GR+EM

Combined gravitational & electromagnetic self-force in electrovac spacetimes

Part I

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I. GR + EM: mass m, charge e, electrovac background $g_{\alpha\beta}$, $F_{\alpha\beta}$

II. Renormalization of fields and self-force of accelerated particles:

Angle average + mass renormalization,
and a singular field

III. A simple way to extend the singular field to subleading order for GR+EM

A. The coupled terms

B. Uniqueness of iterative solution
at subleading order

GR + EM: mass m, charge e,
electrovac background $g_{\alpha\beta}$, $F_{\alpha\beta}$

Particle trajectory $z(t)$ satisfies

$$m u^\beta \nabla_\beta u^\alpha = e F^\alpha{}_\beta u^\beta$$

For a smooth perturbation $g_{\alpha\beta} + h_{\alpha\beta}$, $F_{\alpha\beta} + \delta F_{\alpha\beta}$

m(acceleration – acceleration of test particle)
= self-force f:

Perturbed trajectory satisfies

$$m \bar{u}^\beta \nabla_\beta \bar{u}_\alpha - e F_{\alpha\beta} \bar{u}^\beta = f_\alpha$$

Find fields and self-force to linear order in e and m

$$f_\alpha = f_\alpha^{GR} + f_\alpha^{EM}$$

$$f_\alpha^{GR}[h] = -m q_\alpha{}^\delta (\nabla_\beta h_{\gamma\delta} - \frac{1}{2}\nabla_\delta h_{\beta\gamma}) u^\beta u^\gamma$$

$$f_\alpha^{EM}[\delta A] = e\,(\nabla_\alpha \delta A_\beta - \nabla_\beta \delta A_\alpha) u^\beta$$

($q_\alpha{}^\beta = g_\alpha{}^\beta + u_\alpha u^\beta$ the projection $\perp u^\alpha$)

Renormalize

$$f_\alpha^{\mathrm{ret}} = f_\alpha[h^{\mathrm{ret}}, \delta A^{\mathrm{ret}}]$$

Mino, Sasaki, Tanaka, Quinn, Wald;
Detweiler, Whiting; Gralla, Harte, Wald; Pound;
Reviews: Barack, Poisson, Pound, Vega

The GR+EM system to linear order in m and e:

$$\delta Q = m\left[\frac{\partial}{\partial m}Q\right]_{m=0} + e\left[\frac{\partial}{\partial e}Q\right]_{e=0}$$

Notation

$$h_{\alpha\beta} = \delta g_{\alpha\beta}, \qquad \gamma_{\alpha\beta} = h_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}h \qquad \delta F_{\alpha\beta} = \nabla_{\alpha}\delta A_{\beta} - \nabla_{\beta}\delta A_{\alpha}$$

$$\delta(F_{\alpha}{}^{\beta}) = \delta(g^{\beta\gamma}F_{\alpha\gamma})$$

Coupled system - schematic

$$\delta G = \delta T = muu\delta(x,z) - \delta(FF)$$

$$\delta(\nabla F) = \delta j = eu\delta(x,z)$$

Lorenz gauge:

$$(\Box + \Omega)\,\gamma + F\nabla\delta A = muu\,\delta(x,z)$$

$$(\Box - \mathrm{R})\,\delta A + \nabla(F\gamma) = eu\,\delta(x,z)$$

Details

$$\nabla_\gamma \nabla^\gamma \gamma_{\alpha\beta} + 2\Omega_\alpha{}^\gamma{}_\beta{}^\delta \gamma_{\gamma\delta} + 16\Lambda_{\alpha\beta}{}^{\gamma\delta}\nabla_\gamma \delta A_\delta = -16\pi m \int u_\alpha u_\beta \delta(x, z(\tau)) d\tau$$

$$\nabla_\gamma \nabla^\gamma \delta A_\alpha - R_\alpha{}^\beta \delta A_\beta + 2\nabla^\beta (\Lambda^{\gamma\delta}{}_{\alpha\beta}\gamma_{\gamma\delta}) = -4\pi e \int u_\alpha \delta(x, z(\tau)) d\tau$$

As Tom Linz will discuss, previous work already essentially gives solution to the uncoupled equations

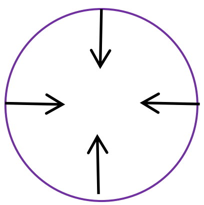
$$\nabla_\gamma \nabla^\gamma \; {}_I\gamma_{\alpha\beta} + 2\Omega_\alpha{}^\gamma{}_\beta{}^\delta \; {}_I\gamma_{\gamma\delta} = -16\pi m \int u_\alpha u_\beta \delta(x, z(\tau)) d\tau$$

$$\nabla_\gamma \nabla^\gamma \; {}_I A_\alpha - R_\alpha{}^\beta \; {}_I A_\beta = -4\pi e \int u_\alpha \delta(x, z(\tau)) d\tau$$

Geodesic motion:

The divergent part of the perturbed metric (and EM field) gives the Coulomb part of the expression f^{ret} for the self force.

