The Laws of Binary Black Hole Mechanics

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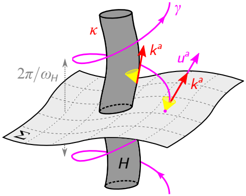
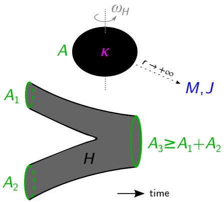
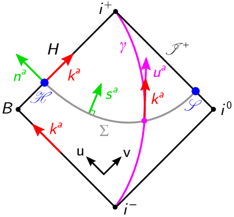






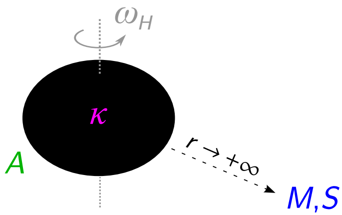
Black hole uniqueness theorem in GR

[Israel (1967); Carter (1971); Hawking (1973); Robinson (1975)]

- The only stationary vacuum black hole solution is the Kerr solution of mass M and angular momentum S

"Black holes have no hair." (J. A. Wheeler)

- Black hole event horizon H characterized by:
  - Angular velocity ω_H
  - Surface gravity κ
  - Surface area A

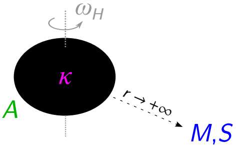


The laws of black hole mechanics

[Hawking (1972); Bardeen *et al.* (1973)]

- Zeroth law of mechanics:

$$\kappa = \text{const.} \quad (\text{over } H)$$

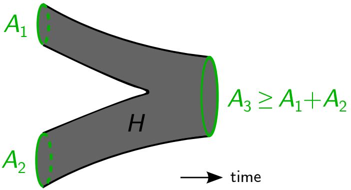


- First law of mechanics:

$$\delta M = \omega_H \, \delta S + \frac{\kappa}{8\pi} \, \delta A$$

- Second law of mechanics:

$$\delta A \geqslant 0$$



Outline

1. The laws of binary black hole mechanics
   - In general relativity
   - In post-Newtonian theory
   - In black hole perturbation theory
   - First laws for generic bound orbits

2. Applications
   - Energy and angular momentum
   - Kerr ISCO frequency shift
   - Horizon surface gravity

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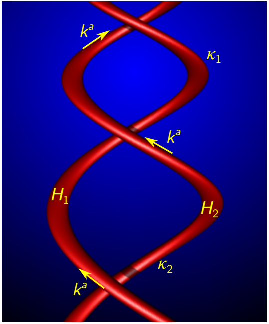
Generalized zeroth law of mechanics

[Friedman, Uryū & Shibata (2002)]

- Black hole spacetimes with *helical* Killing vector field k^a
- On each component H_i of the horizon, the expansion and shear of the null generators vanish
- Generalized rigidity theorem
  ↳ $H = \bigcup_i H_i$ is a Killing horizon
- *Constant* horizon surface gravity

$$\kappa_i^2 = -\frac{1}{2} \, (\nabla^a k^b \, \nabla_a k_b)|_{H_i}$$

- Lack of asymptotic flatness
  ↳ overall scaling of κ_i is free

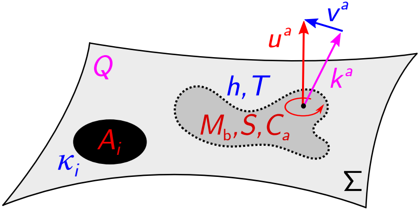


Generalized first law of mechanics

[Friedman, Uryū & Shibata (2002)]

- Spacetimes with black holes + perfect fluid matter sources
- One-parameter family of solutions $\{g_{ab}(\lambda), u^a(\lambda), \rho(\lambda), s(\lambda)\}$
- *Globally* defined Killing field k^a → conseved Noether charge Q

$$\delta Q = \sum_i \frac{\kappa_i}{8\pi}\,\delta A_i + \int_\Sigma \left[\,\bar{h}\,\delta(\mathrm{d}M_\mathrm{b}) + \bar{T}\delta(\mathrm{d}S) + v^a\delta(\mathrm{d}C_a)\right]$$



Issue of asymptotic flatness

[Friedman, Uryū & Shibata (2002)]

- Binaries on circular orbits have a *helical* Killing symmetry k^a
- Helically symmetric spacetimes are *not* asymptotically flat
  [Gibbons & Stewart (1983); Detweiler (1989); Klein (2004)]
- Asymptotic flatness can be recovered if radiation (reaction) can be "turned off" (neglected):
  - Conformal Flatness Condition
  - Post-Newtonian approximation
  - Linear perturbation theory
- For asymptotically flat spacetimes:

