High-precision (MST) solutions for eccentric orbits on Schwarzschild: Part 1

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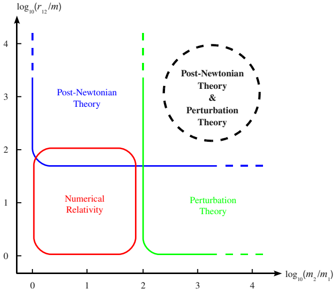
Theoretical motivation

- Three theoretical approaches to compact binary inspiral
  - GSF/perturbation theory
  - PN theory
  - Numerical relativity
- Long history of fruitful comparisons with GSF/perturbation theory & PN theory
  - Poisson (1993)
  - Poisson and Sasaki (1995)
  - Multiple papers by Sasaki, Tagoshi, Tanaka, Shibata, Takasugi, Mano (mid-1990s)
  - Detweiler (2008)
  - Blanchet, Detweiler, Le Tiec, and Whiting (2010,2011)
  - Fujita (2012)
  - Shah, Friedman, and Whiting (2013)
  - Shah (2014)



From: Blanchet, Detweiler, Le Tiec, and Whiting (2011)

Goal: Extend to eccentric orbits on Schwarzschild

Our immediate goal

- Immediate goal: compare eccentric orbit perturbation theory with PN theory

$$\left\langle \frac{dE}{dt} \right\rangle = \frac{32}{5} \left(\frac{\mu}{M} \right)^2 x^5 \left(\mathcal{I}_0 + x\,\mathcal{I}_1 + x^{3/2}\,\mathcal{K}_{3/2} + x^2\,\mathcal{I}_2 + x^{5/2}\,\mathcal{K}_{5/2} + x^3\,\mathcal{I}_3 + x^3\,\mathcal{K}_3 \right)$$

where

$$\mathcal{I}_0 = \frac{1}{(1-e_t^2)^{7/2}} \left(1 + \frac{73}{24}\ e_t^2 + \frac{37}{96}\ e_t^4 \right)$$

$$\mathcal{I}_1 = \frac{1}{(1-e_t^2)^{9/2}} \left(-\frac{1247}{336} + \frac{10475}{672} e_t^2 + \frac{10043}{384} e_t^4 + \frac{2179}{1792} e_t^6 \right)$$

etc

- See Part 2 (Erik Forseth) for details

Start with the Teukolsky equation

- Work with the homogeneous Teukolsky equation ($a = 0$)

$$\left[r^2 f \frac{d^2}{dr^2} - 2(r - M) \frac{d}{dr} + U_{l\omega}(r) \right] R_{lm\omega}(r) = 0,$$

for modes $R_{lm\omega}(r)$ of ψ_4

$$\psi_4 = \frac{1}{r^4} \sum_{lm} \int_{-\infty}^{+\infty} d\omega \, e^{-i\omega t} \, R_{lm\omega}(r) \, {}_{-2}Y_{lm}(\theta, \varphi),$$

- Seek purely downgoing, $R^{\rm in}_{lm\omega}$, and purely outgoing, $R^{\rm up}_{lm\omega}$, solutions

$$R^{\rm in}_{lm\omega}(r) \sim B^{\rm trans}_{lm\omega} r^4 f^2 \, e^{-i\omega r_*} \qquad r \to 2M$$

$$R^{\rm up}_{lm\omega}(r) \sim C^{\rm trans}_{lm\omega} r^3 \, e^{i\omega r_*} \qquad r \to +\infty$$

Use the MST formalism

- Use the Mano, Suzuki, and Takasugi (1996) (MST) method
- Obtain analytic function expansions of $R^{\text{in}}_{lm\omega}$ and $R^{\text{up}}_{lm\omega}$

- $R^{\text{in}}_{lm\omega}$ is an expansion in hypergeometric functions $p_{n+\nu}$

$$R^{\text{in}}_{lm\omega} = e^{-i\omega r_*}\, r^2\, k(r) \sum_{n=-\infty}^{\infty} a_n p_{n+\nu}(r)$$

- The outer solution starts with an expansion in Coulomb wave functions $F_{n+\nu}$

$$R_C^{lm\omega} = r\, h(r) \sum_{n=-\infty}^{\infty} \mathcal{M}(n,\nu)\, a_n F_{n+\nu}(r)$$

