CBPF
Centro Brasileiro de Pesquisas Físicas

Self-force via worldline integration of the Green function

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SF Methods via Field

- Impressive SF results have been obtained in recent years by using methods based on the direct calculation of the field:

$$f_\alpha = \nabla_\alpha \Phi^R$$

Mode-sum Regularization

$$\Box \Phi^{ret/S} = -4\pi\rho$$

$$\Phi^R = \Phi^{ret} - \Phi^S$$

Effective Source Method

$$\Box \Phi^{ret} = \Box(\Phi^R + \Phi^S)$$

$$\Box \Phi^{res} = -4\pi\rho - \Box(W\Phi^P)$$

SF via Green Function

- MiSaTaQuWa eq. for the SF is in terms of a *tail* integral of the retarded Green function over the past world-line

$$f^{\mu}(\tau) = (\text{local terms}) + q^2 \nabla^{\mu} \int_{-\infty}^{\tau^-} G_{ret}(z(\tau), z(\tau'))d\tau'$$

- The *tail* integral contains information about the history dependence of the SF
- This can be understood geometrically in terms of 'backscattering' (generally, fields do not propagate only along null geodesics) and from trapping of null geodesics

Green function

- The retarded Green function is a solution of the wave equation with a δ-source satisfying causality b.c.

$$\Box_x G_{ret}(x,x') = -4\pi\delta_4(x,x')$$

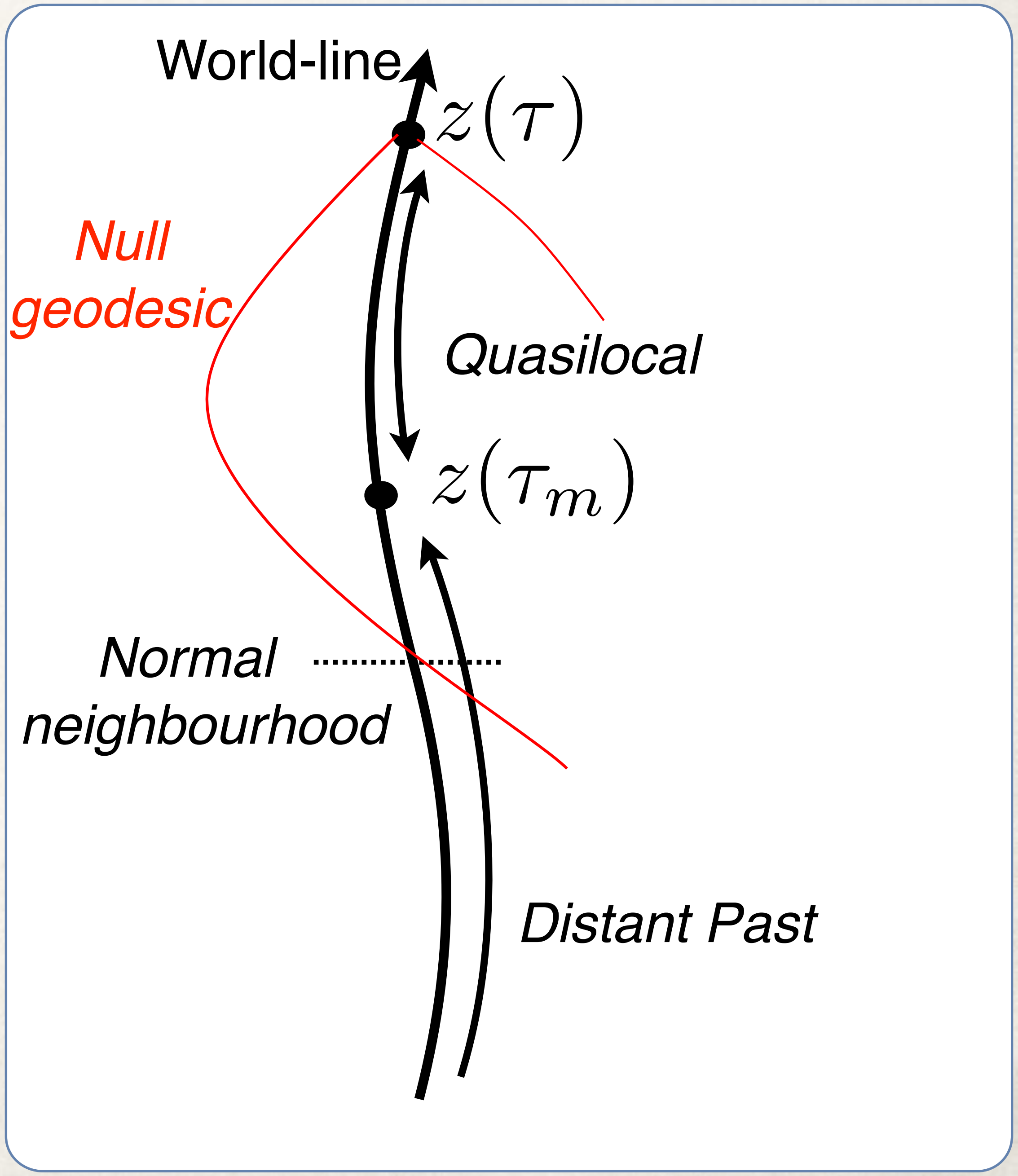
- But G_{ret} has a δ-divergence at $x = x'$. So if G_{ret} were calculated via, eg, a mode-sum, the number of modes required for a certain accuracy would grow as x' approaches $x = z(\tau)$

