Gravitational self-force correction to the Kerr equatorial ISCO

Soichiro Isoyama

Based on the collaboration

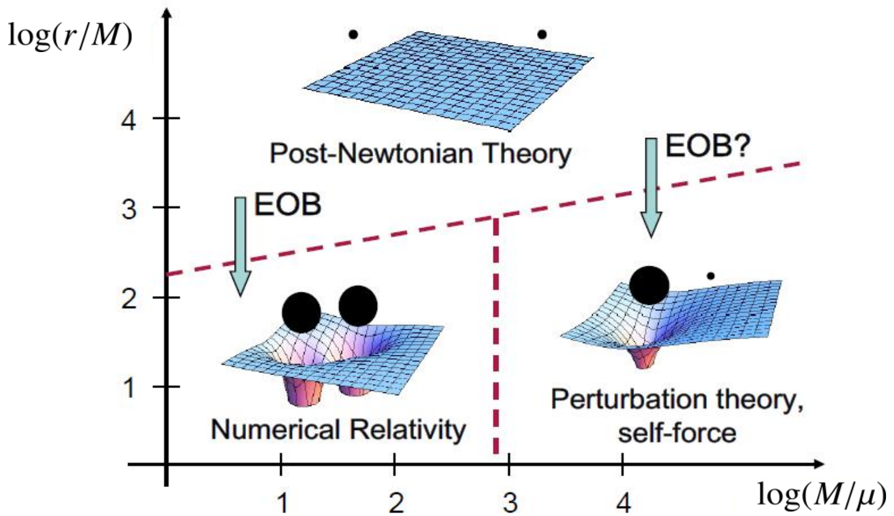
with L. Barack, S. Dolan, R. Fujita, A. Le Tiec, H. Nakano, N. Sago, A. Shah, T. Tanaka and N. Warburton

Why self-force effect @ ISCO?

- Study the 2-body dynamics **in the strong field regime**
- Extract the physical impact of self-forces on the orbital dynamics

➡ ISCO is an unique feature in motion in Kerr spacetime

- Provides a gauge invariant benchmark for other methods.



from slides by L. Barack

What we do and find ?

▪Compute the frequency shift of ISCEO in Kerr due to the conservative GSF (gravitational self-force) effects

➡ **ISCEO** ***:= I***nner most ***S***table ***C***ircular ***E***quatorial ***O***rbits $(\theta = \pi/2)$

conservative = “shut off” the dissipation

✔New ISCEO condition with the redshift variables

✔Identify ISCEO as **the minimum-energy** circular orbit

✔**A new strong-field benchmark** in 2-body problem.

▪ Using the Hamilton formulation for **the circular geodesic** in a locally-defined effective smooth spacetime

ISCEO of test particles (a = 0)

Barack and Sago 0908.1664

- At ISCEO = the restoring radial force **vanishes** under the radial variation onto a slightly eccentric orbit.

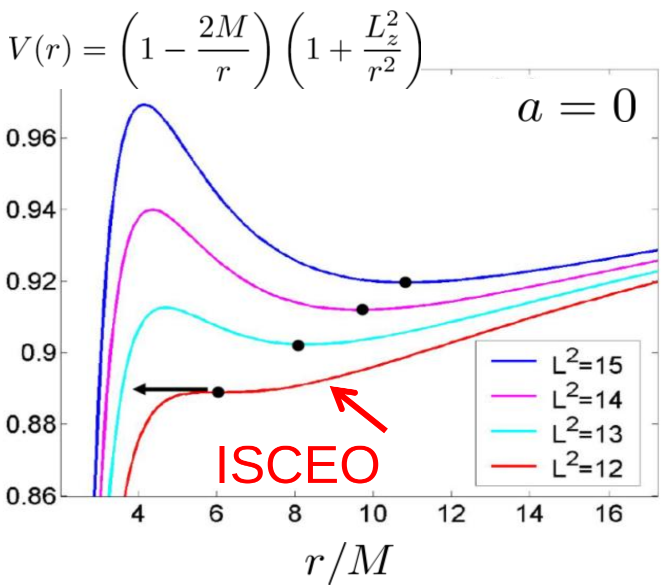
✔Radial EOM: $\frac{\mathrm{d}^2 r}{\mathrm{d}\sigma^2} = -\frac{1}{2}\frac{\partial V}{\partial r}$

✔Linear radial variation

$$r = \underbrace{r_0}_{\text{Circular radius}} + \underbrace{\delta_e r(\sigma)}_{\text{Small eccentricity}} + \mathcal{O}(e^2)$$