- Then R_C is split into $R_C = R_+ + R_-$ (irregular confluent hypergeometric functions) with

$$R^{\text{up}}_{lm\omega} = R_- \sim A_- r^3\, e^{i\omega r_*}$$

Next, find the minimal solution

- The coefficients a_n in the expansions for $R^{\text{in}}_{lm\omega}$ and $R^{\text{up}}_{lm\omega}$ have the same recursion relation
- The MST method introduces a free parameter, the renormalized angular momentum ν
- The minimal solution for a_n (convergent as $n \to +\infty$ and $n \to -\infty$) requires a certain special value for ν
- ν is found as the root of a continued fraction equation
- Well understood procedure [see Sasaki & Tagoshi (2006), Fujita & Tagoshi (2004,2005), Fujita, Hikida, and Tagoshi (2009), Hughes and Throwe (priv comm), Shah, Friedman, and Whiting (2013), Shah (2014)]

Go to Regge-Wheeler and Zerilli master functions

- Skip solving the inhomogeneous Teukolsky equation

- Use the Detweiler-Chandrasekhar transformation (1975) from Teukolsky $R_{lm\omega}$ to Regge-Wheeler $X^{\mathrm{RW}}_{lm\omega}$ master functions

$$X^{\mathrm{RW}}_{lm\omega} = r^3 \left[\left(\frac{d}{dr} \right) - \frac{i\omega}{f} \right]^2 \left(\frac{1}{r^2} R_{lm\omega} \right)$$

  Gives the odd-parity mode function, i.e., $l + m = \text{odd}$

- Even parity: use the second Detweiler-Chandrasekhar transformation

$$X^{\mathrm{Z}}_{lm\omega} = \frac{\left[\lambda(\lambda+1) + \frac{9M^2 f}{r[\lambda r + 3M]} \right] X^{\mathrm{RW}}_{lm\omega} + 3Mf \frac{dX^{\mathrm{RW}}_{lm\omega}}{dr}}{[\lambda(\lambda+1) - 3i\omega M]}, \quad l(l{+}1) = 2(\lambda{+}1)$$

Provides analytic function expansions for $\hat{X}^{\pm}_{lm\omega}$ (odd/even parity)

Usual match across the source

- Summary of standard approach–with extended homogeneous solutions (EHS) [Barack & Sago (2008), Hopper & Evans (2010)]

- Master functions satisfy CPM and ZM equations

$$\left[-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial r_*^2} - V_l(r)\right] \Psi_{lm}(t,r) = S_{lm}(t,r),$$

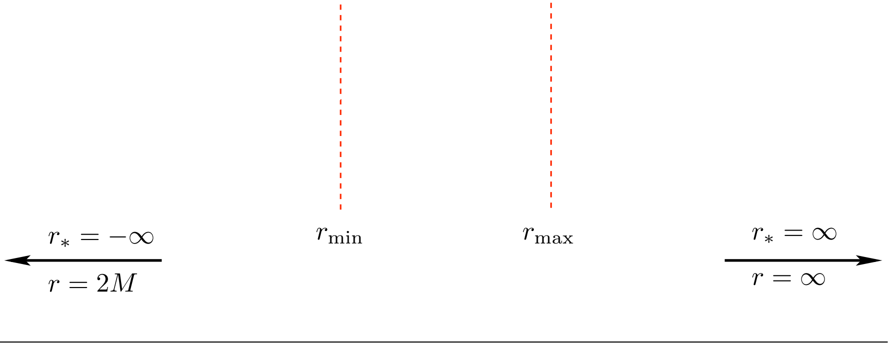
with $\boxed{S_{lm}(t,r) = G_{lm}(t)\ \delta[r - r_p(t)] + F_{lm}(t)\ \delta'[r - r_p(t)]}$