Method of Matched Expansions

- Poisson&Wiseman (Capra2-Dublin'99) suggestion:

$$\int_{-\infty}^{\tau^-} G_{ret}\, d\tau' =$$

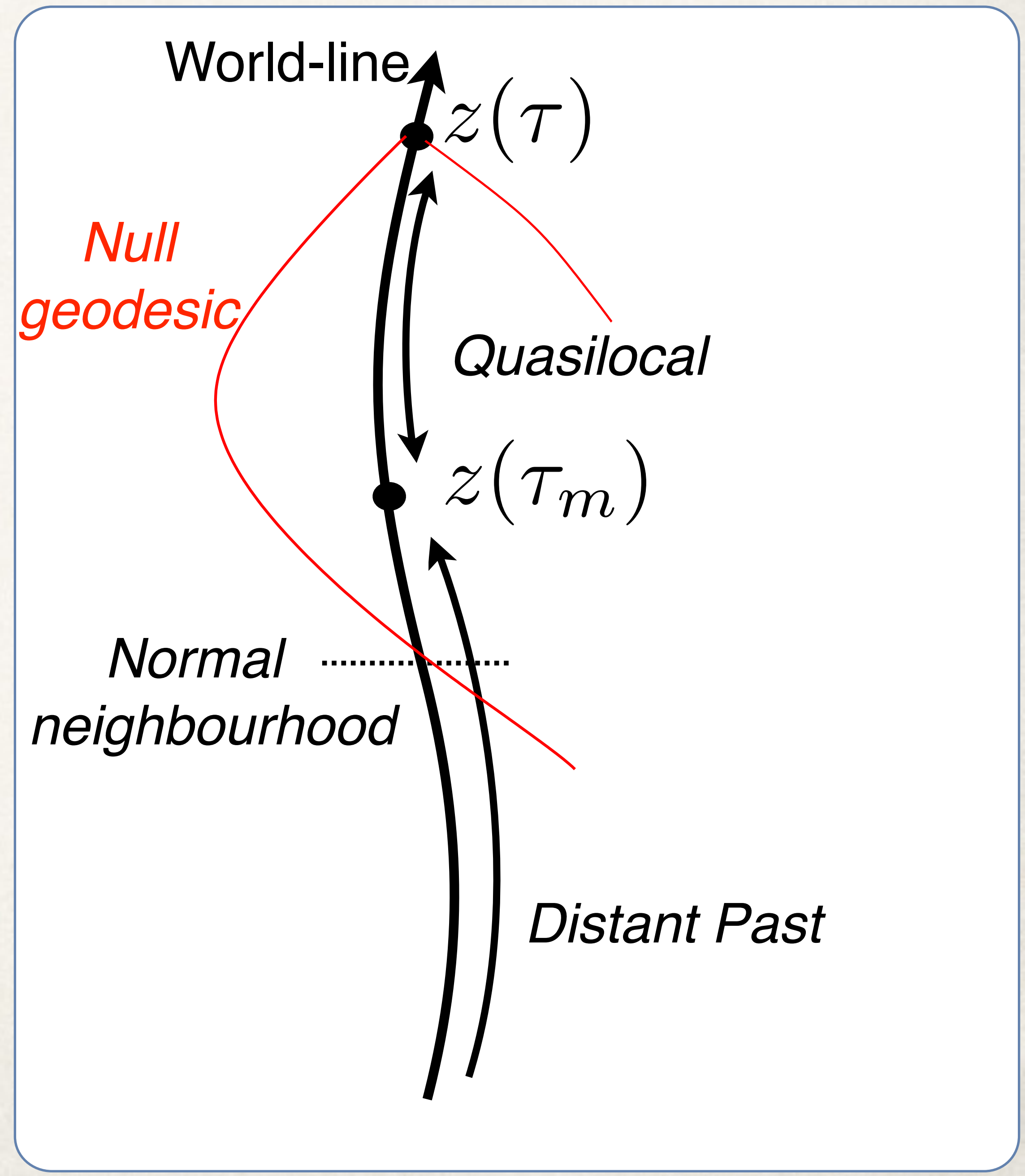
$$\underbrace{\int_{-\infty}^{\tau_m} G_{ret}\, d\tau'}_{\text{DP}} + \underbrace{\int_{\tau_m}^{\tau^-} G_{ret}\, d\tau'}_{\text{QL}}$$



Method of Matched Expansions

- Does a matching time τ_m exist *in practise*?

