SELF-FORCE OF ACCELERATED MOTION

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Self-force

Scalar field satisfies $$(\Box - \xi R - m^2)\Phi = q \int \sqrt{-g}\,\delta_4(x, z(\tau))\,d\tau$$

Regular-singular split: $\Phi_R = \Phi_{ret} - \Phi_S$

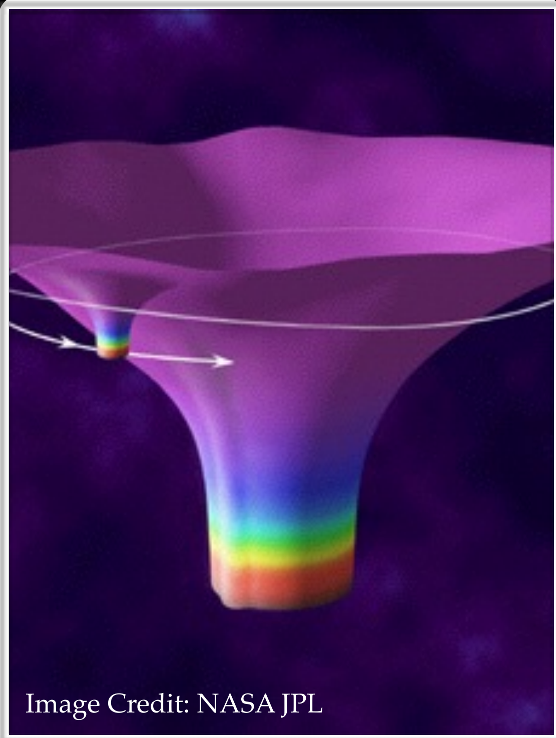


Image Credit: NASA JPL

Detweiler-Whiting Singular field:

$$\Phi_S(x) = q \int_\gamma G^{(\mathrm{S})}(x, z(\tau))\,d\tau$$
$$= \frac{q}{2}\left[\frac{U(x,x')}{\sigma_{c'} u^{c'}}\right]_{x'=x_{\mathrm{ret}}}^{x'=x_{\mathrm{adv}}} + \frac{q}{2}\int_{\tau_{\mathrm{ret}}}^{\tau_{\mathrm{adv}}} V(x, z(\tau))\,d\tau$$

Detweiler-Whiting Green function:

$$G^{(\mathrm{S})} = \frac{1}{2}\left\{U(x,x')\delta[\sigma(x,x')] + V(x,x')\theta[\sigma(x,x')]\right\}$$

Singular Field

The scalar singular field and self-force are

$$\Phi^{(S)}(x) = \left[\frac{U(x,x')}{2\sigma_{a'}u^{a'}}\right]_{x'=x_-}^{x'=x_+} + \int_{\tau_-}^{\tau_+} V(x,z(\tau))d\tau$$

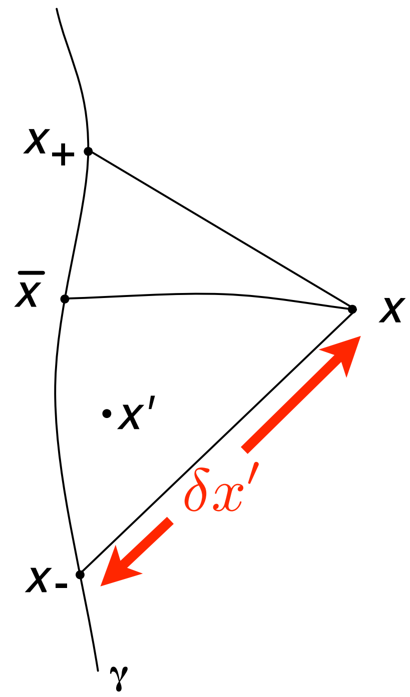
$$f^a = g^{ab}\Phi^{(R)}{}_{,b}.$$

The EM singular field and self-force are

$$A_a^S = \left[\frac{u^{a'}U^a{}_{a'}(x,x')}{2\sigma_{a'}u^{a'}}\right]_{x'=x_-}^{x'=x_+} + \int_{\tau_-}^{\tau_+} V^a{}_{a'}(x,z(\tau))u^{a'}d\tau$$

