High-precision (MST) solutions for eccentric orbits on Schwarzschild: Part II

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MST Approach for Homogeneous Teukolsky Solutions

- "Mano-Suzuki-Takasugi" (1996) – semi-analytic formalism for obtaining homogeneous solutions to the Teukolsky equation
- Solutions R_{in}, R_{up} given as expansions in hypergeometric functions
- Expansion coefficients depend on numerically-determined "renormalized angular momentum" ν

MST Approach for Homogeneous Teukolsky Solutions

- Ingoing solution given in series of ordinary hypergeometric functions:

$$R^{\nu}_{\text{in}}(x) = e^{i\epsilon\kappa x}(-x)^{-s-i(\epsilon+\tau)/2}(1-x)^{i(\epsilon-\tau)/2} \times \sum_{n=-\infty}^{\infty} a^{\nu}_{n}\,{}_2F_1(n+\nu+1-i\tau, -n-\nu-i\tau; 1-s-i\epsilon-i\tau; x)$$

- Upgoing solution given in series of confluent hypergeometric functions:

$$R^{\nu}_{\text{up}}(z) = 2^{\nu}e^{-\pi\epsilon}e^{-i\pi(\nu+1+s)}e^{iz}z^{\nu+i(\epsilon+\tau)/2}(z-\epsilon\kappa)^{-s-i(\epsilon+\tau)/2} \times \sum_{n=-\infty}^{\infty} i^{n}\frac{(\nu+1+s-i\epsilon)_n}{(\nu+1-s+i\epsilon)_n}b^{\nu}_{n}(2z)^{n} \times \Psi(n+\nu+1+s-i\epsilon, 2n+2\nu+2; -2iz)$$

- See Sasaki/Tagoshi (2003/2006)

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Renormalized Angular Momentum ν

- $\{a_n\}$'s and $\{b_n\}$'s chosen to satisfy same recurrence relation
- ν determined by requiring that series converge, gives minimal solution of recurrence relation
- In practice, ν found as numeric root of a continued fraction equation
- ν can become complex above $\omega \simeq 0.36$
- Low-frequency expansion: $\nu = \ell + c_2\omega^2 + c_4\omega^4 + c_6\omega^6 + \mathcal{O}(\omega^8) + \cdots$

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Transformations to Regge-Wheeler/Zerilli Variables

- Chandrasekhar Transformation (1975) yields odd-parity Regge-Wheeler solution from Teukolsky variable:

$$X^{\mathrm{RW}}_{\ell m\omega} = r^3 \left[\left(\frac{d}{dr} \right) - \frac{i\omega}{f} \right]^2 \left(\frac{1}{r^2} R_{\ell m\omega} \right)$$

- Further transformation to obtain Zerilli (even-parity) solution:

$$X^{\mathrm{Z}}_{\ell m\omega} = \frac{\left[\lambda(\lambda+1) + \frac{9M^2 f}{r[\lambda r + 3M]}\right] X^{\mathrm{RW}}_{\ell m\omega} + 3Mf \frac{dX^{\mathrm{RW}}_{\ell m\omega}}{dr}}{[\lambda(\lambda+1) - 3i\omega M]}$$

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Advantages of Regge-Wheeler/Zerilli Formalism

- Wave equations with short-range potential:

$$\left[-\partial_t^2 + \partial_{r_*}^2 - V_\ell\right] \Psi_{\ell m}(t,r) = S_{\ell m}(t,r)$$

- For $\ell + m$ even, we use Zerilli-Moncrief (ZM) variable and source term, and for $\ell + m$ odd we use Cunningham-Price-Moncrief (CPM) variable/source
- Experience with extended homogeneous solutions (EHS) method for eccentric orbits (Hopper & Evans 2010, due to Barack, Ori, and Sago 2008)
- Substantial group infrastructure available for metric perturbations in RWZ

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Fourier Harmonic Decomposition

- Frequency domain (FD) decomposition of Ψ:

$$\Psi_{\ell m}(t,r) = \sum_n X_{\ell mn}(r) e^{-i\omega_{mn}t}$$

- FD solutions have leading wave behavior at horizon and infinity:

$$X^{\pm}_{\ell mn}(r_* \to \pm\infty) = A^{\pm}_{\ell mn} e^{\pm i\omega_{mn} r_*}$$

- For convenience, we divide off asymptotic amplitudes $A^{\pm}$ to obtain $\hat{X}$'s, which are normalized to unity asymptotically