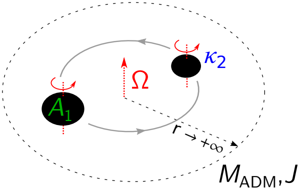
$$k^a \to t^a + \Omega\,\phi^a \quad \text{and} \quad \delta Q = \delta M_{\text{ADM}} - \Omega\,\delta J$$

Application to black hole binaries

[Friedman, Uryū & Shibata (2002)]

- Rigidity theorem → black holes are in a state of corotation
- Conformal flatness condition → asymptotic flatness recovered
  ↳ preferred normalization of κ_i
- For binary black holes the generalized first law reduces to

$$\delta M_{\mathrm{ADM}} = \Omega\,\delta J + \sum_i \frac{\kappa_i}{8\pi}\,\delta A_i$$



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First law for point-particle binaries

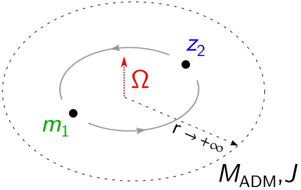
[Le Tiec, Blanchet & Whiting (2012)]

- For balls of dust, the generalized first law reduces to

$$\delta Q = \int_\Sigma z\,\delta(\mathrm{d}M_\mathrm{b}) + \cdots\,, \quad \text{where} \quad z = -k^a u_a$$

- Conservative PN dynamics → asymptotic flatness recovered
- Two *spinless* compact objects modelled as point masses m_i and moving along circular orbits obey the first law

$$\delta M_\mathrm{ADM} = \Omega\,\delta J + \sum_i z_i\,\delta m_i$$



Extension to spinning binaries

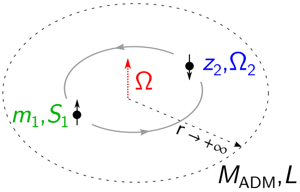
[Blanchet, Buonanno & Le Tiec (2013)]

- ADM Hamiltonian $H(\mathbf{x}_i, \mathbf{p}_i, \mathbf{S}_i; m_i)$ of two point particles with rest masses m_i and canonical spins $\mathbf{S}_i$ [Steinhoff *et al.* (2008)]
- Redshift observables and spin precession frequencies:

$$\frac{\partial H}{\partial m_i} = z_i \quad \text{and} \quad \frac{\partial H}{\partial \mathbf{S}_i} = \boldsymbol{\Omega}_i$$

- First law for aligned spins ($J = L + \sum_i S_i$) and circular orbits:

$$\delta M_{\mathrm{ADM}} = \Omega \, \delta L + \sum_i \left(z_i \, \delta m_i + \Omega_i \, \delta S_i \right)$$



Corotating point particles

[Blanchet, Buonanno & Le Tiec (2013)]

- A point particle with rest mass m_i and spin S_i is given an *irreducible* mass μ_i and a proper rotation frequency ω_i *via*

$$\delta m_i = \omega_i \, \delta S_i + c_i \, \delta \mu_i \quad \text{and} \quad m_i^2 = \mu_i^2 + S_i^2/(4\mu_i^2)$$

- The first law of binary point-particle mechanics becomes

$$\delta M_{\text{ADM}} = \Omega \, \delta J + \sum_i \left[z_i c_i \, \delta \mu_i + (z_i \, \omega_i + \Omega_i - \Omega) \, \delta S_i \right]$$

- Comparing with the first law for *corotating* black holes, $\delta M_{\text{ADM}} = \Omega \, \delta J + \sum_i (4\mu_i \kappa_i) \, \delta \mu_i$, the corotation condition is

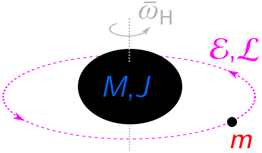
$$z_i \, \omega_i = \Omega - \Omega_i \quad \longrightarrow \quad \omega_i(\Omega)$$

Outline

Rotating black hole + orbiting moon

- Kerr black hole of mass M and spin S perturbed by a moon of mass $m \ll M$:

$$g_{ab}(\varepsilon) = \bar{g}_{ab} + \varepsilon\,\mathcal{D}g_{ab} + \mathcal{O}(\varepsilon^2)$$



- Perturbation $\mathcal{D}g_{ab}$ obeys the linearized Einstein equation with point-particle source

$$\mathcal{D}G_{ab} = 8\pi\,\mathcal{D}T_{ab} = 8\pi\, m \int_\gamma \mathrm{d}\tau\, \delta_4(x,y)\, u_a u_b$$