$$f^a = g^{ab}u^c A^{(R)}{}_{[c,b]}.$$

$$2\sigma = \sigma_{a'}\sigma^{a'}$$



The gravitational singular field and self-force are

$$\bar{h}^S_{ab} = \left[\frac{u^{a'}u^{b'}U^{ab}{}_{a'b'}(x,x')}{2\sigma_{a'}u^{a'}}\right]_{x'=x_-}^{x'=x_+} + \int_{\tau_-}^{\tau_+} V^{ab}{}_{a'b'}(x,z(\tau))u^{a'}u^{b'}d\tau \qquad \text{and} \qquad f^a = k^{abcd}\bar{h}^{(R)}_{bc;d},$$

where

$$\sigma(x,x') = \tfrac{1}{2}g_{ab}(x)\delta x^{a'}\delta x^{b'} + A_{abc}(x)\delta x^{a'}\delta x^{b'}\delta x^{c'} + \ldots$$

$$k^{abcd} \equiv \frac{1}{2}g^{ad}u^b u^c - g^{ab}u^c u^d - \frac{1}{2}u^a u^b u^c u^d + \frac{1}{4}u^a g^{bc}u^d + \frac{1}{4}g^{ad}g^{bc}.$$

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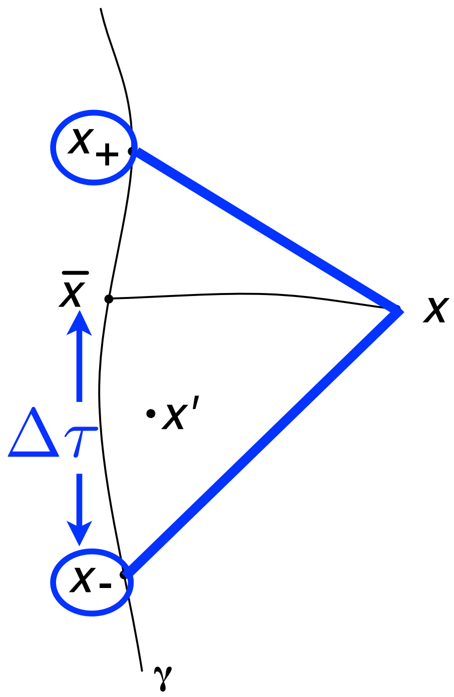
$$A^S_a = \left[\frac{u^{a'}U^a{}_{a'}(x,x')}{2\sigma_{a'}u^{a'}}\right]_{x'=x_-}^{x'=x_+} + \int_{\tau_-}^{\tau_+} V^a{}_{a'}(x,z(\tau))u^{a'}d\tau$$

$$f^a = g^{ab}u^c A^{(R)}{}_{[c,b]}.$$

$$\sigma = 0$$



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where

$$x^{a'}(\tau) = x^{\bar{a}} + u^{\bar{a}}\Delta\tau + \frac{1}{2!}\dot{u}^{\bar{a}}\Delta\tau^2 + \cdots$$

$$k^{abcd} \equiv \frac{1}{2}g^{ad}u^b u^c - g^{ab}u^c u^d - \frac{1}{2}u^a u^b u^c u^d + \frac{1}{4}u^a g^{bc}u^d + \frac{1}{4}g^{ad}g^{bc}.$$

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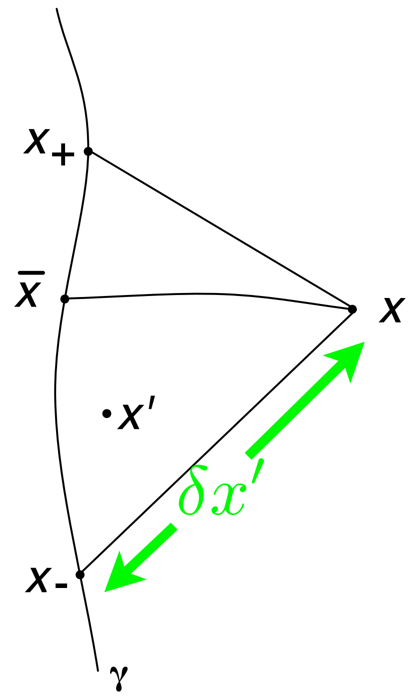
$$f^a = g^{ab}u^c A^{(\mathrm{R})}{}_{[c,b]}.$$