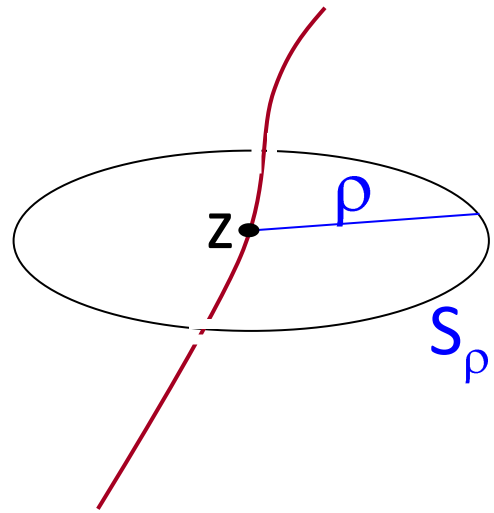
It points radially inward (outward for EM)



and averages to zero.

In fact,

$$f_\alpha^{ren} = \lim_{\rho\to 0}\langle f_\alpha^{ret}\rangle_\rho = \frac{1}{4\pi}\lim_{\rho\to 0}\int_{S_\rho} d\Omega\, f_\alpha^{ret}$$



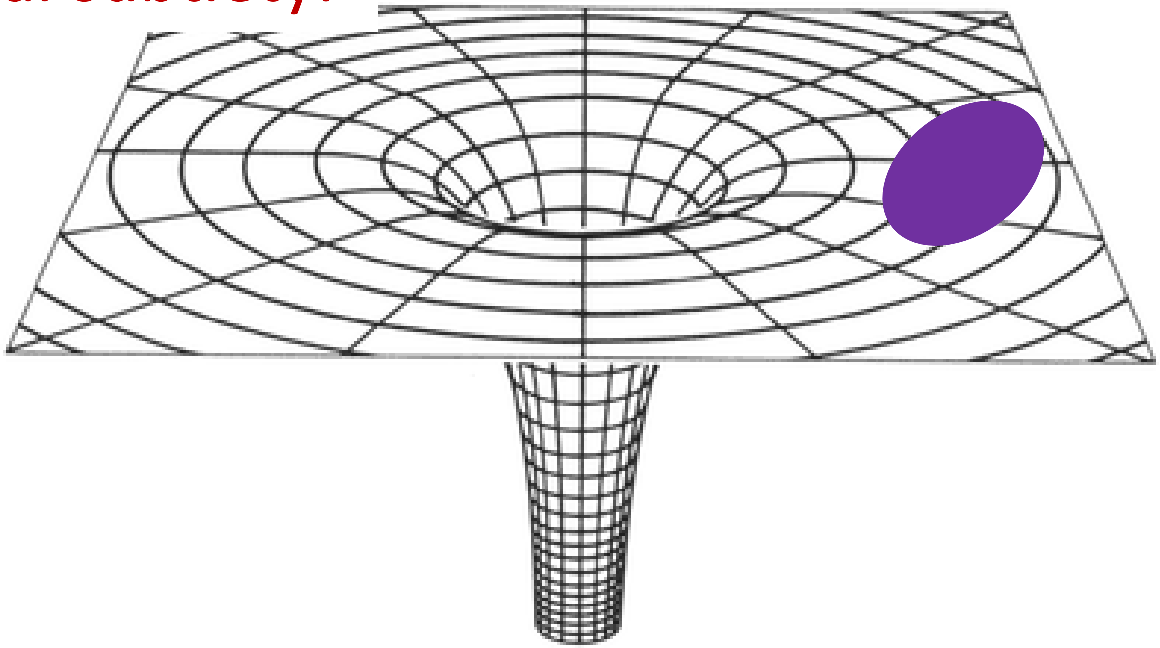
S_ρ a sphere of geodesic radius ρ about point z on the trajectory;

indices in Riemann normal coordinates.