- Particle has energy $\mathcal{E} = -m\, t^a u_a$ and ang. mom. $\mathcal{L} = m\,\phi^a u_a$
- Physical $\mathcal{D}g_{ab}$: retarded solution, no incoming radiation, perturbations $\mathcal{D}M_{\rm B} = \mathcal{E}$ and $\mathcal{D}J = \mathcal{L}$ [Keidl *et al.* (2010)]

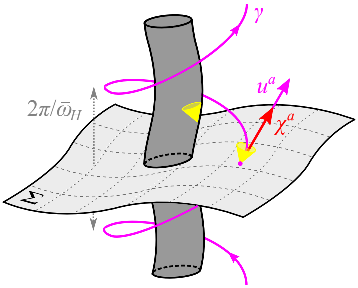
Rotating black hole + corotating moon

- We choose for the geodesic γ the unique equatorial, circular orbit with azimuthal frequency $\bar{\omega}_H$, i.e., the *corotating* orbit
- Gravitational radiation-reaction is $\mathcal{O}(\varepsilon^2)$ and neglected
  ↳ the spacetime geometry has a helical symmetry
- In adapted coordinates, the helical Killing field reads

$$\chi^a = t^a + \bar{\omega}_H \, \phi^a$$

- Conserved orbital quantity associated with symmetry:

$$z \equiv -\chi^a u_a = m^{-1} \left(\mathcal{E} - \bar{\omega}_H \, \mathcal{L} \right)$$



Zeroth law for a black hole with moon

[Gralla & Le Tiec (2013)]

- Because of helical symmetry and corotation, the expansion and shear of the *perturbed* future event horizon H vanish
- Rigidity theorems then imply that H is a Killing horizon
  [Hawking (1972); Chruściel (1997); Friedrich *et al.* (1999); etc]
- The horizon-generating Killing field must be of the form

$$k^a(\varepsilon) = t^a + \big(\underbrace{\bar{\omega}_H + \varepsilon\, \mathcal{D}\omega_H}_{\substack{\text{circular orbit} \\ \text{frequency } \Omega}}\big)\phi^a + \mathcal{O}(\varepsilon^2)$$

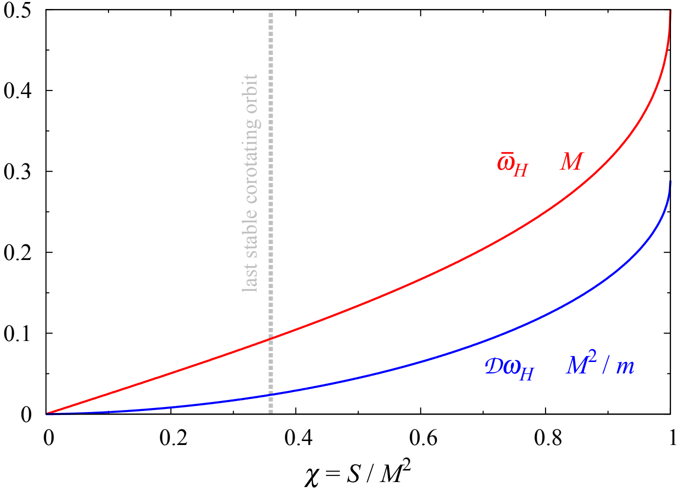
- The surface gravity κ is defined in the usual manner as

$$\kappa^2 = -\frac{1}{2}\,(\nabla^a k^b\, \nabla_a k_b)|_H$$

- Since $\kappa =$ const. over *any* Killing horizon [Bardeen *et al.* (1973)], we have proven a zeroth law for the *perturbed* event horizon

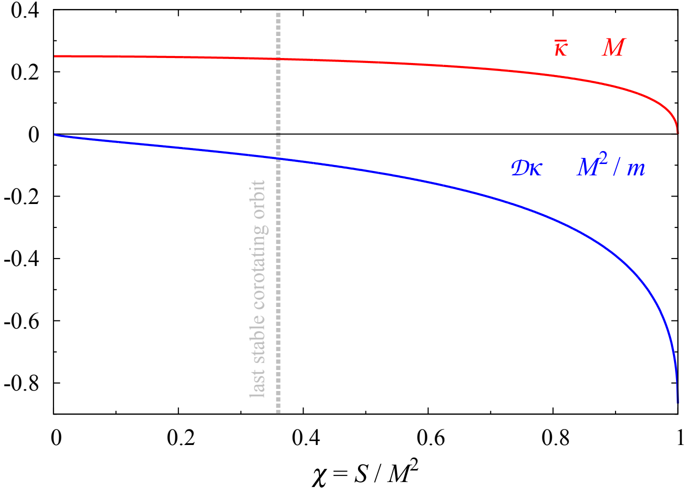
Angular velocity *vs* black hole spin

[Gralla & Le Tiec (2013)]



