The not-so-nonlinear nonlinearity of Einstein's equation

Abraham Harte

Max Planck Institute for Gravitational Physics
Albert Einstein Institute
Potsdam, Germany

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Motivation

- Einstein's equation is fundamentally geometric.
- Perturbation theory typically ignores this:

  Variables like $h_{ab} = g_{ab} - \hat{g}_{ab}$ are analytically simple but geometrically awkward.
- Might other choices be better?

Find a geometrically simple way to generically deform one metric into another.

Example I: Conformal transformations

If

$$g_{ab} = \Omega^2 \hat{g}_{ab},$$

Ω has a simple geometrical interpretation:

Lengths of vectors are locally rescaled while angles and causal structure are preserved.

The **trace** (only!) of the vacuum Einstein equation is linear in Ω:

$$\hat{R} = 0 \qquad \Rightarrow \qquad R \propto \hat{\Box}\Omega = 0.$$

This isn't general enough in 4D...

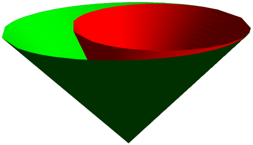
Example II: Kerr-Schild deformations

A less obvious deformation is of Kerr-Schild type:

$$g_{ab} = \hat{g}_{ab} + \ell_a \ell_b.$$

Let ℓ_a be null wrt $\hat{g}_{ab}$. It is then null wrt g_{ab} as well.

This deforms light cones. Rays tangent to ℓ^a are preserved while other null directions change.



Kerr-Schild is simple

Einstein's equation is linear in these variables!

Theorem [Xanthopoulos, 1978]

If $\hat{R}_{ab} = 0$ and $h_{ab} = \ell_a \ell_b$ satisfies the vacuum Einstein equation **linearized** about $\hat{g}_{ab}$, the metric $g_{ab} = \hat{g}_{ab} + h_{ab}$ satisfies the vacuum Einstein equation **exactly**.

Generalizations with matter are also known:
[Xanthopoulos, 1986] and [Mastronikola and Xanthopolous 1988].

A Kerr-Schild example

Gravitational pp-waves are Kerr-Schild solutions.

Written in KS form, Einstein's equation on the natural waveform is

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) H(u, x, y) = 0.$$

This is exact.

Trying to model the same thing using textbook gauges is vastly more complicated.

Why does Einstein's equation simplify so much for KS metrics?

1. Metric inverses are trivial. Letting $\ell^a := \hat{g}^{ab}\ell_b$,

$$g_{ab} = \hat{g}_{ab} + \ell_a \ell_b \qquad \Rightarrow \qquad g^{ab} = \hat{g}^{ab} - \ell^a \ell^b.$$

2. Volume elements are unchanged: $g/\hat{g} = 1$.

3. h_{ab} is triply-degenerate:

$$h_{ab}\ell^b = h_{ab}x^b = h_{ab}y^b = 0.$$

- Kerr-Schild metrics include many interesting things:

  Kerr black holes, some gravitational waves, ultrarelativistic systems, . . .

- But there are only 3 degrees of freedom.

- Solutions are algebraically special (at least with $\hat{g}_{ab}$ flat).

Something more general is needed. . .

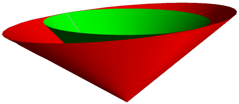
Kerr-Schild metrics squish null cones using **one** vector field.

Try two vector fields instead. If ℓ_a and n_a are both null wrt $\hat{g}_{ab}$, let

$$g_{ab} = \hat{g}_{ab} + \ell_{(a} n_{b)}.$$

Then n_a and ℓ_a are also null wrt to g_{ab}.

Rays tangent to ℓ^a *and* n^a are preserved while the rest of the null cone is sheared.



Extended Kerr-Schild as a generic perturbation

Deform causal structure using ℓ_a and n_a, and then deform outside of null cones using a conformal factor Ω:

$$\boxed{g_{ab} = \Omega^2(\hat{g}_{ab} + \ell_{(a} n_{b)})}$$

Ordinary KS is recovered when $\Omega \equiv 1$ and $n_a \propto \ell_a$.

This has the required $(2 \times 3 - 1) + 1 = 6$ degrees of freedom!

Deformation theorem (Llosa and Soler, 2005)

Given any analytic Lorentzian metric g_{ab}, there exist (non-unique) 2-forms $F_{ab} = F_{[ab]}$ and scalar fields ϕ such that

$$\hat{g}_{ab} = \phi g_{ab} \pm g^{cd} F_{ac} F_{bd}$$

is a flat metric.

Writing F_{ab} in terms of its principal null directions recovers the xKS deformation $g_{ab} = \Omega^2(\hat{g}_{ab} + \ell_{(a} n_{b)})$ [Llosa and Carot, 2009].

So this is a sufficiently general ansatz.

Why do this?