✔**Stability condition**

$$\frac{d^2\delta_e r}{dr^2} = -\Omega_r^2 \delta_e r$$



- The stability demands $\Omega_r = 0$ ⇒ $r_{\text{ISCEO}} = 6M$

With conservative GSF

Barack and Sago 0908.1664, Warburton and Barack: 1103.0287

- Similar argument also holds for an accelerated orbit in Kerr spacetime subjected to the **conservative GSF**

✔Radial EOM: $$m\frac{\mathrm{d}^2 r}{\mathrm{d}\sigma^2} = -\frac{m}{2}\frac{\partial V}{\partial r} + \underline{F^r_{(\mathrm{con})}}$$

Conservative GSF (symmetric under time reversal)

- Radial variations demands GSF for small eccentric orbits

Barack and Sago1002.2386

$$F^\mu = \lim_{r\to 6M}\lim_{e\to 0} F^\mu[e\,, r(\sigma)]$$

➡ **In Kerr,** no GSF code for eccentrics orbit is available

Can we find a bypass to GSF corrected ISCEO in Kerr?

Motion in effective spacetime

- Consider geodesics in locally-defined effective spacetime

$$g_{\alpha\beta} = g_{(0)\alpha\beta} + \underline{h^{R,\mathrm{sym}}_{\alpha\beta}}$$

Regularized time-symmetric part

- Describe **geodesics** in 8-dim effective phase spacetime

Mino+ 9606018, Detweiler 0202086, Poud 0907.5197, Harte 1103.0543

$$H[x^\mu, p_\mu; \gamma] := \frac{1}{2\mu} g^{\alpha\beta}_{R,\mathrm{sym}}[x^\mu, p_\mu; \gamma]\, p_\alpha p_\beta = \underline{H_{(0)}(x^\mu, p_\mu)} + H_{\mathrm{int}}[x^\mu, p_\mu; \gamma]$$

"Background Kerr part"

✔**Interaction Hamiltonian** takes care of GSF effects

$$\boxed{H_{\mathrm{int}}[x^\mu, p_\mu; \gamma] := -\frac{1}{2\mu} h^{\alpha\beta}_{R,\mathrm{sym}}[x^\mu, p_\mu; \gamma]\, p_\alpha p_\beta}$$

Orbits in effective spacetime

- Describe circular geodesics **in the effective spacetime**

✔Momenta are linked to 4-velocities and constrained

$$u^\mu := \frac{\mathrm{d}x^\mu}{\underline{\mathrm{d}\tau}} = \left.\frac{\partial H}{\partial p_\mu}\right|_\gamma = \frac{g^{\mu\nu}p_\nu}{\mu} \qquad H|_\gamma = \left.\frac{1}{2\mu}g^{\mu\nu}p_\mu p_\nu\right|_\gamma = -\frac{\mu}{2}$$

Proper time of the orbit in effective space time

✔Circularity conditions $\quad p^r = 0 \quad \frac{\mathrm{d}p_r}{\mathrm{d}\tau} = -\left.\frac{\partial H}{\partial r}\right|_\gamma = 0$

✔"Constants of motion" of the orbits exists due to **the translation invariance** of back ground Kerr geodesic

$$\frac{\mathrm{d}p_t}{\mathrm{d}\tau} = \left.\frac{\partial H_{\mathrm{int}}}{\partial t}\right|_\gamma = 0 \quad \frac{\mathrm{d}p_\phi}{\mathrm{d}\tau} = 0 \quad \Rightarrow \quad p_t =: -\mu\mathcal{E} \qquad p_\phi =: \mu\mathcal{L}$$

Notion ! all quantities are defined **in effective space time**

✔3 relevant phase spacetime coordinates: $\zeta^I := \{r,\, \mathcal{E},\, \mathcal{L}\}$

Quite similar to the circular geodesic in Kerr

ISCEO in effective spacetime

- Specifies the ISCEO by **the stationary perturbation** to a nearby **non-geodesic circular orbit** with fixed $(\mathcal{E},\mathcal{L})$

  ✔Radial variation changes “the constants of motion” at $\mathcal{O}(e^2)$

- The perturbation varies both “field” and “source” orbits.