Surface gravity *vs* black hole spin

[Gralla & Le Tiec (2013)]



First law for a black hole with moon

[Gralla & Le Tiec (2013)]

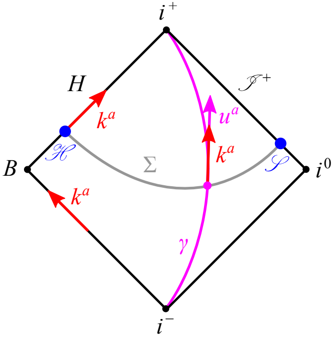
- Adapting [Iyer & Wald (1994)] to non-vacuum perturbations of non-stationary spacetimes we find (with $Q_{ab} \equiv -\varepsilon_{abcd} \nabla^c k^d$)

$$\int_{\partial\Sigma} (\delta Q_{ab} - \Theta_{abc} k^c) = 2\,\delta \int_\Sigma \varepsilon_{abcd} G^{de} k_e - \int_\Sigma \varepsilon_{abcd} k^d G^{ef} \delta g_{ef}$$

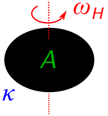
- Applied to nearby BH with moon spacetimes, this gives the first law

$$\delta M_\mathrm{B} = \Omega\,\delta J + \frac{\kappa}{8\pi}\,\delta A + z\,\delta m$$

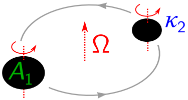
- Features variations of the Bondi mass and angular momentum



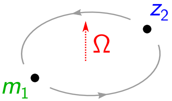
Black holes and point particles



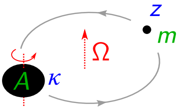
$$\delta M - \omega_H \, \delta S = \frac{\kappa}{8\pi} \, \delta A \qquad\qquad M - 2\omega_H S = \frac{\kappa A}{4\pi}$$



$$\delta M - \Omega \, \delta J = \sum_i \frac{\kappa_i}{8\pi} \, \delta A_i \qquad\qquad M - 2\Omega J = \sum_i \frac{\kappa_i A_i}{4\pi}$$



$$\delta M - \Omega \, \delta J = \sum_i z_i \, \delta m_i \qquad\qquad M - 2\Omega J = \sum_i z_i m_i$$



$$\delta M - \Omega \, \delta J = \frac{\kappa}{8\pi} \, \delta A + z \, \delta m \qquad\qquad M - 2\Omega J = \frac{\kappa A}{4\pi} + z m$$

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Particle Hamiltonian first law

- Geodesic motion of test mass m in Kerr geometry $\bar{g}_{ab}$ derives from canonical Hamiltonian

$$\bar{H}(x^\mu, p_\mu) = \frac{1}{2m}\,\bar{g}^{ab}(x)p_a p_b$$

- Hamilton-Jacobi equation completely separable [Carter (1968)]
- Canonical transformation $(x^\mu, p_\mu) \to (q_\alpha, J_\alpha)$ to *generalized* action-angle variables [Schmidt (2002); Hinderer & Flanagan (2008)]

$$\frac{\mathrm{d}J_\alpha}{\mathrm{d}\tau} = -\frac{\partial \bar{H}}{\partial q_\alpha} = 0\,, \quad \frac{\mathrm{d}q_\alpha}{\mathrm{d}\tau} = \frac{\partial \bar{H}}{\partial J_\alpha} \equiv \omega_\alpha$$

- Varying $\bar{H}(J_\alpha)$ yields a particle Hamiltonian first law valid for *generic* bound orbits [Le Tiec (2014)]

$$\delta\mathcal{E} = \Omega_\varphi\,\delta\mathcal{L} + \Omega_r\,\delta J_r + \Omega_\theta\,\delta J_\theta + \langle z\rangle\,\delta m$$

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$$\delta\mathcal{E} = \Omega\,\delta\mathcal{L} + z\,\delta m$$

Inclusion of conservative GSF effects

[Isoyama *et al.* (in preparation)]

- Geodesic motion of self-gravitating mass m in *time-symmetric* regular metric $g_{ab} + h^{\mathrm{R}}_{ab}$ derives from canonical Hamiltonian

$$\mathcal{H}[x^\mu, p_\mu; \gamma] = \bar{H}(x^\mu, p_\mu) + H_{\mathrm{int}}[x^\mu, p_\mu; \gamma]$$