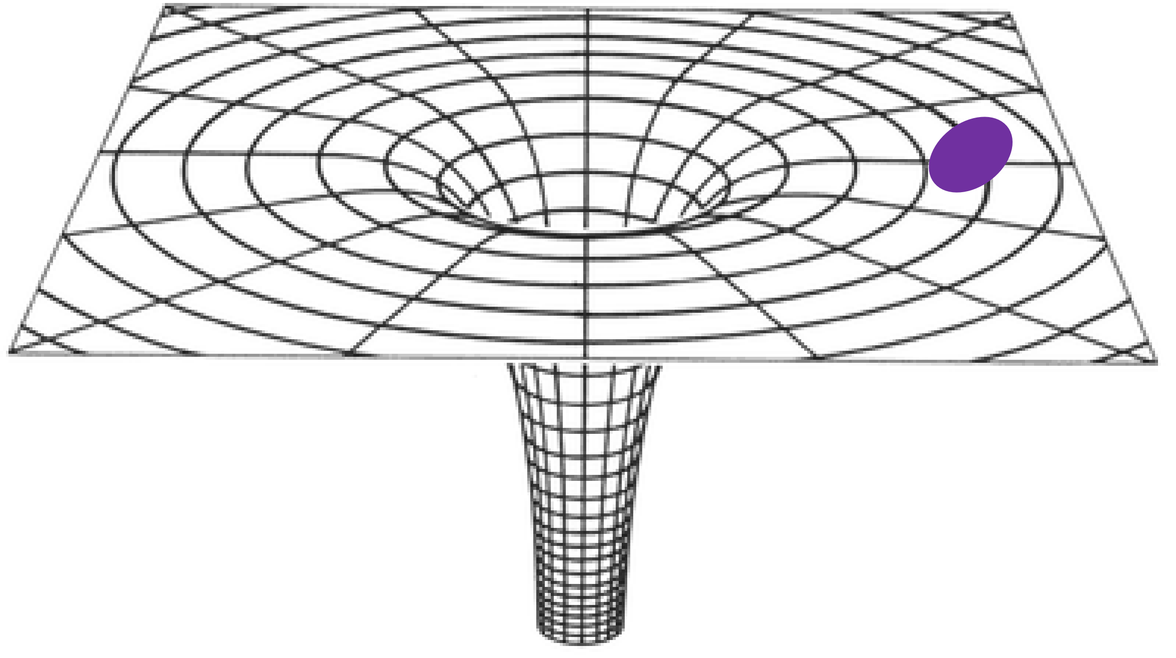
(or use exp to pull f back to tangent space at z.)

Gralla '11

An additional subtlety:



As one shrinks a finite body to zero size, with m and e also shrinking to zero, the force from the internal multipole moments coupled to the perturbed field vanishes faster than m or e.



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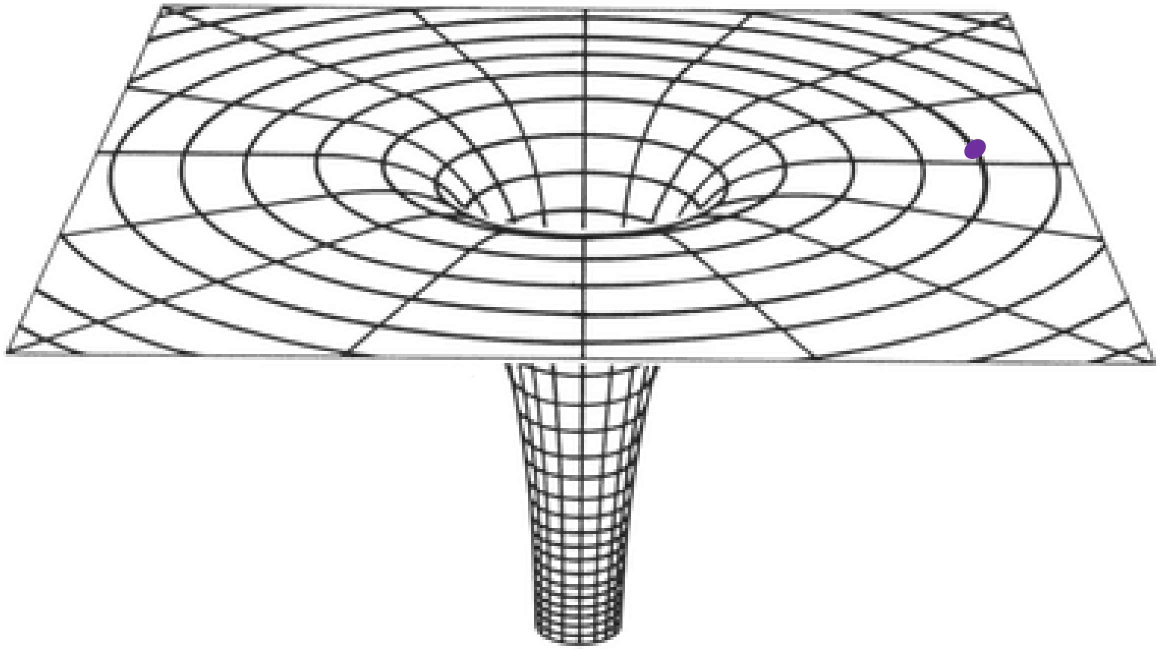