- Anderson&Wiseman'05: weak field approx. in DP in Schwarzschild. Found "poor" convergence in the DP mode-sum



QL - Hadamard form

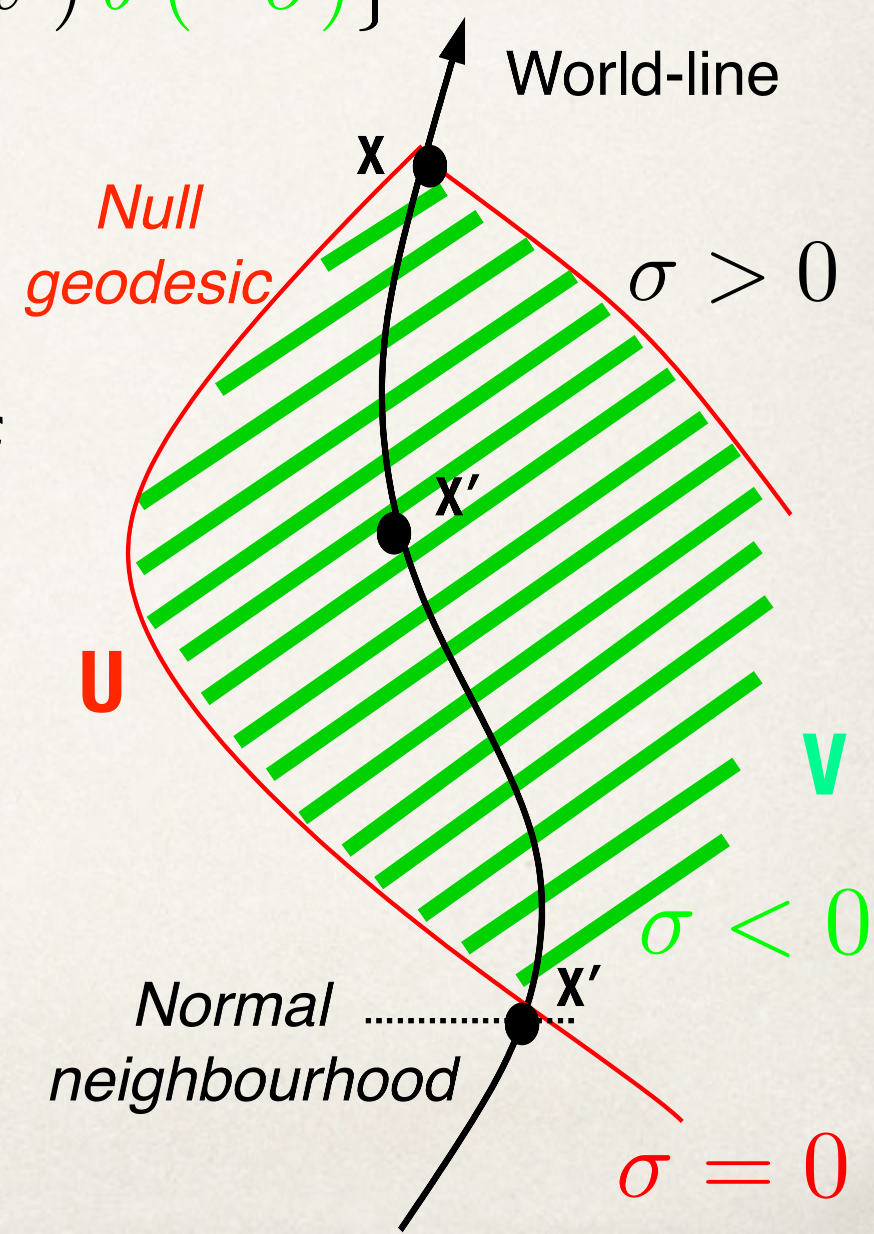
- Hadamard form is valid in a normal nbd (unique geod. joining x & x')

$$G_{ret}\left(x,x'\right)=\theta\left(\Delta t\right)\left\{U\left(x,x'\right)\delta\left(\sigma\right)+V\left(x,x'\right)\theta\left(-\sigma\right)\right\}$$