- It is still possible to perform a canonical transformation $(x^\mu, p_\mu) \to (q_\alpha, J_\alpha)$ to generalized action-angle variables
- Varying $\mathcal{H}(J_\alpha)$ yields a first law valid for *generic* bound orbits

$$\delta\mathcal{E} = \Omega_\varphi\, \delta\mathcal{L} + \Omega_r\, \delta J_r + \Omega_\theta\, \delta J_\theta + \langle z \rangle\, \delta m$$

- The actions J_α, fundamental frequencies Ω_α, and averaged redshift $\langle z \rangle$ include conservative GSF corrections from H_{int}

Inclusion of conservative GSF effects

Next talk by Tanaka [Isoyama *et al.* (in preparation)]

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ADM mass, Bondi mass, binding energy

- Conservation of mass-energy

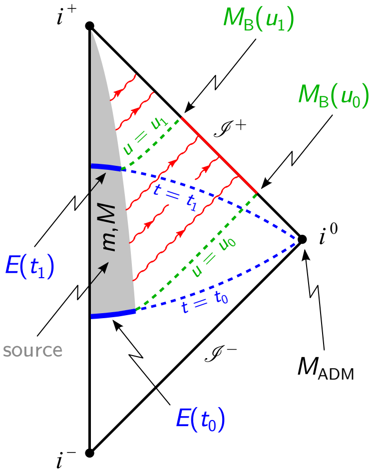
$$M_{\mathrm{ADM}} = M_{\mathrm{B}}(u) + \int_{-\infty}^{u} \mathcal{F}(u')\,\mathrm{d}u'$$

- Bondi-Sachs mass loss formula

$$\frac{\mathrm{d}M_{\mathrm{B}}}{\mathrm{d}u} = -\mathcal{F}(u)$$

- Binding energy of the binary

$$E(t) = M_{\mathrm{B}}(u) - (M + m)$$



Binding energy beyond the test-mass limit

[Le Tiec, Barausse & Buonanno (2012)]

- The binding energy E is a function of $x \equiv [(M+m)\Omega]^{2/3}$
- In the extreme mass ratio limit $q \equiv m/M \ll 1$,

$$z = \sqrt{1-3x} + q\, z_{\mathrm{GSF}}(x) + \mathcal{O}(q^2)$$
$$\frac{E}{\mu} = \left(\frac{1-2x}{\sqrt{1-3x}} - 1\right) + q\, E_{\mathrm{GSF}}(x) + \mathcal{O}(q^2)$$

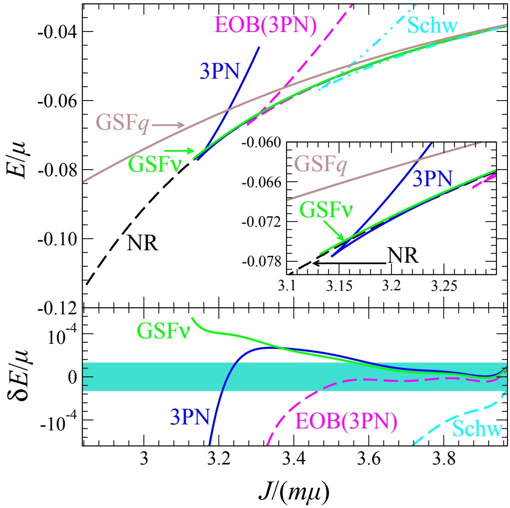
- The exact conservative self-force effect $z_{\mathrm{GSF}}(x)$ is known
  [Detweiler (2008); Shah *et al.* (2011); Akcay *et al.* (2012); etc]
- The first law provides a relationship $E \leftrightarrow z$, which implies

$$E_{\mathrm{GSF}}(x) = \frac{1}{2}\, z_{\mathrm{GSF}}(x) - \frac{x}{3}\, z'_{\mathrm{GSF}}(x) + f(x)$$

- A similar result holds for the total angular momentum J

Binding energy *vs* angular momentum

[Le Tiec, Barausse & Buonanno (2012)]



Outline

Innermost stable circular orbit (ISCO)

- The innermost stable circular orbit is identified by a vanishing restoring radial force under small-e perturbations:

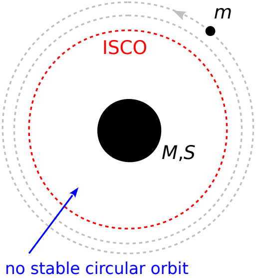
$$\frac{\partial^2 H}{\partial r^2} = 0 \quad \longrightarrow \quad \Omega_{\text{ISCO}}$$