$$g_{ab';c}\sigma^c = 0$$



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where

$$\Delta^{\frac{1}{2}}(x,x') = \left(-\left[-g(x)\right]^{-\frac{1}{2}}\left|-\sigma_{a'b}(x,x')\right|\left[-g(x')\right]^{-\frac{1}{2}}\right)^{\frac{1}{2}}$$

$$k^{abcd} \equiv \frac{1}{2}g^{ad}u^b u^c - g^{ab}u^c u^d - \frac{1}{2}u^a u^b u^c u^d + \frac{1}{4}u^a g^{bc}u^d + \frac{1}{4}g^{ad}g^{bc}.$$

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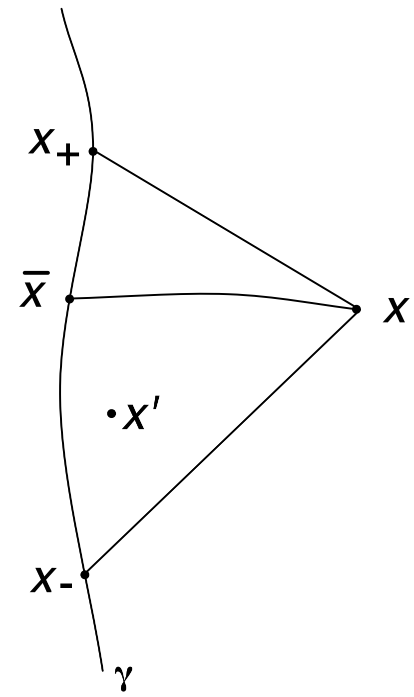
$$f^a = g^{ab}\Phi^{(\mathrm{R})}{}_{,b}.$$

The EM singular field and self-force are

$$A_a^{\mathrm{S}} = \left[\frac{u^{a'}U^a{}_{a'}(x,x')}{2\sigma_{a'}u^{a'}}\right]_{x'=x_-}^{x'=x_+} + \int_{\tau_-}^{\tau_+} V^a{}_{a'}(x,z(\tau))\,u^{a'}\,d\tau$$

$$f^a = g^{ab}u^c A^{(\mathrm{R})}{}_{[c,b]}.$$

$$V^{AB'}(x,x') = \sum_{n=0}^{\infty} V_n{}^{AB'}(x,x')\,\sigma^n(x,x')$$



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where

$$\sigma^{;\alpha'}\left(\Delta^{-1/2}V_n^{AB'}\right)_{;\alpha'} + (n+1)\,\Delta^{-1/2}V_n^{AB'} + \frac{1}{2n}\Delta^{-1/2}\mathcal{D}^{B'}{}_{C'}V_{n-1}^{AC'} = 0$$

$$k^{abcd} \equiv \frac{1}{2}g^{ad}u^b u^c - g^{ab}u^c u^d - \frac{1}{2}u^a u^b u^c u^d + \frac{1}{4}u^a g^{bc}u^d + \frac{1}{4}g^{ad}g^{bc}.$$

Mode Sum

$$
\begin{aligned}
f_a^l\left(r_0, t_0\right) &= \lim_{\Delta r \to 0} \sum_m f_a^{lm}\left(r_0 + \Delta r, t_0\right) Y^{lm}\left(\alpha_0, \beta_0\right) \\
&= \frac{2l+1}{4\pi} \lim_{\Delta r \to 0} \int f_a\left(r_0 + \Delta r, t_0, \alpha, \beta\right) P_l\left(\cos\alpha\right) d\Omega
\end{aligned}
$$

$$
\begin{aligned}
\sin\theta\cos\phi &= \cos\alpha \\
\sin\theta\sin\phi &= \sin\alpha\sin\beta \\
\cos\theta &= \sin\alpha\cos\beta
\end{aligned}
$$

