Universe

- Massless limit in even D de Sitter leads to a constant G_d tail.

$$\mathcal{G}^{(\text{Tail})}_{\text{even } d}[x,x'] = \frac{\pi\Theta[-Z-1]}{(2\pi)^{\frac{d}{2}}} \frac{(d-2)!}{2^{\frac{d-2}{2}}\left(\frac{d-2}{2}\right)!}$$

- TT GWs in 4D spatially flat FLRW obey massless scalar eqn. w.r.t. bg. FLRW metric.

- This may lead to a pseudo-memory effect for TT GWs in 4D de Sitter universe. (Y.-Z.C., unpublished)

$$g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = a^2[\eta]\left(\eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu + D^{(\text{TT})}_{ij}\mathrm{d}x^i\mathrm{d}x^j\right)$$

4D de Sitter Universe

- TT GWs in 4D spatially flat FLRW obey massless scalar eqn. w.r.t. bg. FLRW metric.
- This may lead to a pseudo-memory effect for TT GWs in 4D de Sitter universe.
- Asymptotics: at late times, $\eta_{\mu\nu}$ suffers a permanent "DC" shift proportional to spacetime volume integral of $T_{ij}^{(TT)}$.

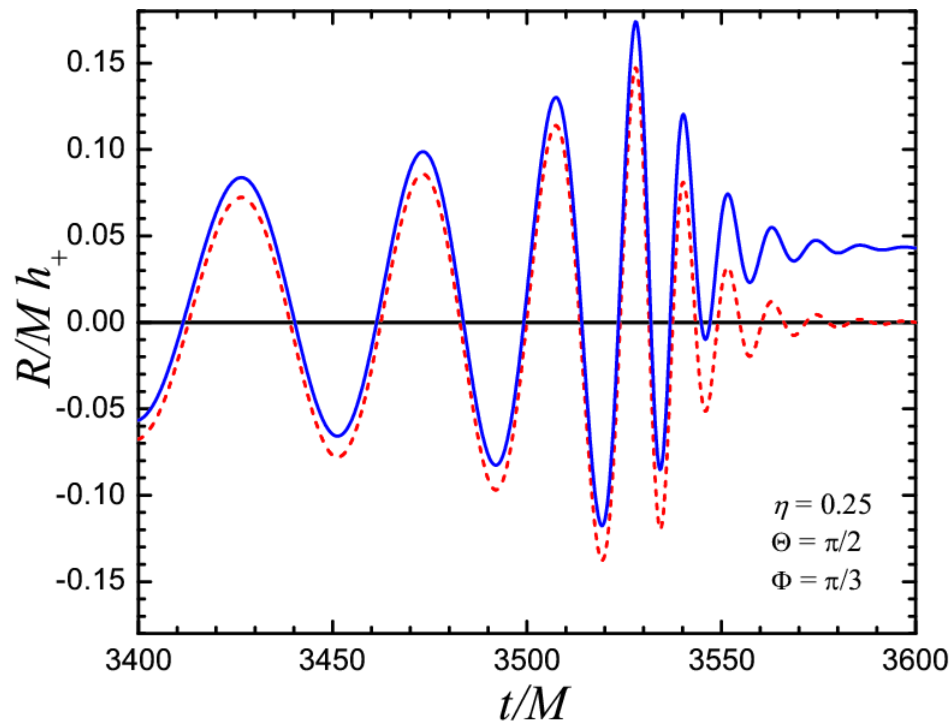


Fig. 1 of M.Favata, Class. Quant. Grav. 27, 084036 (2010) [arXiv:1003.3486 [gr-qc]].

$$g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = a^2[\eta]\left(\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + D_{ij}^{(\mathrm{TT})}\mathrm{d}x^i \mathrm{d}x^j\right)$$

Schwarzschild G From 6D Flat G?

- **Heuristics**
- Is there a (probably 2D) source that the 6D Minkowski experimentalists could set up, to fool the observers living in 4D Schwarzschild spacetime that they are detecting the field of a point source at x' in their world?