- The minimum energy circular orbit is the most bound orbit along a sequence of circular orbits:

$$\frac{\partial E}{\partial \Omega} = 0 \quad \longrightarrow \quad \Omega_{\text{MECO}}$$

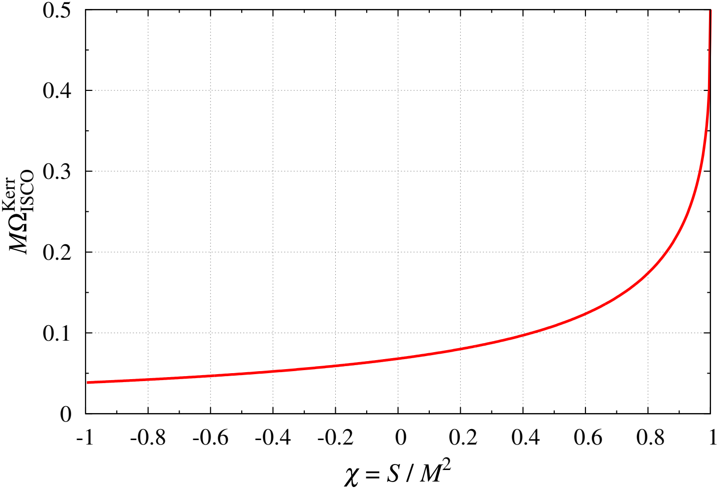
- For Hamiltonian systems [Buonanno *et al.* (2003)]

$$\Omega_{\text{ISCO}} = \Omega_{\text{MECO}}$$



Kerr ISCO frequency *vs* black hole spin

[Bardeen, Press & Teukolsky (1972)]



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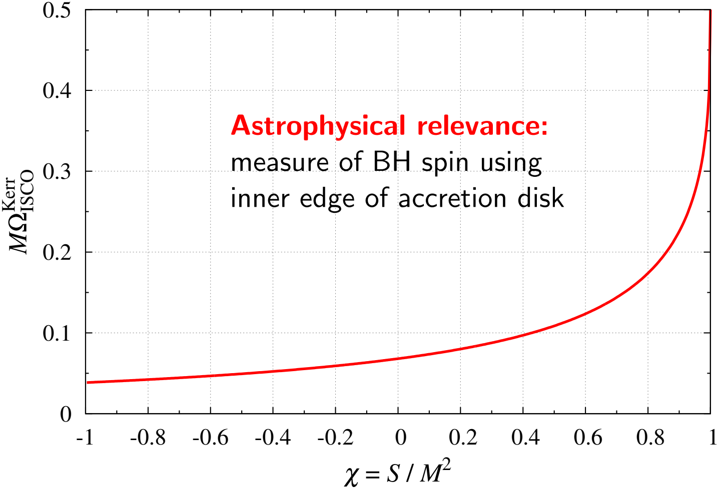


Frequency shift of the Kerr ISCO

[Isoyama *et al.* (2014)]

- The orbital frequency of the Kerr ISCO is shifted under the effect of the conservative self-force:

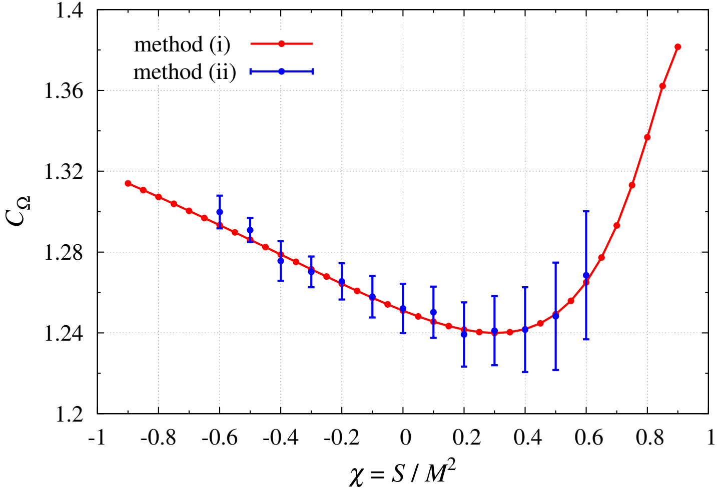
$$(M+m)\Omega_{\mathrm{ISCO}} = \underbrace{M\Omega^{\mathrm{Kerr}}_{\mathrm{ISCO}}(\chi)}_{\substack{\text{test mass}\\ \text{result}}} \left\{ 1 + \underbrace{q\, C_\Omega(\chi)}_{\substack{\text{conservative}\\ \text{GSF effect}}} + \mathcal{O}(q^2) \right\}$$