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Particular Solution from EHS

- Use $\hat{X}$'s to define

$$\Psi^{\pm}_{\ell m}(t,r) = \sum_n C^{\pm}_{\ell mn} \hat{X}^{\pm}_{\ell mn}(r) e^{-i\omega_{mn} t}$$

where

$$C^{\pm}_{\ell mn} = \frac{1}{W_{\ell mn}} \int_{r_{\min}}^{r_{\max}} dr \, \frac{1}{f(r)} \hat{X}^{\mp}_{\ell mn}(r) Z_{\ell mn}(r)$$

- Then particular solution to TD equation is abutment of $\Psi^{\pm}$'s:

$$\Psi_{\ell m}(t,r) = \Psi^{+}_{\ell m}(t,r)\theta(r-r_p) + \Psi^{-}_{\ell m}(t,r)\theta(r_p - r)$$

(method of EHS)

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Circular Orbits as Test Case

- For circular orbits, only one frequency contributes

$$\Psi^{\pm}_{\ell m}(t,r) = C^{\pm}_{\ell m0}\hat{X}^{\pm}_{\ell m0}(r)e^{-im\Omega_\phi t}$$

- Normalization coefficients are found algebraically:

$$C^{\pm} = \frac{1}{W}\left[-\left(G + \frac{2M}{r_p^2 f_p}F\right)\hat{X}^{\mp}(r_p) + \frac{F}{f_p}\partial_r \hat{X}^{\mp}(r_p)\right]$$

where F and G are "fully-evaluated" source terms:

$$S_{\ell m}(t,r) = G_{\ell m}(t)\delta(r - r_p) + F_{\ell m}(t)\delta'(r - r_p)$$

evaluated at $t = \phi_p = 0$

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Confirmation of Circular Orbit Energy Fluxes

- Radiated energy fluxes given by

$$\langle \dot{E}^{\pm\infty}_{\ell m} \rangle = \frac{1}{64\pi} \frac{(\ell+2)!}{(\ell-2)!} (m\Omega_\phi)^2 \left| C^{\pm}_{\ell m} \right|^2$$

- Flux given to 5.5PN relative order in e.g. Sasaki and Tagoshi (2003/2006):

$$\langle \dot{E}^{\infty} \rangle = \frac{32\mu^2 M^3}{5 r_p^5} \left[1 - \frac{1247}{336} x + 4\pi x^{3/2} - \frac{44711}{9072} x^2 - \frac{8191}{672} x^{5/2} + \left(\frac{6643739519}{69854400} - \frac{1712}{105}\gamma + \frac{16}{3}\pi^2 - \frac{3424}{105}\ln 2 - \frac{1712}{210}\ln x \right) x^3 + \cdots + (\cdots) x^{11/2} \right]$$

$$x \equiv (M\Omega_\phi)^{2/3}$$

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- Numerically compute flux to very high order for, say, $r_p = 10^{10}$: $\langle \dot{E}^{\infty} \rangle = 6.399999997 \cdots \times 10^{-50}$
- Subtract Newtonian term and get residual $\delta = -2.3751577 \cdots \times 10^{-59}$
- Subtract 1PN correction: $\delta = 8.0424456 \cdots \times 10^{-64}$

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- Subtract through 5.5PN correction: $\delta = 2.6583024 \cdots \times 10^{-105}$
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Circular Orbits - Conservative Test

- Example: Detweiler's gauge-invariant redshift quantity
- Defined via

$$\Delta U = -U \times \left(\frac{1}{2} \bar{u}^\alpha \bar{u}^\beta h^{\mathrm{R}}_{\alpha\beta} \right) = -U \times H^{\mathrm{R}}$$

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Detweiler's Gauge-Invariant Redshift

- Singular structure of field known to go as

$$H^{\text{sing}(\ell)} = B - \frac{D}{(2\ell-1)(2\ell+3)} + \mathcal{O}(\ell^{-4})$$

- B known analytically:

$$B = 2\sqrt{\frac{r_p - 3M}{r_p^2(r_p-2M)}}\,{}_2F_1\left(\frac{1}{2}, \frac{2}{2}, 1, \frac{M}{r-2M}\right)$$