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That is, at order r^0 , f^{ret} has a term of the form

$$\frac{P_{\alpha\beta_1\cdots\beta_n} x^{\beta_1}\cdots x^{\beta_n}}{r^n} \qquad\qquad r^2 = q_{\alpha\beta} x^\alpha x^\beta = \rho^2 + O(\rho^4)$$

with n odd. No internal multipoles should appear, and the angle average removes them:

$$\left\langle \frac{P_{\alpha\beta_1\cdots\beta_n} x^{\beta_1}\cdots x^{\beta_n}}{r^n} \right\rangle = 0.$$

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$$\left\langle \frac{P_{\alpha\beta_1\cdots\beta_n} x^{\beta_1}\cdots x^{\beta_n}}{r^n} \right\rangle = 0.$$

But when the particle is accelerating, there are terms at subleading order - order $1/r$ - of the form

GR $\qquad$ EM

$$Cm^2\frac{a_\alpha}{r} \qquad Ce^2\frac{a_\alpha}{r}$$

Accelerated motion

In this more general case, because the remaining term in the singular field is proportional to the acceleration, removing it can be regarded as mass renormalization:

$$\mathrm{f}_\alpha^{ren} = \lim_{\rho\to 0}\langle \mathrm{f}_\alpha^{ret}\rangle_\rho = \lim_{\rho\to 0}\left[\frac{1}{4\pi}\int_{S_\rho} d\Omega\, \mathrm{f}_\alpha^{ret} - m^{\mathrm{sing}}(\rho) a_\alpha\right]$$

(Linz, JF, Wiseman)

Why is this equivalent to subtracting the singular field of most CAPRA talks?

Equations in Lorenz gauge are a coupled hyperbolic system of the form

$$\Box \psi^A + Q^{A\alpha}\nabla_\alpha \psi + R^A{}_B \psi^B = S^A \delta(x,z)$$

This holds for the coupled GR+EM system, with

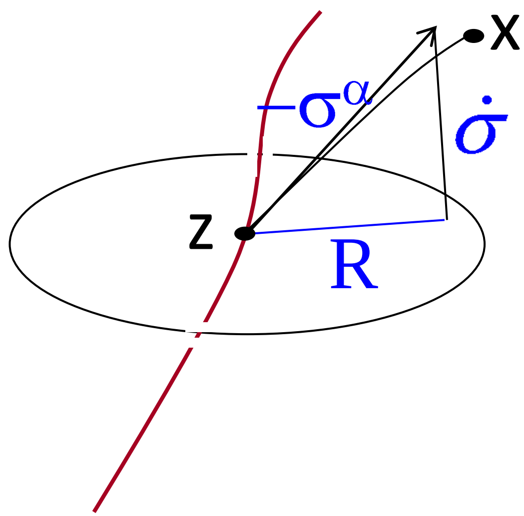
$$\psi^A = (\gamma_{\alpha\beta}, \delta A_\alpha)$$

Highest derivative operator (the symbol) for every component is $g^{\alpha\beta}\partial_\alpha \partial_\beta$ so characteristics are null geodesics, and retarded solution in convex normal neighborhood has Hadamard form.