- Fourier transform field and source; seek solution to inhomogeneous equation

$$\left[\frac{d^2}{dr_*^2} + \omega_{mn}^2 - V_l(r)\right] X_{lmn}(r) = Z_{lmn}(r)$$

$$\omega_{mn} \equiv m\Omega_\varphi + n\Omega_r$$

Standard (non-EHS) Green function solution



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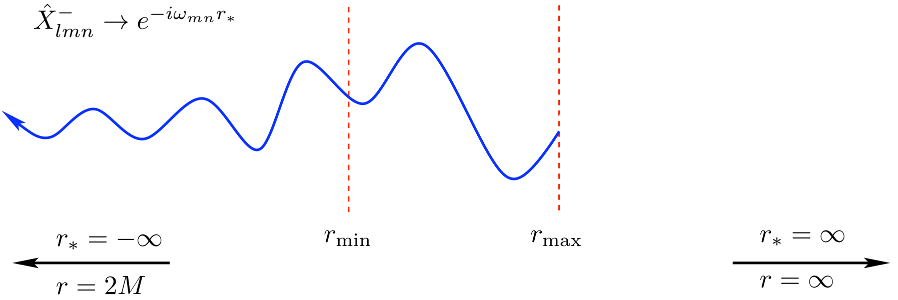

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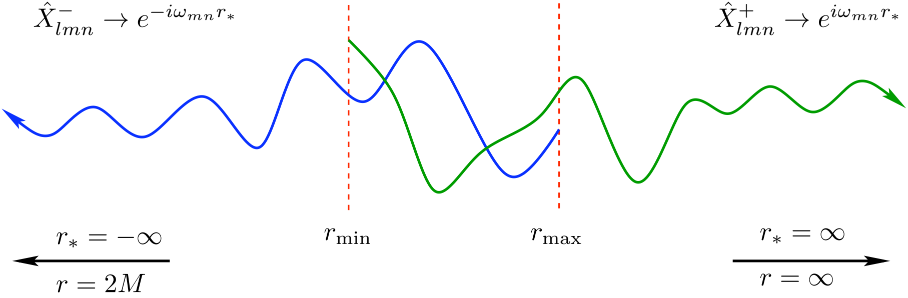

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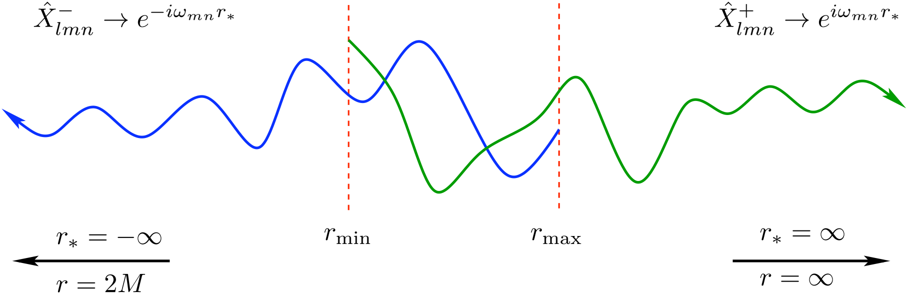
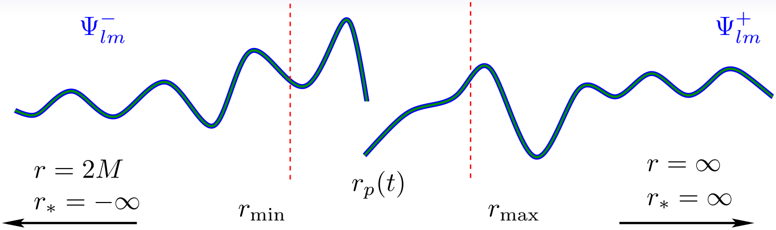


$$X^{\rm std}_{lmn}(r) = \hat{X}^{+}_{lmn}(r) \int_{r_{\min}}^{r} \frac{\hat{X}^{-}_{lmn}(r') Z_{lmn}(r')}{f(r') W_{lmn}} dr' + \hat{X}^{-}_{lmn}(r) \int_{r}^{r_{\max}} \frac{\hat{X}^{+}_{lmn}(r') Z_{lmn}(r')}{f(r') W_{lmn}} dr'$$

Standard EHS method



- Use Green function to find normalization coefficients

$$\boxed{C^{\pm}_{lmn} = W^{-1}_{lmn} \int_{r_{\min}}^{r_{\max}} \hat{X}^{\mp}_{lmn}(r') Z_{lmn}(r') dr'/f(r')}$$