- Fit out higher order regularization parameters numerically

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- *Analytically* known PN terms:

$$\Delta U = \frac{-1}{r_p} + \frac{-2}{r_p^2} + \frac{-5}{r_p^3} + \frac{-3872 + 123\pi^2}{96 r_p^4} + \frac{-592384 - 196608\gamma + 10155\pi^2 - 393216\log(2)}{7680 r_p^5}$$
$$+ \frac{64\log(r_p)}{5 r_p^5} + \frac{-956\log(r_p)}{105 r_p^6} + \frac{-13696\pi}{525 r_p^{6.5}} + \frac{-51256\log(r_p)}{567 r_p^7} + \frac{81077\pi}{3675 r_p^{7.5}} + \frac{27392\log^2(r_p)}{525 r_p^8}$$
$$+ \frac{82561159\pi}{467775 r_p^{8.5}} + \frac{-27016\log^2(r_p)}{2205 r_p^9} + \frac{-11723776\pi\log(r_p)}{55125 r_p^{9.5}} + \frac{-4027582708\log^2(r_p)}{9823275 r_p^{10}}$$
$$+ \frac{99186502\pi\log(r_p)}{1157625 r_p^{10.5}} + \frac{23447552\log^3(r_p)}{165375 r_p^{11}}$$

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$$\Delta U = \frac{-1}{r_p} + \frac{-2}{r_p^2} + \frac{-5}{r_p^3} + \frac{-3872 + 123\pi^2}{96 r_p^4} + \frac{-592384 - 196608\gamma + 10155\pi^2 - 393216\log(2)}{7680 r_p^5} + \frac{64\log(r_p)}{5 r_p^5} + \frac{-956\log(r_p)}{105 r_p^6} + \frac{-13696\pi}{525 r_p^{6.5}} + \frac{-51256\log(r_p)}{567 r_p^7} + \frac{81077\pi}{3675 r_p^{7.5}} + \frac{27392\log^2(r_p)}{525 r_p^8} + \frac{82561159\pi}{467775 r_p^{8.5}} + \frac{-27016\log^2(r_p)}{2205 r_p^9} + \frac{-11723776\pi\log(r_p)}{55125 r_p^{9.5}} + \frac{-4027582708\log^2(r_p)}{9823275 r_p^{10}} + \frac{99186502\pi\log(r_p)}{1157625 r_p^{10.5}} + \frac{23447552\log^3(r_p)}{165375 r_p^{11}}$$

- We compared numeric result at e.g. $r_p = 10^{10}$:
$-1.000000000200000000050000000276879\cdots \times 10^{-10}$

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with

$$\alpha_5 = -243.17681446\cdots$$

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etc.

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- But we aren't going to do circular orbits any better....

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Eccentric Orbits - Formalism

- Matching coefficients for $\{\ell mn\}$ FD modes is given by EHS method:

$$C^{\pm}_{\ell mn} = \frac{1}{W_{\ell mn}} \int_{r_{\min}}^{r_{\max}} dr \frac{1}{f(r)} \hat{X}^{\mp}_{\ell mn}(r) Z_{\ell mn}(r)$$

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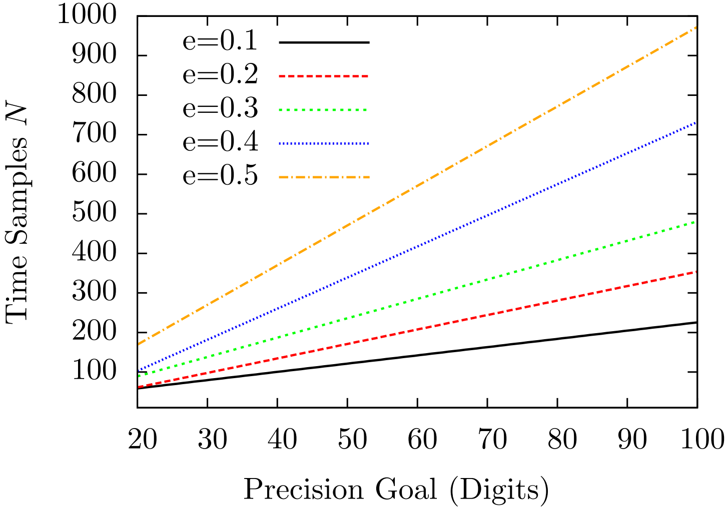
Sampling the Orbit

- How to determine sampling N?