With $S_{GR}{}^{\alpha\beta} = mu^{\alpha}u^{\beta}$ $S_{EM}{}^{\alpha} = eu^{\alpha}$

the components of the field have the form,

$$\psi^{A}(x) = \left[\frac{U^{A}}{\dot{\sigma}}\right]_{ret} + \int_{-\infty}^{\tau_{ret}-\varepsilon} G^{A}[x,z(\tau)]d\tau$$



Have field-independent factor

$$\frac{1}{(\sigma^\alpha u_\alpha)_{ret/adv}} = \frac{1}{r}[1-\frac{1}{2}(1-\frac{t^2}{r^2})a_\alpha x^\alpha + P_{\alpha\beta\gamma\delta\varepsilon}\frac{x^\alpha x^\beta x^\gamma x^\delta}{r^2}$$

$$\pm P_{\alpha\beta\gamma\delta\varepsilon}(a,\dot{a},u)\frac{x^\alpha x^\beta x^\gamma x^\delta x^\varepsilon}{r^3}] + O(\varepsilon^2)$$

multiplied by smooth tensor

$$U^A = [S^A + \partial_\alpha U^A x^\alpha + \partial_\alpha \partial_\beta U^A x^\alpha x^\beta + O(\varepsilon^3)]$$

to give field of form

$$\psi = \frac{1}{r}[S + \partial_\alpha U\, x^\alpha - \frac{1}{2}(1-\frac{t^2}{r^2})a_\alpha x^\alpha + \partial_\alpha \partial_\beta U\, x^\alpha x^\beta$$

$$-\frac{1}{2}(1-\frac{t^2}{r^2})a_\alpha x^\alpha U_\beta x^\beta + P_{\alpha\beta\gamma\delta}(a,\dot{a},u)\frac{x^\alpha x^\beta x^\gamma x^\delta}{r^2}$$

$$\pm P_{\alpha\beta\gamma\delta\varepsilon}(a,\dot{a},u)\frac{x^\alpha x^\beta x^\gamma x^\delta x^\varepsilon}{r^3}] \pm \int_{\mp\infty}^{\tau_{adv/ret}} G\, d\tau$$

$$\psi = \frac{1}{r}[S + \partial_\alpha U\, x^\alpha - \frac{1}{2}(1 - \frac{t^2}{r^2})a_\alpha x^\alpha + \partial_\alpha \partial_\beta U\, x^\alpha x^\beta$$

$$- \frac{1}{2}(1 - \frac{t^2}{r^2})a_\alpha x^\alpha U_\beta x^\beta + P_{\alpha\beta\gamma\delta}(a, \dot{a}, u)\frac{x^\alpha x^\beta x^\gamma x^\delta}{r^2}$$

$$\pm P_{\alpha\beta\gamma\delta\varepsilon}(a, \dot{a}, u)\frac{x^\alpha x^\beta x^\gamma x^\delta x^\varepsilon}{r^3}] \pm \int_{\mp\infty}^{\tau_{adv/ret}} G d\tau + O(\varepsilon^2)$$

Magenta terms even in x, implying gradient odd in x, whence angle-average vanishes.

Red terms proportional to acceleration

For the term $\partial_\alpha U x^\alpha / r$ this follows from the calculation.

Finally, the Hadamard form as usual implies that the integral contributes to f_{ret} a term of form $V^A x^\alpha / r$ whose angle-average vanishes and that contributes the DW term to the singular field.

Solution of the GR+EM system

$$(\Box + \Omega)\,\gamma + F\nabla\delta A = muu\,\delta(x,z)$$

$$(\Box - \mathrm{R})\,\delta A + \nabla(F\gamma) = eu\,\delta(x,z)$$

As Tom Linz will discuss, previous work already essentially gives solution to the uncoupled equations.

Solve iteratively:

$$\gamma = {}_{I}\gamma + {}_{II}\gamma$$

$$\delta A = {}_{I}A + {}_{II}A$$

First the uncoupled equations :

${}_{I}\gamma =$ the purely gravitational perturbation of accelerated mass in electrovac background

${}_{I}A =$ the purely electromagnetic perturbation of accelerated charge in electrovac background

$$(\Box + \Omega)\,{}_{I}\gamma = muu\,\delta(x,z)$$

$$(\Box - \mathrm{R})\,{}_{I}A = eu\,\delta(x,z)$$

This includes the leading, Coulomb term, in the two fields.

Next find the subleading contributions by substituting $_I\gamma,\ \ _IA$ for $\gamma,\ \ \delta A$ in the mixed terms

$$(\Box+\Omega)_{II}\gamma=-F\nabla\delta A$$

$$(\Box-R)_{II}A=-\nabla(F_I\gamma)$$

The RHS is $O(r^{-2})$, giving a solution at $O(r^0)$, subleading, and allowing us to use leading-order operator:

$$\Box_{II}\gamma = -F\nabla_I A$$

$$\Box_{II} A = -\nabla(F_I\gamma)$$

The leading term is the leading term of the Hadamard expansion. To show that the subleading term found by this iteration is the subleading term of the Hadamard expansion we show that it is unique at subleading order.