Mode Sum

$$f_a^l(r_0, t_0) = \frac{2l+1}{4\pi}\left[\epsilon^{-2} \lim_{\Delta r \to 0} \int \frac{B_a^{(1)}}{\rho^3} P_l(\cos\alpha)\, d\Omega \right. F_{a[-1]}^l(r_0, t_0)$$

$$\left. + \epsilon^{n-3} \sum_{n=2} \int \rho_0^{n-3} c_{a(n)}(r_0, \beta) P_l(\cos\alpha)\, d\Omega \right]$$

$$w_1 = 2\sin\left(\frac{\alpha}{2}\right)\cos\beta$$

$$w_2 = 2\sin\left(\frac{\alpha}{2}\right)\sin\beta$$

Singular field contribution:

$$f_a(r, t, \alpha, \beta) = \sum_{n=1} \frac{\mathcal{B}_a^{(3n-2)}}{\rho^{2n+1}} \epsilon^{n-3}$$

where $\mathcal{B}_a^{(k)} = b_{a_1 a_2 \ldots a_k}(\bar{x}) \Delta x^{a_1} \Delta x^{a_2} \ldots \Delta x^{a_k}$

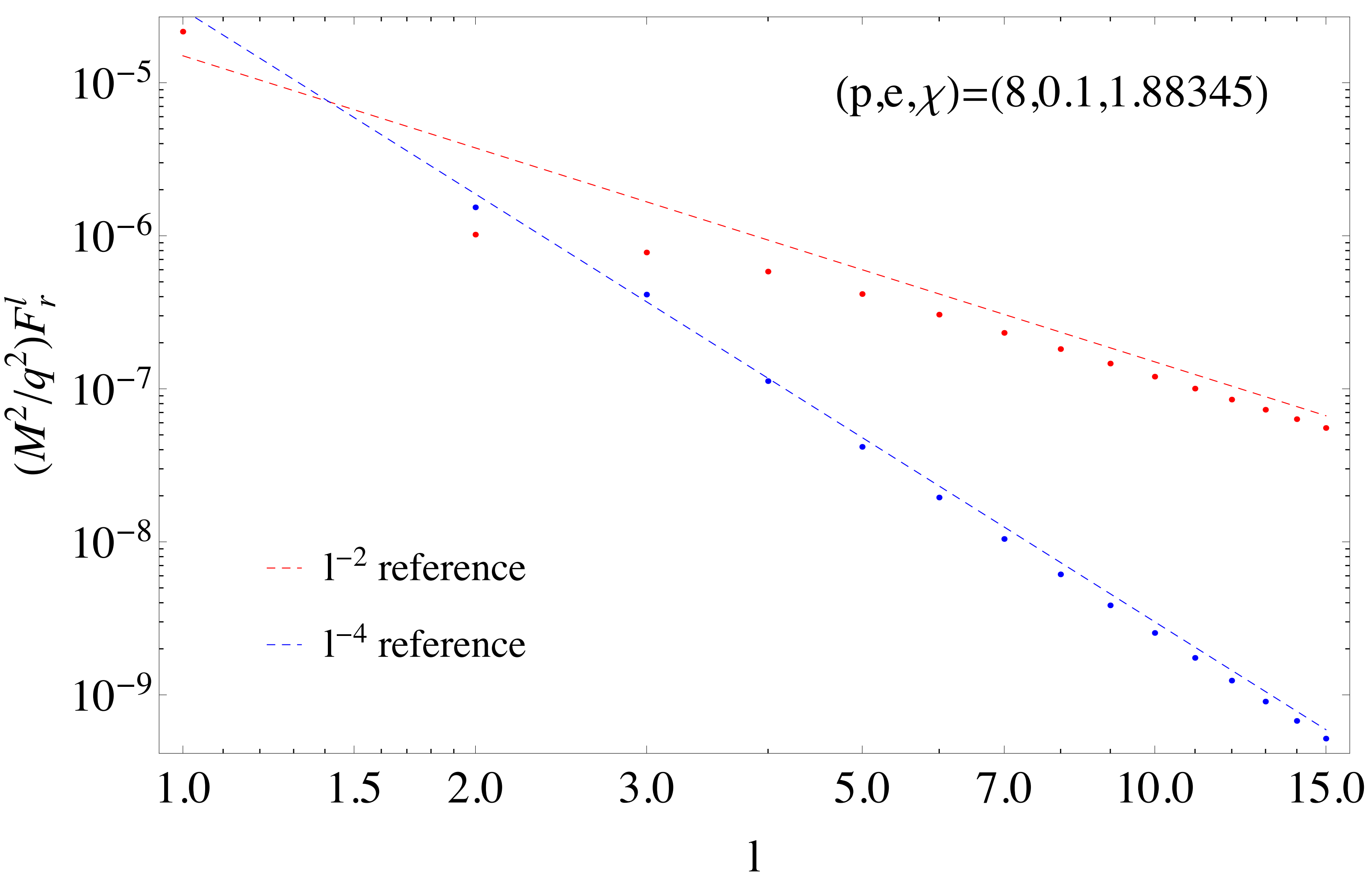
$$\rho^2 = (g_{\bar{a}\bar{b}} u^{\bar{a}} \Delta x^b)^2 + g_{\bar{a}\bar{b}} \Delta x^a \Delta x^b$$