- Then the TD extended homogeneous solutions

$$X^{\pm}_{lmn} = C^{\pm}_{lmn} \hat{X}^{\pm}_{lmn}, \qquad \Psi^{\pm}_{lm}(t,r) = \sum_n X^{\pm}_{lmn} e^{-i\omega_{mn} t}$$

$$\Psi^{\rm EHS}_{lm}(t,r) = \Psi^{+}_{lm}(t,r)\,\theta\left[r - r_p(t)\right] + \Psi^{-}_{lm}(t,r)\,\theta\left[r_p(t) - r\right]$$

The difficulty with combining MST and EHS

- Applying high-precision MST expansion for $\hat{X}^{\pm}_{lmn}$ encounters difficulties computing $C^{\pm}_{lmn}$

$$C^{\pm}_{lmn} = W^{-1}_{lmn} \int_{r_{\min}}^{r_{\max}} \hat{X}^{\mp}_{lmn}(r') Z_{lmn}(r') dr' / f(r')$$

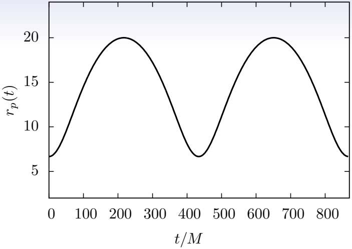
- $Z_{lmn}(r)$ is not smooth
- An ODE solver would invoke a tremendous number of function calls

- S. Hughes uses MST expansions in his radiative approximation code, but only to skirt long integrations in external regions

Use of MST in ODE integration will devolve to accuracy of ODE solver and be very costly

Why it should be possible to use MST

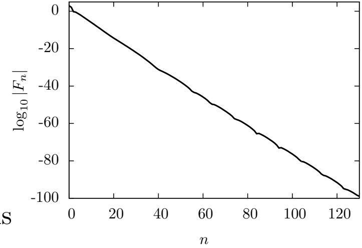
- Orbital motion is periodic and **smooth**
  e.g., $r = r_p(t)$



- Consider $dt/d\chi = I(\chi)$
- MST code needs $t = t(\chi)$ to high precision
- ODE integrators struggle at 10^{-40}

- Instead recognize as a quadrature with periodic integrand

$$t(\chi) = \int_0^\chi I(\chi')d\chi'$$



- Periodicity $\implies$ plane-wave basis functions

Spectrum falls exponentially
Spectral solution is trivial and arbitrarily precise

Fourier (spectral) representation of TD source terms

- Set a precision goal $\mathcal{P}$ (digits) or fractional error $\epsilon_g \simeq 10^{-\mathcal{P}}$
- $T^{\mu\nu}$ source terms/derived quantities have amplitudes that are smooth functions of time [e.g., $G_{lm}(t)$ and $F_{lm}(t)$]
- Non-periodic factor: source functions advance in phase, $e^{-im\Omega_\varphi t}$
- Factoring out leaves periodic functions, e.g., $\bar{G}(t) = e^{+im\Omega_\varphi t} G(t)$
- All such functions can be Fourier transformed (DFT)

$$\mathcal{G}_n = \frac{1}{N} \sum_{k=0}^{N-1} \bar{G}(t_k)\, e^{in\Omega_r t_k}$$

For some $N(\mathcal{P})$, the ratio of least-to-most significant amplitude is $\simeq \epsilon_g$

Effectively band-limited functions

- Typically we transform from TD to FD with a Fourier series

$$\Psi_{lm}(t,r) = \sum_{n=-\infty}^{+\infty} X_{lmn}(r) e^{-i\omega_{mn} t} \simeq \sum_{n=-N_1}^{+N_2} X_{lmn}(r) e^{-i\omega_{mn} t}$$

with

$$X_{lmn}(r) = \frac{1}{T_r} \int_0^{T_r} dt \; \Psi_{lm}(t,r) e^{i\omega_{mn} t}$$

- Alternate view: If effectively band-limited (EBL), use DFT

$$\bar{G}_{lm}(t_k) = \sum_{n=-N/2}^{N/2-1} \mathcal{G}_{lmn} \, e^{-in\Omega_r t_k}, \qquad \mathcal{G}_{lmn} = \frac{1}{N} \sum_{k=-N/2}^{N/2-1} \bar{G}_{lm}(t_k) \, e^{in\Omega_r t_k}$$