In the general form of the Hadamard expansion at subleading order, each component of each field has form

$$P^A{}_\alpha \frac{x^\alpha}{r} - \frac{S^A}{2} a_\alpha \frac{x^\alpha t^2}{r^3}$$

Expression has only monopole and dipole contributions, and the second term is independent of the field. The calculation therefore can find only a contribution to P_α. The calculation would be ambiguous if there were solutions to the homogeneous wave equation of the form $\tilde{P}_\alpha \frac{x^\alpha}{r}$, but the dipole solutions are x and x/r^3

$$\Box \frac{x^\alpha}{r} = -2\frac{x^\alpha}{r^3} \neq 0$$

and the monopole (t/r) has a delta-fn source.
Given the constraints of the Hadamard expansion, the solution is therefore unique.

Uniqueness at sub-subleading order, O(ε).

With the first-order solution known, the only subleading term in the Hadamard expansion that depends on the field is (proportional to) $\partial_\alpha \partial_\beta U^A \dfrac{x^\alpha x^\beta}{r}$

This is in general a sum of terms of the form

$$\left(Ar + \tilde{A}\frac{t^2}{r}\right) + A_\alpha \frac{r^\alpha t}{r} + A^{STF}_{\alpha\beta} \frac{r^\alpha r^\beta}{r} = \text{monopole+dipole+quadrupole}$$

A homogeneous solution has this form only if $\Box$ kills each term. This immediately implies that the dipole and quadrupole terms vanish. In the monopole term

$$\Box\left(Ar + \tilde{A}\frac{t^2}{r}\right) = 0 \Rightarrow \tilde{A} t^2 \nabla^2 \frac{1}{r} = -\tilde{A} 4\pi t^2 \delta(\vec{r}) = 0 \Rightarrow \tilde{A} = 0$$

Then A must also vanish, and the subleading solution is unique.

At sub-subleading order, O(ε), there is an ambiguity that, at Tom will discuss, leaves the renormalized self-force and the regularization parameters unchanged.

At O(ε), using no information about $\partial_\alpha U^A,\ \partial_\alpha\partial_\beta U^A, V$, the Hadamard expansion allows terms proportional to $P_{\alpha\beta\gamma\delta}\dfrac{x^\alpha x^\beta x^\gamma x^\delta}{r^3}$, with powers of t up to t^3.

This allows a homogeneous solution of the form

$$C_\alpha x^\alpha\left(\frac{t^3}{r^3}+3\frac{t}{r}\right).$$

Using the known value of the subleading term may eliminate the ambiguity.

Have

$$-\sigma^{\alpha}{}_{ret} = x^{\alpha} - z_{ret}{}^{\alpha} + R^{\alpha}{}_{\cdots}(x^{\cdot} - z_{ret}{}^{\cdot})(x^{\cdot} - z_{ret}{}^{\cdot})$$

$$z_{ret}{}^{\alpha} = u^{\alpha}\tau + \frac{1}{2}a^{\alpha}\tau_{ret}{}^{2} + O(\tau^{3})$$

$$\frac{1}{(\sigma^{\alpha}u_{\alpha})_{ret}} = \frac{1}{r}[1 - \frac{1}{2}(1 - \frac{t^{2}}{r^{2}})a_{\alpha}x^{\alpha} + S_{\alpha\beta\gamma\delta\varepsilon}\frac{x^{\alpha}x^{\beta}x^{\gamma}x^{\delta}}{r^{2}}$$

$$+ S_{\alpha\beta\gamma\delta\varepsilon}(a,\dot{a},u)\frac{x^{\alpha}x^{\beta}x^{\gamma}x^{\delta}x^{\varepsilon}}{r^{2}}] + O(\varepsilon^{2})$$

$$U^{A} = u^{A}[1 + \partial_{\alpha}U\, x^{\alpha} + \partial_{\alpha}\partial_{\beta}U\, x^{\alpha}x^{\beta} + O(\varepsilon^{3})]$$

$$\Rightarrow$$

$$\psi^{A} = \frac{u^{A}}{r}\left[1 + \partial_{\alpha}U\, x^{\alpha} - \frac{1}{2}(1 - \frac{t^{2}}{r^{2}})a_{\alpha}x^{\alpha}\right]$$