$$= \underbrace{\frac{\dot{t}_0^2}{1 + r_0^2 \dot{\phi}_0^2}}_{\nu^2} \Delta r^2 + \underbrace{\left(r_0^2 + r_0^4 \dot{\phi}_0^2\right)}_{\zeta^2} \left[\Delta w_1^2 + \underbrace{\frac{\dot{r}_0 \dot{\phi}_0}{f(r_0)\left(1 + r_0^2 \dot{\phi}_0\right)}}_{-\mu} \Delta r\right]^2 + r_0^2 \Delta w_2^2$$

$$\Delta w_1 \to \Delta w_1 + \mu \Delta r \Rightarrow \rho(r, t_0, \alpha, \beta) = \nu^2 \Delta r + \zeta^2 \Delta w_1^2 + r_0^2 \Delta w_2^2$$

$$F_{a[-1]}^l(r_0, t_0) = (l + 1/2) \frac{b_{a_r} sgn(\Delta r)}{\zeta \nu r_0}$$

Eccentric Scalar



Circular Orbits

Defining the 4-velocity

$$\Omega_\varphi \equiv \Omega_\varphi(t) = \frac{d\varphi}{dt} \quad \Rightarrow u^\varphi(t) = \Omega_\varphi(t) u^t(t) \quad \Rightarrow u^t(t) = \left(f_0 - r_0^2 \Omega_\varphi(t)^2\right)^{-1/2}$$