EBL periodic functions would allow eliminating integrals over time

Nyquist-Shannon sampling theorem

- Nyquist-Shannon sampling theorem tells us:
  - If a function is band-limited within $|\omega| \lesssim 2\pi B = \frac{N}{2}\Omega_r$,
  - Then it is completely determined by giving its values at a set of points equally spaced by $\Delta t = \frac{1}{2B}$.
  - For periodic functions this requires a finite number N of points spaced by $\Delta t = \frac{1}{2B} = \frac{1}{N}T_r$

Implication: It should be possible to sample source functions at a few $\sim$ 100's of points and know everything about them

- Integrals over time are replaced by finite sums – a special case of Newton-Cotes (Fourier) quadratures

Implications for MST and EHS

- Return to central problem – finding matching coefficients $C^{\pm}_{lmn}$

$$C^{\pm}_{lmn} = W^{-1}_{lmn} \int_{r_{\min}}^{r_{\max}} \hat{X}^{\mp}_{lmn}(r') Z_{lmn}(r') dr'/f(r')$$

- Put in the source and exchange order of t and r integration

$$C^{\pm}_{lmn} = \frac{1}{W_{lmn}\, T_r} \int_0^{T_r} dt\, \bar{E}^{\pm}_{lmn}(t)\, e^{in\Omega_r t}$$

where

$$\bar{E}^{\pm}_{lmn}(t) = \int_{r_{\min}}^{r_{\max}} dr' \, \frac{1}{f(r')} \hat{X}^{\mp}_{lmn}(r')\, \bar{S}_{lm}(t, r')$$

Simple constructive procedure for matching

- Evaluate the delta and derivative of delta terms

$$\bar{E}^{\pm}_{lmn}(t) = \frac{\bar{G}_{lm}(t)}{f_p}\,\hat{X}^{\mp}_{lmn}(r_p(t)) + \frac{2M}{r_p^2}\frac{\bar{F}_{lm}(t)}{f_p^2}\,\hat{X}^{\mp}_{lmn}(r_p(t)) - \frac{\bar{F}_{lm}(t)}{f_p}\,\partial_r\hat{X}^{\mp}_{lmn}(r_p(t))$$

- Each $\bar{E}^{\pm}_{lmn}(t)$ is a smooth, periodic function

- Replace time integral with sum over sample points

$$C^{\pm}_{lmn} = \frac{1}{NW_{lmn}}\sum_{k=0}^{N-1}\bar{E}^{\pm}_{lmn}(t_k)\,e^{in\Omega_r t_k}$$

Samples of $\bar{E}^{\pm}_{lmn}(t_k)$ at high precision $\implies$ $C^{\pm}_{lmn}$ at high precision

Simple constructive procedure–finite, modest number of function calls

Conclusions

- MST and EHS can be combined
- Utilizes small sample sets in TD and DFT
- Provides high-precision solution for eccentric orbits on Schwarzschild

- Computing the overlap between GSF and PN theory for p and e
- Have determined $\dot{E}$ up to 2.5PN (relative) order tails for range of e
- Have confirmed various eccentricity "enhancement" functions

- Working on 2.5PN and 3PN terms and then beyond known results
- Next: conservative GSF effects

- May very likely speed up double/quad precision RWZ codes
- May very likely speed up Lorenz gauge codes
- May be extensible to Kerr with 2D DFTs

Metric perturbation reconstruction

- Even-parity sector:

$$K^{\ell m}(t,r) = f\partial_r \Psi^{\ell m}_{\rm even} + A\,\Psi^{\ell m}_{\rm even} - \frac{r^2 f^2}{(\lambda+1)\Lambda} Q^{tt}_{\ell m},$$

$$h^{\ell m}_{rr}(t,r) = \frac{\Lambda}{f^2}\left[\frac{\lambda+1}{r}\Psi^{\ell m}_{\rm even} - K^{\ell m}\right] + \frac{r}{f}\partial_r K^{\ell m},$$

$$h^{\ell m}_{tr}(t,r) = r\partial_t\partial_r \Psi^{\ell m}_{\rm even} + rB\,\partial_t \Psi^{\ell m}_{\rm even} - \frac{r^2}{\lambda+1}\left[Q^{tr}_{\ell m} + \frac{rf}{\Lambda}\partial_t Q^{tt}_{\ell m}\right],$$