  ✔Interaction Hamiltonian is **symmetric** w.r.t. its arguments

$$H_{\rm int}[\zeta^I;\gamma(\zeta^I_\gamma)] \approx \int_{-\infty}^{+\infty} \mathrm{d}\tau' G^{R,{\rm sym}\,\alpha\beta}{}_{\rho'\sigma'}[z(\tau)\,, z_{(\gamma)}(\tau')]\, u_\alpha(\tau)u_\beta(\tau)\, u^{\tau'}_{(\gamma)}(\tau')u^{\sigma'}_{(\gamma)}(\tau')$$

- Restoring force vanishes if

$$\underbrace{\left(\frac{\partial}{\partial r}+\frac{\partial}{\partial r_\gamma}\right)}_{\text{Stationary perturbation}} \underbrace{\left.\frac{\partial H[\zeta^I;\gamma(\zeta^I_\gamma)]}{\partial r}\right|_{\rm isceo}}_{\text{Restoring force}} = 0$$

Stationary perturbation

Restoring force

Link to the redshift variable

- Replace the perturbation into circular non-geodesics with that **within the sequence of circular geodesics**

⇨ expresses every quantities as a function of $\Omega = u^\phi / u^t$

- Introduce a redshift variables Detweiler. 0804.3529

$$(u^t)^{-1} := z(\Omega) = z_{(0)}(\Omega) + \eta\, z_{(1)}(\Omega) + \mathcal{O}(\eta^2) \qquad \eta = \mu/M \ll 1$$

✔e.g. normalization of 4-velocities with redshift variable

$$g_{\alpha\beta} u^\alpha u^\beta = -1 \quad \Rightarrow \quad z = \mathcal{E} - \Omega \mathcal{L}$$

✔varying the "on-shell" Hamiltonian with fixed frequency:

$$\delta H(\zeta^I) = \cancel{\frac{\partial H}{\partial r}} \delta r + \cdots = 0 \quad \Rightarrow \quad \frac{H_{\rm int}}{\mu} = \eta \frac{z_{(1)}}{z_{(0)}}$$

Circular condition

ISCEO with a redshift variable

- Vanishing restoring force leads **a simple condition** in terms of the modified redshift variables at ISCEO

i) Identity operator: $$\frac{\mathrm{d}}{\mathrm{d}\Omega} = r' \frac{\partial}{\partial r} + \mathcal{E}' \frac{\partial}{\partial \mathcal{E}} + \mathcal{L}' \frac{\partial}{\partial \mathcal{L}} \qquad {}' := \frac{\mathrm{d}}{\mathrm{d}\Omega}$$

ii) "On shell" conditions $$\frac{\mathrm{d}H}{\mathrm{d}\Omega} = \frac{\mathrm{d}^2 H}{\mathrm{d}\Omega^2} = 0$$

iii) Circular conditions $$p^r = \frac{\mathrm{d}p_r}{\mathrm{d}\tau} = 0$$

$$\left(\frac{\partial}{\partial r} + \frac{\partial}{\partial r_\gamma}\right) \frac{\partial H[\zeta^I; \gamma(\zeta^I_\gamma)]}{\partial r}\Bigg|_{\mathrm{isceo}} = 0 \quad \Rightarrow \quad \tilde{z}''(\Omega_{\mathrm{isceo}}) = 0$$

✔**Modified** redshift variables

$$\tilde{z}(\Omega) := z_{(0)}(\Omega) + \underline{\frac{1}{2}\,\eta\, z_{(1)}(\Omega)} + \mathcal{O}(\eta^2)$$

GSF correction with extra ½.

The frequency shift of ISCEO

- Parameterize the frequency shift of ISCEO due to the conservative GSF:

$$(M+\mu)\,\Omega_{\rm isceo} := \underbrace{M\Omega^{(0)}_{\rm isceo}(q)}_{\text{Background Kerr result}}\left\{1+\underbrace{\eta\,C_\Omega(q)}_{\text{Conservative GSF correction}}+\mathcal{O}(\eta^2)\right\}$$

- Substitution to ISCEO condition gives the desired shift

$$\boxed{\tilde{z}''(\Omega_{\rm isceo})=0 \quad\Longrightarrow\quad C_\Omega(q) = -\frac{1}{2}\,\frac{z''_{(1)}(\Omega^{(0)}_{\rm isceo})}{\Omega^{(0)}_{\rm isceo}\,z'''_{(0)}(\Omega^{(0)}_{\rm isceo})}}$$

✔Agree with the ISCEO shift in Schwarzschild case

Le Tiec+ 1111.5609

✔**What is the meaning of ISCEO condition?**

ISCEO from the binary 1st law

c.f. talks by Le Tiec and Takahiro

- For circular orbits, **perturbative binary 1st law** leads

$$\delta E = \Omega\,\delta L + z\,\delta\mu \qquad \{M, a\}\ \text{fixed}$$

✔Define the binding energy and the angular momentum to satisfy the perturbative 1st law

$$E := \mu u_t \left(1 - \frac{H_{\rm int}}{2\mu}\right) \qquad L := \mu u_\phi \left(1 - \frac{H_{\rm int}}{2\mu}\right)$$