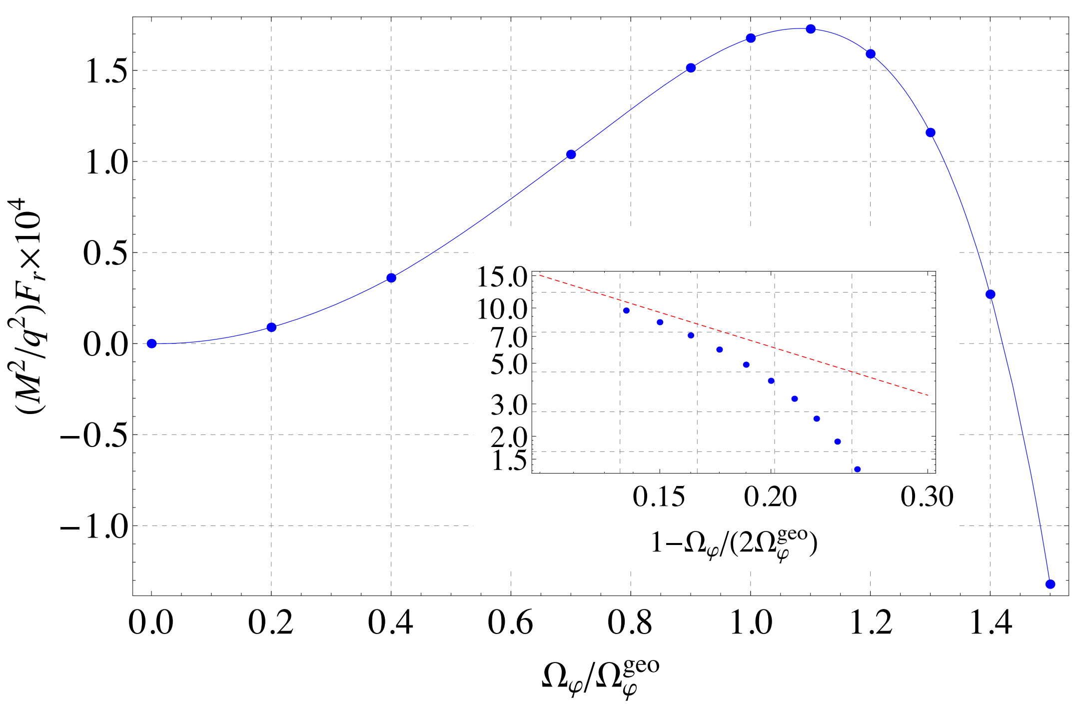
The usual …

$$\mathcal{E}(t) \equiv -u_t(t) = f_0 u^t(t)$$

$$\mathcal{L}(t) \equiv u_\varphi = r_0^2 \Omega_\varphi(t) u^t(t)$$

$$\Omega_\varphi^{geo} = \left(\frac{M}{r_0^3}\right)^{1/2}$$

$$0 \leq |\Omega_\varphi(t)| < \sqrt{f_0/r_0^2} = 2\Omega_\varphi^{geo}$$



Eccentric Orbits

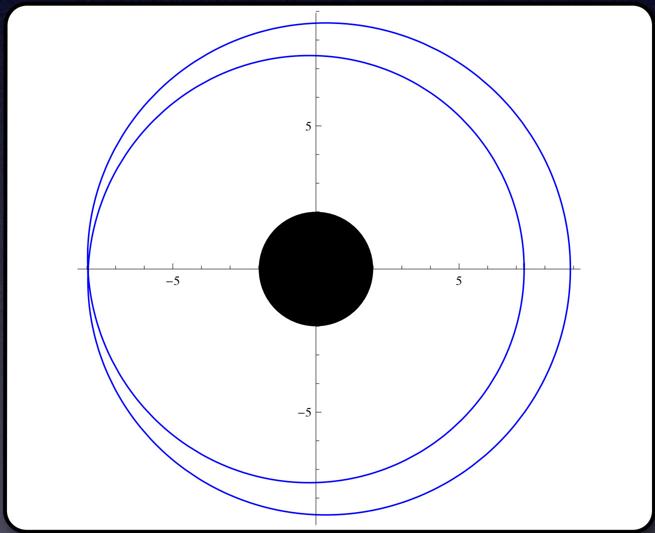
Orbital parameters

$$p = \frac{2r_{\max}r_{\min}}{M(r_{\max}+r_{\min})}$$

$$e = \frac{r_{\max}-r_{\min}}{r_{\max}+r_{\min}}$$

Relativistic anomaly parameter

$$r_p(\chi) = \frac{p}{1+e\cos\chi}$$



4-Velocity

$$\frac{d\varphi_p}{d\chi} = \sqrt{\frac{p}{p-6-2e\cos\chi}}$$

$$\frac{dr_p}{d\chi} = \frac{pe\sin(\chi)}{(1+e\cos(\chi))^2}$$

$$\frac{d\tau}{d\chi} = \left[f_p\left(\frac{dt_p}{d\chi}\right)^2 - \frac{1}{f_p}\left(\frac{dr_p}{d\chi}\right)^2 - r_p^2\left(\frac{d\varphi_p}{d\chi}\right)^2\right]^{1/2}$$

Eccentric Orbits

Specific Energy and Angular momentum for non-geodesic motion

$$\mathcal{E}(\chi) = f_p \frac{dt_p}{d\chi} \Big/ \frac{d\tau}{d\chi}, \quad \mathcal{L}(\chi) = r_p^2 \frac{d\varphi_p}{d\chi} \Big/ \frac{d\tau}{d\chi}$$

4-Acceleration $a^\alpha = u^\beta \nabla_\beta u^\alpha$

$$a^t = (u^t)'/(d\tau/d\chi) + \frac{2}{f_p r_p^2} u^t u^r, \quad a^\varphi = (u^\varphi)'/(d\tau/d\chi) + \frac{2}{r_p} u^\varphi u^r,$$

$$a^r = (u^r)'/(d\tau/d\chi) + \frac{f_p}{r_p^2}(u^t)^2 - \frac{(u^r)^2}{f_p r_p^2} - f_p r_p (u^\varphi)^2.$$