$$h^{\ell m}_{tt}(t,r) = f^2 h^{\ell m}_{rr} + fQ^{\sharp}_{\ell m},$$

$$\Lambda(r) \equiv \lambda + \frac{3M}{r}, \qquad \lambda \equiv \frac{(\ell+2)(\ell-1)}{2}.$$

$$A \equiv \frac{1}{r\Lambda}\left[\lambda(\lambda+1) + \frac{3M}{r}\left(\lambda + \frac{2M}{r}\right)\right],\ B \equiv \frac{1}{rf\Lambda}\left[\lambda\left(1 - \frac{3M}{r}\right) - \frac{3M^2}{r^2}\right]$$

- Q's are the even-parity source terms $\propto \delta\left[r - r_p(t)\right]$

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Metric perturbation reconstruction

- Even-parity sector:
- Assume $\Psi = \Psi^+\theta(z) + \Psi^-\theta(-z)$, $\qquad z \equiv r - r_p(t)$
- We find all metric perturbation amplitudes have the form

$$M(t,r) = M^+(t,r)\,\theta(z) + M^-(t,r)\,\theta(-z) + M^S(t)\,\delta(z),$$

$$K^\pm(t,r) = f\partial_r\Psi^\pm + A\Psi^\pm, \qquad K^S(t) = 0,$$

$$h_{rr}^\pm(t,r) = \frac{\Lambda}{f^2}\left[\frac{\lambda+1}{r}\Psi^\pm - K^\pm\right] + \frac{r}{f}\partial_r K^\pm, \qquad h_{rr}^S(t) = \frac{r_p}{f_p}[\![K]\!]_p,$$

$$h_{tr}^\pm(t,r) = r\partial_t\partial_r\Psi^\pm + rB\,\partial_t\Psi^\pm, \qquad h_{tr}^S(t) = \mathcal{E}^2\frac{\dot{r}_p}{f_pU_p^2}q^\sharp$$

$$h_{tt}^\pm(t,r) = f^2h_{rr}^\pm, \qquad h_{tt}^S(t) = f_p^2h_{rr}^S + f_pq^\sharp.$$

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- Assume $\Psi = \Psi^{+}\theta(z) + \Psi^{-}\theta(-z), \qquad z \equiv r - r_p(t)$
- We find all metric perturbation amplitudes have the form

$$M(t,r) = M^{+}(t,r)\,\theta(z) + M^{-}(t,r)\,\theta(-z) + M^{S}(t)\,\delta(z),$$

$$K^{\pm}(t,r) = f\partial_r\Psi^{\pm} + A\Psi^{\pm}, \qquad K^{S}(t) = 0,$$

$$h^{\pm}_{rr}(t,r) = \frac{\Lambda}{f^2}\left[\frac{\lambda+1}{r}\Psi^{\pm} - K^{\pm}\right] + \frac{r}{f}\partial_r K^{\pm}, \qquad h^{S}_{rr}(t) = \frac{r_p}{f_p}[\![K]\!]_p,$$

$$h^{\pm}_{tr}(t,r) = r\partial_t\partial_r\Psi^{\pm} + rB\,\partial_t\Psi^{\pm}, \qquad h^{S}_{tr}(t) = \mathcal{E}^2\frac{\dot r_p}{f_p U_p^2}q^{\sharp}$$

$$h^{\pm}_{tt}(t,r) = f^2 h^{\pm}_{rr}, \qquad h^{S}_{tt}(t) = f_p^2 h^{S}_{rr} + f_p q^{\sharp}.$$

- q's are the even-parity source terms with $\delta(z)$ "divided off."

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Metric perturbation reconstruction

- Odd-parity sector:

$$h_t^{\ell m}(t,r) = \frac{f}{2}\partial_r\left(r\Psi_{\text{odd}}^{\ell m}\right) - \frac{r^2 f}{2\lambda}P_{\ell m}^t,$$

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- P's are odd-parity source terms $\propto \delta\left[r - r_p(t)\right]$
- Again, assume $\Psi = \Psi^+\theta(z) + \Psi^-\theta(-z), \qquad z \equiv r - r_p(t)$
- Both of the odd-parity amplitudes are C^{-1}, with no singular terms:

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