- Based on ability to Fourier-represent certain orbital quantities (like $dt/d\chi$ and hence T_r) given a precision goal

Sampling the Orbit



Check on Method with Small p Orbit

- Compared $\langle \dot{E}^{\infty} \rangle$, $\langle \dot{E}^{\mathrm{H}} \rangle$ with double-precision C code at semi-latus rectum $p = 100$
- For the $(\ell, m) = (2, 2)$ mode and $e = 0.01$, we have
  - MST (Mathematica):
    - Infinity: $6.154421591814879645507076321 2 \cdots \times 10^{-10}$
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- Agree as well as we can expect!

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Comparison with PN Results

- But our MST solutions are far more accurate....
- Compare with PN literature. Blanchet (2014) gives:

$$\langle \dot{E} \rangle = \frac{32}{5} \left(\frac{\mu}{M} \right)^2 x^5 \left(\mathcal{I}_0 + x\mathcal{I}_1 + x^{3/2}\mathcal{K}_{3/2} + x^2\mathcal{I}_2 + x^{5/2}\mathcal{K}_{5/2} + x^3\mathcal{I}_3 + x^3\mathcal{K}_3 \right)$$

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Peters-Mathews
Enhancement

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1.5PN tail

etc.

Enhancement Functions

$$\mathcal{I}_0 = \frac{1}{(1-e_t^2)^{7/2}} \left(1 + \frac{73}{24} e_t^2 + \frac{37}{76} e_t^4\right)$$

$$\mathcal{I}_1 = \frac{1}{(1-e_t^2)^{9/2}} \left(-\frac{1247}{336} + \frac{10475}{672} e_t^2 + \frac{10043}{384} e_t^4 + \frac{2179}{1792} e_t^6\right)$$

$$\mathcal{K}_{3/2} = 4\pi\varphi(e_t) = 4\pi \left(1 + \frac{2335}{192} e_t^2 + \mathcal{O}(e_t^4)\right)$$

$$\begin{aligned}\mathcal{I}_2 =& \frac{1}{(1-e_t^2)^{11/2}} \left(-\frac{203471}{9072} - \frac{3807197}{18144} e_t^2 - \frac{268447}{24192} e_t^4 + \frac{1307105}{16128} e_t^6 + \frac{86567}{64512} e_t^8\right) \\ &+ \frac{1}{(1-e_t^2)^5} \left(\frac{35}{2} + \frac{6425}{48} e_t^2 + \frac{5065}{64} e_t^4 + \frac{185}{96} e_t^6\right), \\ \mathcal{K}_{5/2} =& -\frac{8191}{672} \pi \psi(e_t) \\ &\vdots\end{aligned}$$

e_t and φ

- e_t is "time eccentricity"
- We use Darwin eccentricity – needed to be able to translate
- We developed an expansion for e_t in powers of our e:

$$e_t^2 = e^2 \left(1 - 6x + \frac{\left[-15e^4 + \left(15\sqrt{1-e^2} - 19\right) + \left(34 - 15\sqrt{1-e^2}\right) e^2\right] x^2}{-e^4 + 2e^2 - 1} + \cdots \right)$$

- Also developed expansion for enhancement function $\varphi(e_t)$ to arbitrary order in e_t:

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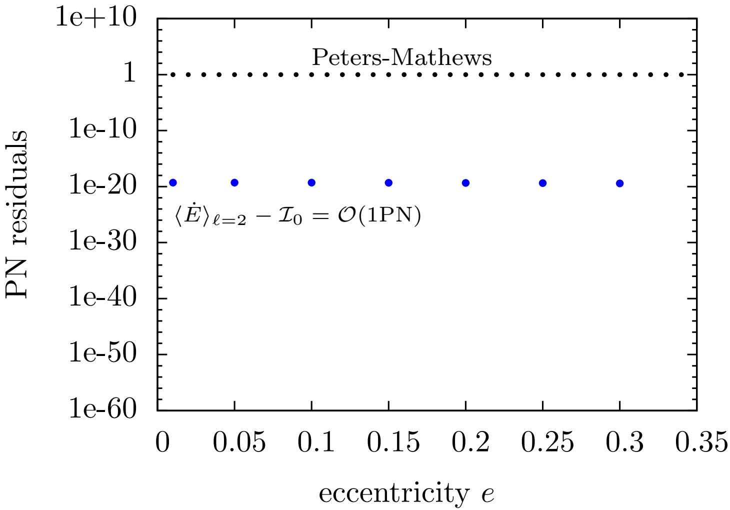
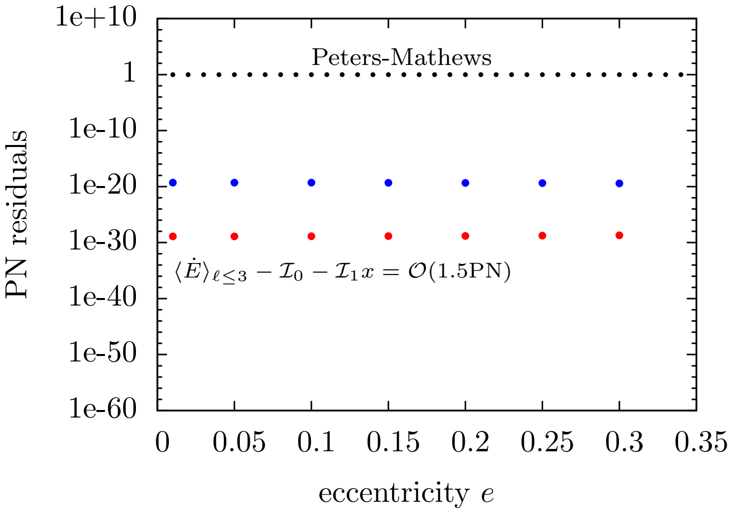
$\ell = 2$ quadrupole flux