Orbital frequency

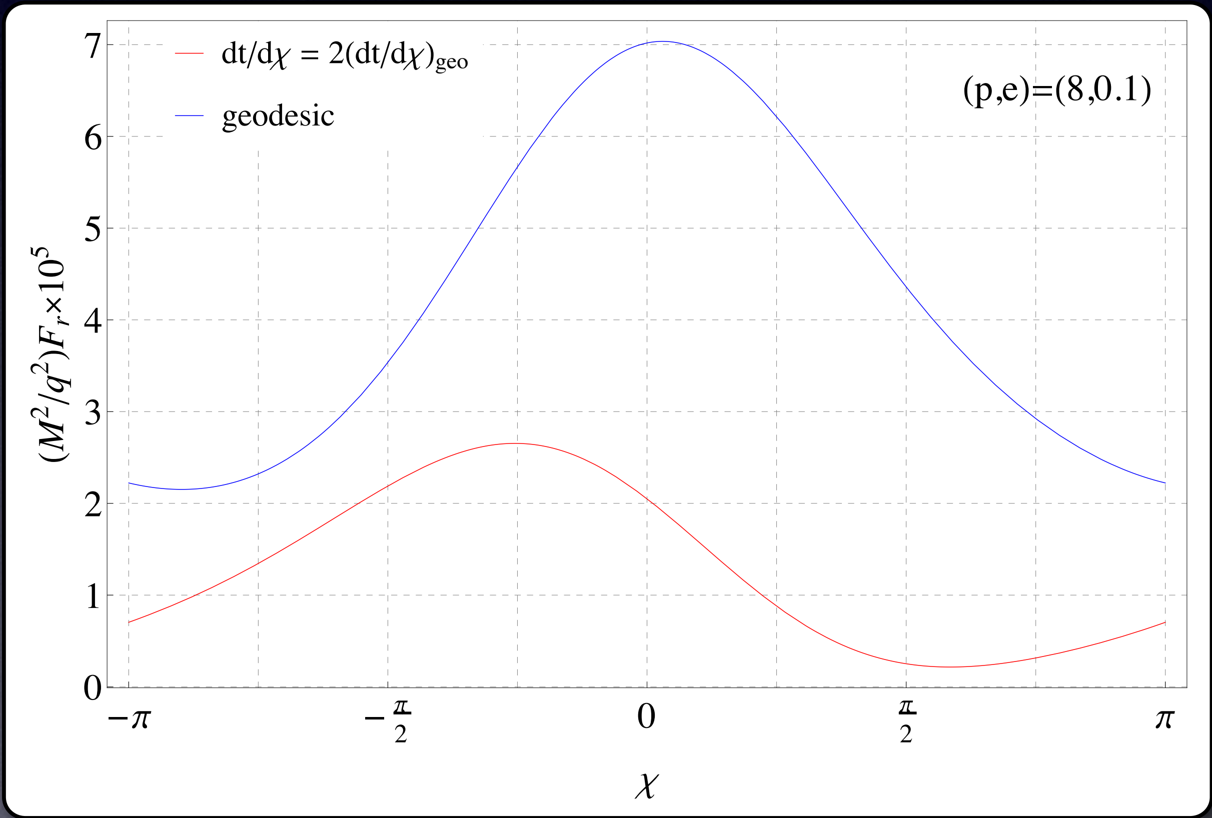
$$\Delta\varphi = \int_0^{2\pi} \frac{d\varphi_p}{d\chi}\, d\chi = 4\left(\frac{p}{p-2-2e}\right) K\left(\frac{4e}{p-6-2e}\right)$$

$$T_r = \int_0^{2\pi} \frac{dt_p}{d\chi}\, d\chi \quad \Rightarrow \quad \Omega_r = \frac{2\pi}{T_r}, \quad \Omega_\varphi = \frac{\Delta\varphi}{T_r}.$$