✔The modified redshift function is related as

$$\mu\,\tilde{z} = E - \Omega L$$

- The ISCEO condition is exactly equal to the minimum-energy circular orbit (MECO) condition. cf. Buonanno+ 0205122 .

$$\tilde{z}''(\Omega_{\rm isceo}) = 0 \iff E'(\Omega_{\rm meco}) = 0$$

Numerical implementation

c.f. talks byShah and Warburton

- ISCEO shifts only needs the GSF correction to **the redshift variables** **for circular orbits** (and its derivative)

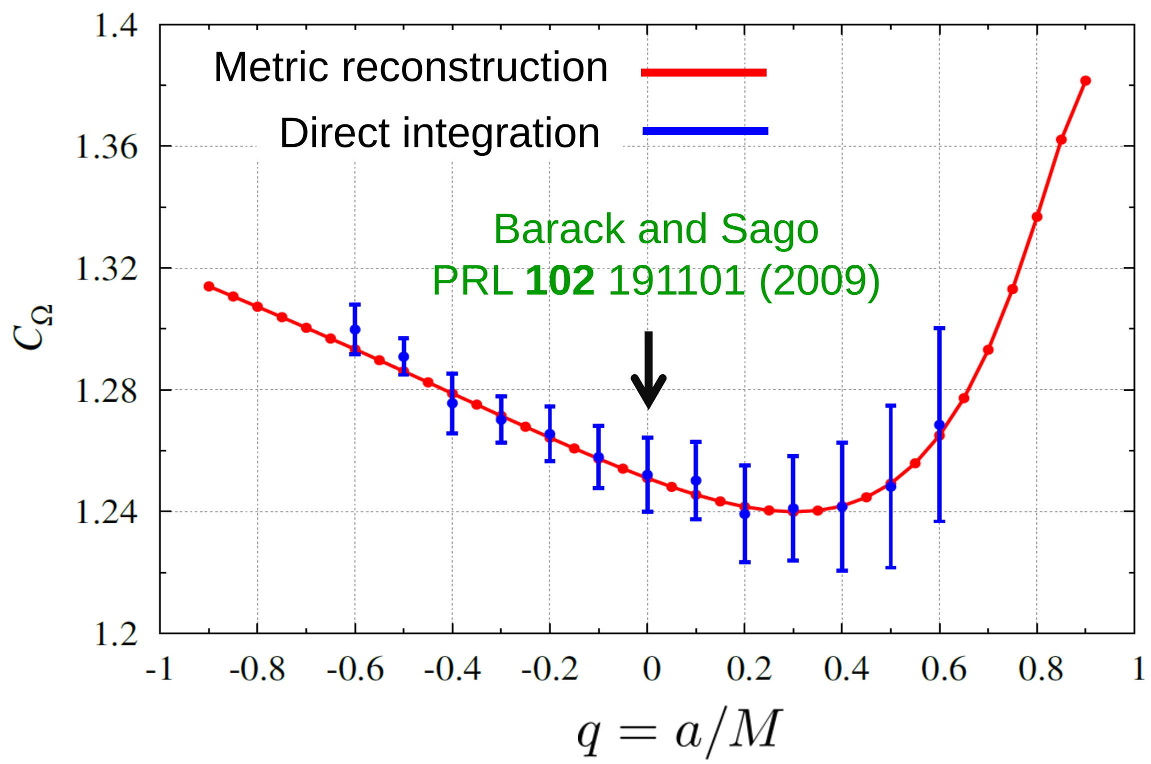
$$C_\Omega(q) = -\frac{1}{2}\frac{z''_{(1)}(\Omega^{(0)}_{\rm isceo})}{\Omega^{(0)}_{\rm isceo} z'''_{(0)}(\Omega^{(0)}_{\rm isceo})}$$

i)**Metric reconstruction in frequency-domain**, radiation gauge, mode-sum regularization Barack and Ori 9912010, Shah+ 1207.5595

ii)**Direct time-domain integration** of Einstein eq., Lorenz gauge, m-mode regularization Dolan+ 1107.0012, 1211.4686, Heffernan+ 1211.6446

Frequency shift of Kerr ISCEO

$$\Omega_{\rm isceo} := \underbrace{\Omega^{(0)}_{\rm isceo}(q)}_{\text{Kerr value}} \left\{ 1 + \underbrace{\eta C_\Omega(q)}_{\text{Conservative GSF correction}} + O(\eta^2) \right\} \qquad \eta = \mu/M \ll 1$$