- The frequency shift can be computed from a stability analysis of slightly eccentric orbits near the Kerr ISCO
- Combining the Hamiltonian first law with the MECO conditio $\partial E/\partial\Omega = 0$ yields the same result:

$$C_\Omega = \frac{1}{2}\, \frac{z''_{\mathrm{GSF}}(\Omega^{\mathrm{Kerr}}_{\mathrm{ISCO}})}{\mathcal{E}''(\Omega^{\mathrm{Kerr}}_{\mathrm{ISCO}})}$$

ISCO frequency shift *vs* black hole spin

[Isoyama *et al.* (2014)]



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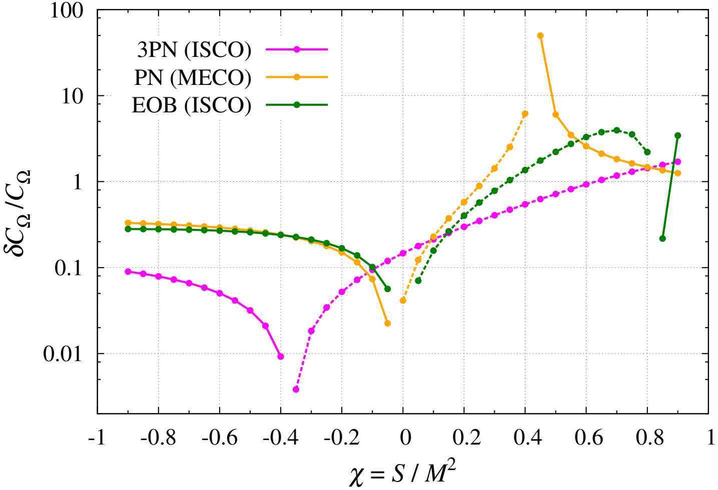
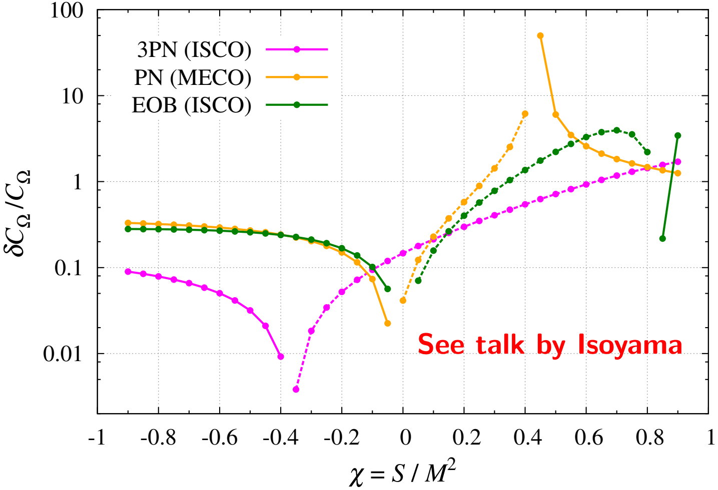


ISCO frequency shift *vs* black hole spin

[Isoyama *et al.* (2014)]



Outline

Surface gravity and redshift observable

[Blanchet, Buonanno & Le Tiec (2013)]

- First law for corotating black holes

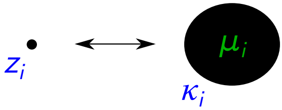
$$\delta M_{\mathrm{ADM}} = \Omega\,\delta J + \sum_i (4\mu_i\kappa_i)\,\delta\mu_i$$

- First law for corotating point particles

$$\delta M_{\mathrm{ADM}} = \Omega\,\delta J + \sum_i z_i c_i\,\delta\mu_i$$

- Analogy between BH surface gravity and particle redshift

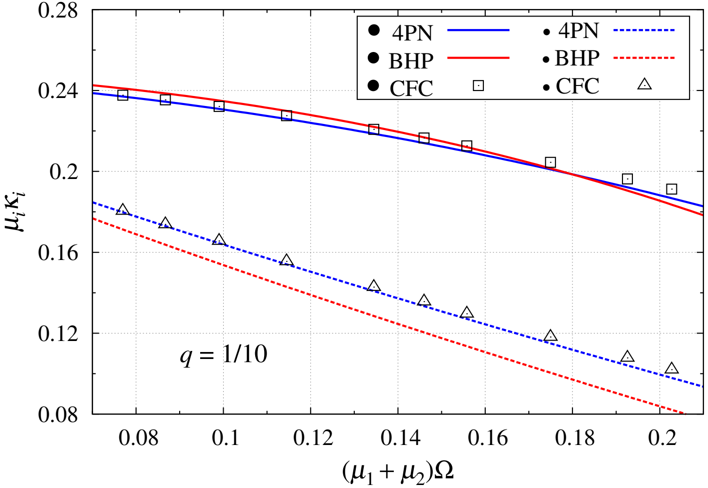
$$4\mu_i\kappa_i \longleftrightarrow z_i c_i$$