$\ell \leq 3$ quadrupole and octopole flux



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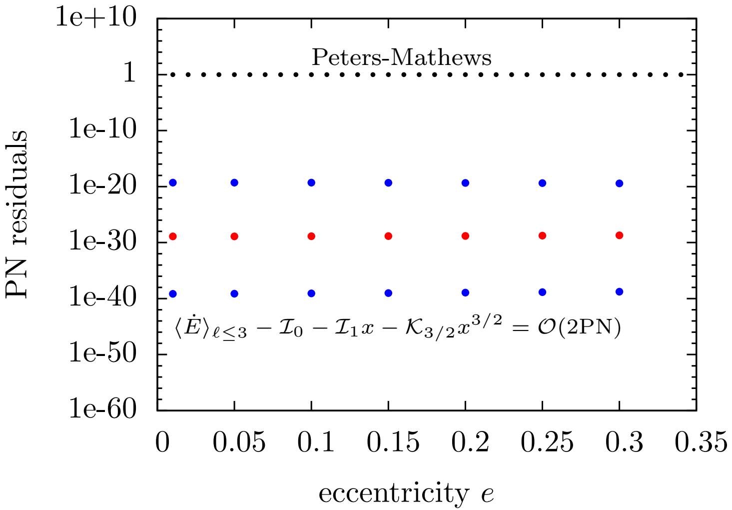
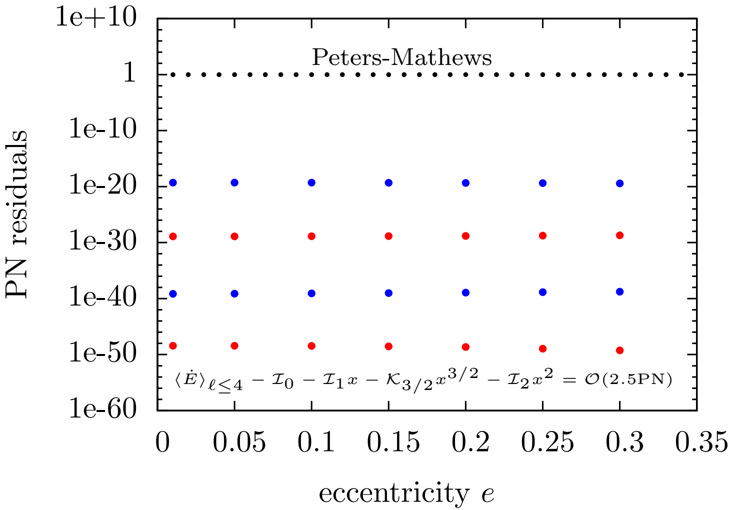


$\ell \leq 4$ flux

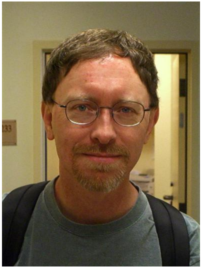


What's Next?

- Match through 3PN order, determine unknown parameters beyond 3PN
- Conservative dynamics for eccentric orbits
- Kerr?

Thanks!

Dr. Charles Evans (Advisor)



Dr. Seth Hopper (Collaborator)

UNC THE GRADUATE SCHOOL

ROYSTER SOCIETY OF FELLOWS

NORTH CAROLINA space grant