Many simplifications of KS are only slightly more complicated with xKS:

Letting $h_{ab} = \ell_{(a} n_{b)}$ and $h = \hat{g}^{ab} h_{ab} =$ "non Kerr-Schildness,"

1. Metric inverses are simple:

$$g_{ab} = \Omega^2 (\hat{g}_{ab} + h_{ab}) \quad \Rightarrow \quad g^{ab} = \Omega^{-2} \left(\hat{g}^{ab} - (1 + h/2)^{-1} h^{ab} \right).$$

2. Volume elements deform via the linearized expression:

$$\frac{\sqrt{-g}}{\sqrt{-\hat{g}}} = \Omega^4 \left(1 + h/2\right).$$

3. xKS perturbations are proportional to projection operators:

$$h_{ac} h^c{}_b = \frac{1}{2} h h_{ab}.$$

4. h_{ab} is doubly-degenerate when $h \neq 0$:

$$\exists x^a,\, y^a \quad \text{such that} \quad h_{ab} x^b = h_{ab} y^b = 0.$$

5. (Conformal) Killing vectors in g_{ab} are also (conformal) Killing vectors in $\hat{g}_{ab}$ when this metric is flat [Llosa and Carot, 2009].

All of this simplifies Einstein's equation dramatically.

Einstein's equation in tensorial form

Setting $\Omega \equiv 1$ for simplicity,

$$\mathfrak{h}^{ab} := \hat{g}^{ab} - \left(\frac{\sqrt{-g}}{\sqrt{-\hat{g}}} \right) g^{ab} = h^{ab} - \frac{1}{2} h \hat{g}^{ab}.$$

So trace-reversed h_{ab} satisfies a (generalized) Landau-Lifshitz equation.

But g^{ab} is simple, so **nonlinearities are only polynomial in h_{ab}!**

Newman-Penrose formalism

- xKS singles out null vector fields ℓ^a, n^a.
- This is just what the Newman-Penrose formalism is built for.
- NP replaces the metric by a complex null tetrad.
- Derivatives of this tetrad are related to curvature components.

If $\ell^a \not\propto n^a$, let $k^a \propto n^a$ such that

$$(\ell^a, k^a, m^a, \bar{m}^a)$$

is a normalized NP tetrad for $\hat{g}_{ab}$.

Perturbed NP tetrad

If $g_{ab} = \hat{g}_{ab} + 2(1-H)\ell_{(a}k_{b)}$ for some $H = 1 + h/2$, the tetrad $(H^{-1}\ell^a, k^a, m^a, \bar{m}^a)$ is normalized and null wrt g_{ab}.

Just a simple rescaling!

- Now apply Einstein's equation to this adapted null tetrad.

- This results in equations for the **background** tetrad and for H.

- Once these are solved, recovering the metric is trivial:

$$\begin{aligned} g_{ab} &= 2[m_{(a}\bar{m}_{b)} - \ell_{(a}k_{b)}] + 2(1-H)\ell_{(a}k_{b)} \\ &= 2[m_{(a}\bar{m}_{b)} - H\ell_{(a}k_{b)}]. \end{aligned}$$

Perturbed spin coefficients

First derivatives of the tetrad are traditionally denoted using $\alpha, \beta, \ldots$.

Collecting these into $\mathbf{S}$, perturbations $\hat{\mathbf{S}} \to \mathbf{S}$ are simple:

$$\mathbf{S} = H^{-1}[\mathbf{L}(H)\hat{\mathbf{S}} + \mathbf{E}\hat{\mathcal{D}}H],$$
$$\mathbf{L}(H) = \mathbf{L}_0 + H\mathbf{L}_1 + H^2\mathbf{L}_2.$$

$\mathbf{L}_i$ and $\mathbf{E}$ are constant, nearly diagonal matrices.

Simple H-dependent linear transformation $\oplus$ first derivatives of H.

Implementing Einstein's equation

Vacuum Einstein now looks like

$$0 = A^{Ij}{}_{\alpha\beta} \mathcal{D}_I S_j + B^{ij}{}_{\alpha\beta} S_i S_j$$

Or more schematically,

$$0 \sim \hat{\mathcal{D}}^2 H + \hat{\mathcal{D}} \hat{S} + \hat{S} \hat{\mathcal{D}} H + (\hat{\mathcal{D}} H)^2 + \hat{S}^2.$$

Low-order polynomial in H and quadratic in $\hat{\mathbf{S}}$. Solve for these!

Or start from a known background frame $\hat{\mathbf{S}}^*$ and solve for the Lorentz transformation $\hat{\mathbf{S}}^* \to \hat{\mathbf{S}}$.

Conclusions

- A generalization of the Kerr-Schild ansatz includes all metrics.
- Nonlinearities of Einstein's equation simplify significantly.
- Common spacetimes like Kerr satisfy linear equations.
- GR seems simplest in terms of Newman-Penrose variables.
- Possibly useful for perturbation theory and finding new exact solutions.