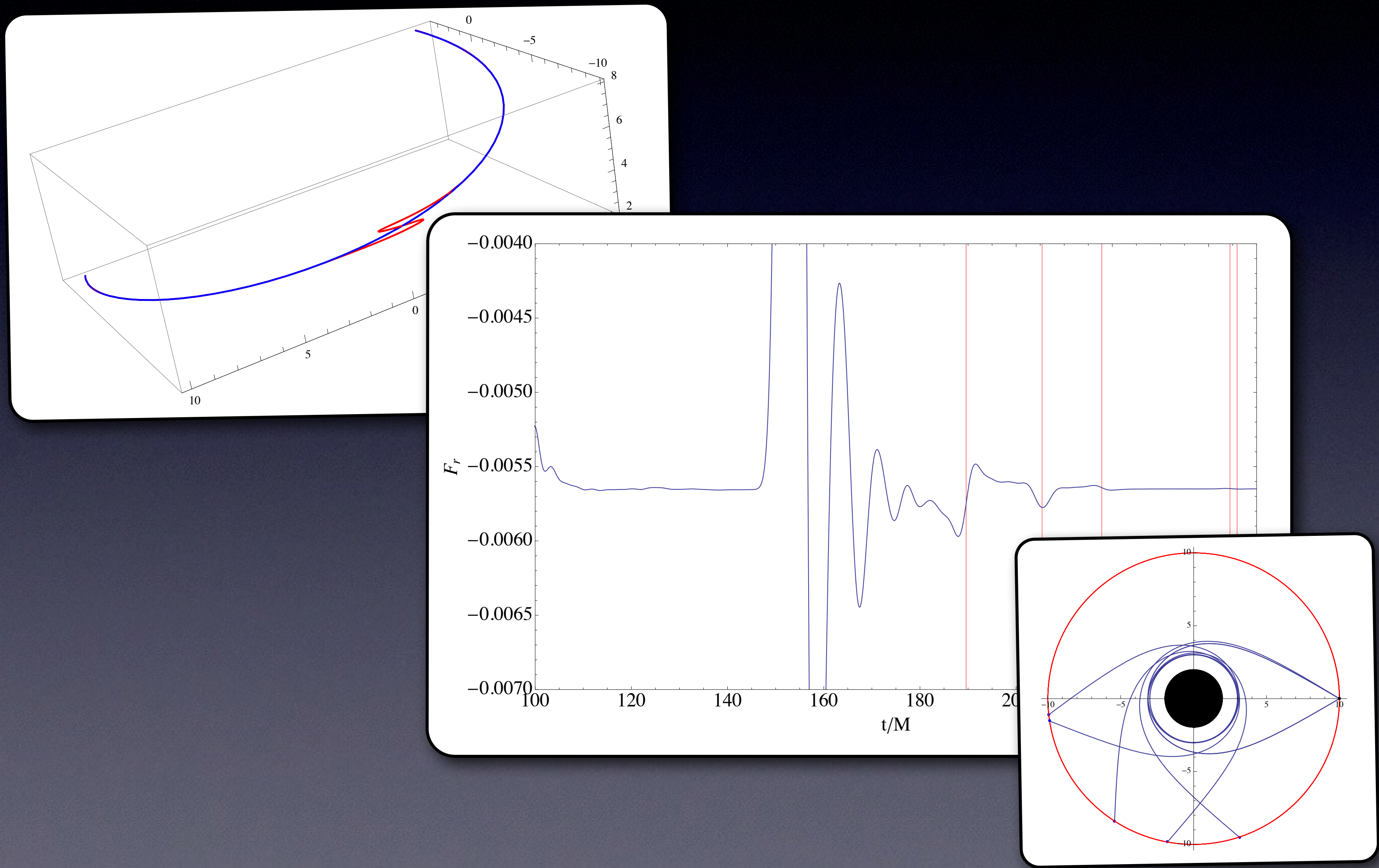
Eccentric Orbits

Relationship between t_p and χ

$$\left(\frac{dt_p}{d\chi}\right)_{geo} = \frac{Mp^2}{(p-2-2e\cos\chi)(1+e\cos\chi)^2}\sqrt{\frac{(p-2-2e)(p-2+2e)}{p-6-2e\cos\chi}}$$



Self-force Memory



Conclusions/ Further work

- Localised non-uniform acceleration for bound eccentric orbits requires time domain code
- Kerr: Uniformly accelerated circular orbits are relatively easy in frequency domain - possible investigation tool for light cone ripples (JT)?
- Cosmic censorship application?

Thanks for listening!