- New *invariant* relations for NR/BHP/PN comparison: $\kappa_i(\Omega)$

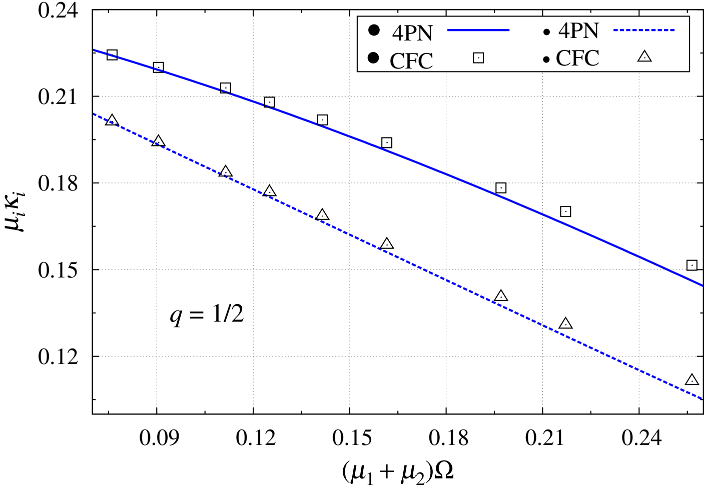
Surface gravity *vs* orbital frequency

[Grandclément & Le Tiec (work in progress)]



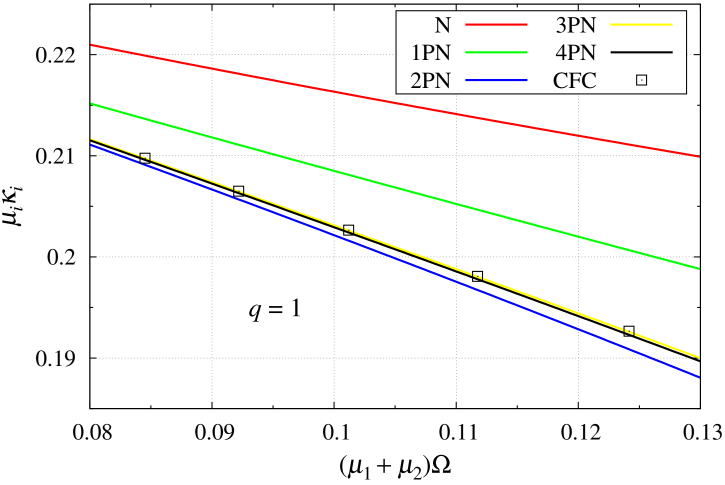
Surface gravity *vs* orbital frequency

[Grandclément & Le Tiec (work in progress)]



Surface gravity *vs* orbital frequency

[Grandclément & Le Tiec (work in progress)]



Summary and prospects

- The classical laws of black hole mechanics can be extended to binary systems of compact objects
- First laws of mechanics come in a variety of different forms:
  - Context: exact GR, perturbation theory, PN theory
  - Objects: black holes, point particles
  - Orbits: circular, generic bound
  - Derivation: geometric, Hamiltonian
- Combined with the first law, the redshift $z(\Omega)$ provides crucial information about the binary dynamics:
  - Binding energy E, total angular momentum J
  - Innermost stable circular orbit frequency Ω_{ISCO}
  - Horizon surface gravity κ

Summary and prospects

- Exploit the Hamiltonian first law for a particle in Kerr:
  - Innermost spherical orbit [Tanaka's & Isoyama's talks]
  - Marginally bound orbits [Colleoni's talk]
- Extend PN Hamiltonian first law for two spinning particles:
  - Non-aligned spins and generic precessing orbits
  - Contribution from quadrupole moments [Dolan's talk]
- Explore the surface gravity in corotating black holes:
  - Compute $\mathcal{D}\kappa$ from a direct analysis of h_{ab}^{ret} [Shah's talk]
  - Perturbative prediction for κ with $q \to \nu$ [Focused discussion]
- Redshift at second order $\to \mathcal{O}(q^2)$ corrections in $E(\Omega), J(\Omega)$ [Pound's talk]