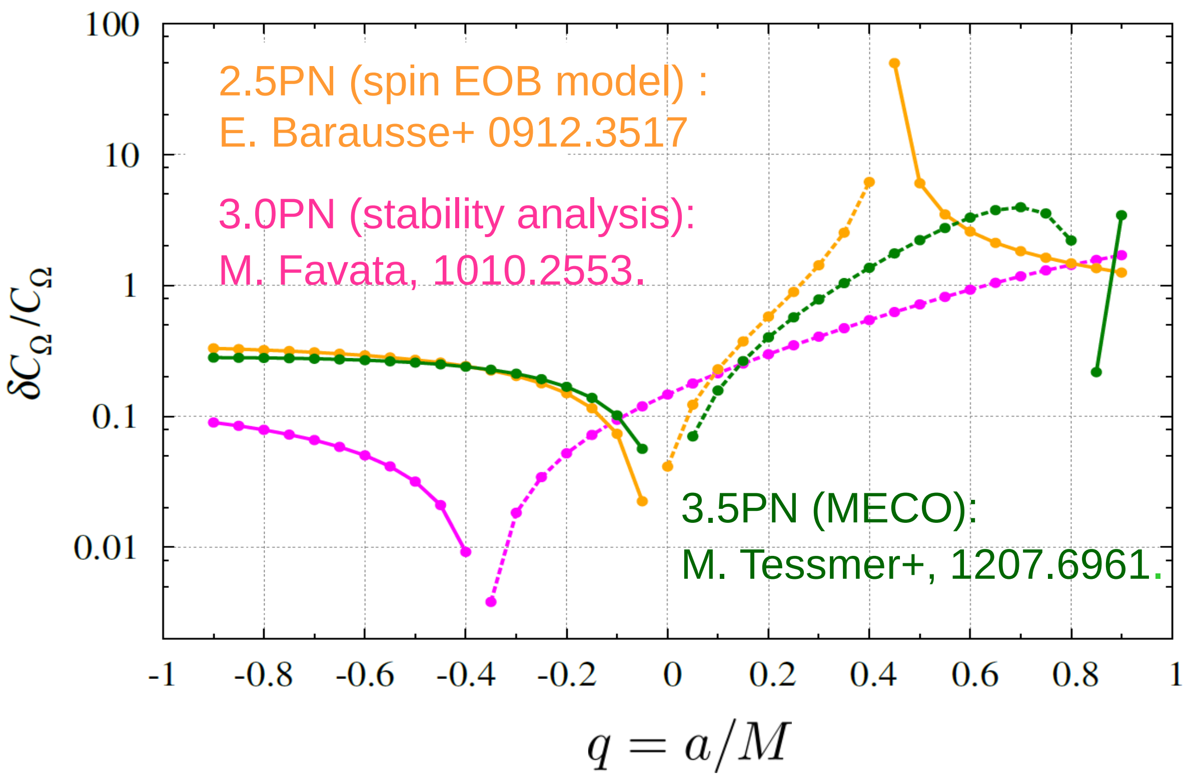


✔ All positive and "weak" dependence on the Kerr spin

Benchmark results

▪Check the predictions from previous pN-approximation and effective one-body (EOB) models

✔ Errors in ISCEO shifts: $\delta C_{\Omega}^{(\mathrm{PN/EOB})}/C_{\Omega} := 1 - C_{\Omega}^{\mathrm{PN/EOB}}/C_{\Omega}$



✔ Relatively large discrepancy in co-rotating case

Summary of the talk

▪We compute **the frequency shift of ISCEO** due to the conserved GSF effects with Hamiltonian formulation.

✔studied the circular **geodesic** in **the effective space time**

✔specified ISCEO in terms of **the redshift variables**

✔ISCEO is equivalent to **minimum-energy circular orbit**

✔tested the accuracy of pN / EOB predictions

Our results provide very accurate strong-field benchmark **for spin effects** in 2-body problem.

A future prospect

▪Two avenues for the conserved dynamics beyond **the circular equatorial orbits** in Kerr spacetime

✔Eccentric equatorial orbits.

✔Circular **inclined** (spherical) orbits:

▪Spherical orbits has non-vanishing Carter constant; the self-force impacts on this orbit is **largely unexplored.**

✔GSF corrected ISCO condition for spherical orbits (preliminary):

$$n^i n^j \left(\frac{\partial^2 \tilde{z}(\Omega^k_{\rm isco})}{\partial \Omega^i \partial \Omega^j} \right)_{J_\mu} = 0 \qquad i = (r, \theta)$$

⇨ "Minimizing" the binding energy ??

終
(Fin.)