- U & V: regular biscalars

- It renders regularization trivial (subtraction of the Detweiler-Whiting singular GF):

$$\int_{\tau_m}^{\tau^-} G_{ret}\, d\tau' = \int_{\tau_m}^{\tau^-} V d\tau'$$



QL - Hadamard form

- Calculate V with, eg, coordinate expansion (Heffernan,Ottewill&Wardell'13)

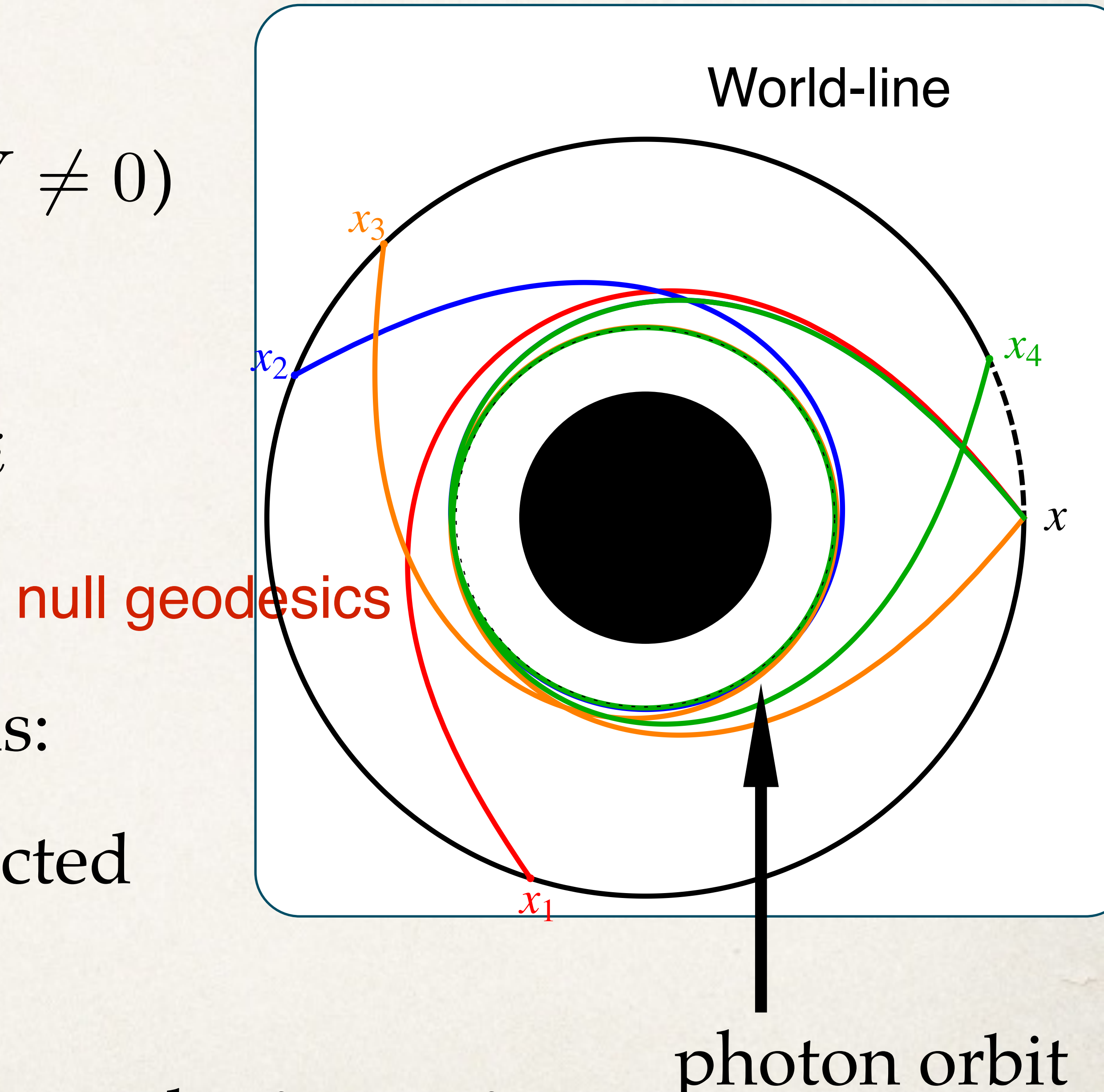
$$V(x,x') = \sum_{i,j,k,n=0}^{\infty} v_{ijkn}(r,\theta)\ (t-t')^i(\theta-\theta')^j(\varphi-\varphi')^n(r-r')^k$$

- Improve accuracy&domain of validity via knowledge of singularity structure at 1st light-crossing and use of Padé approximants

Tail contribution to the SF

- Contributions:
  1. Backscattering due to potential (ie, $V \neq 0$)
  2. Trapping of null surfaces:

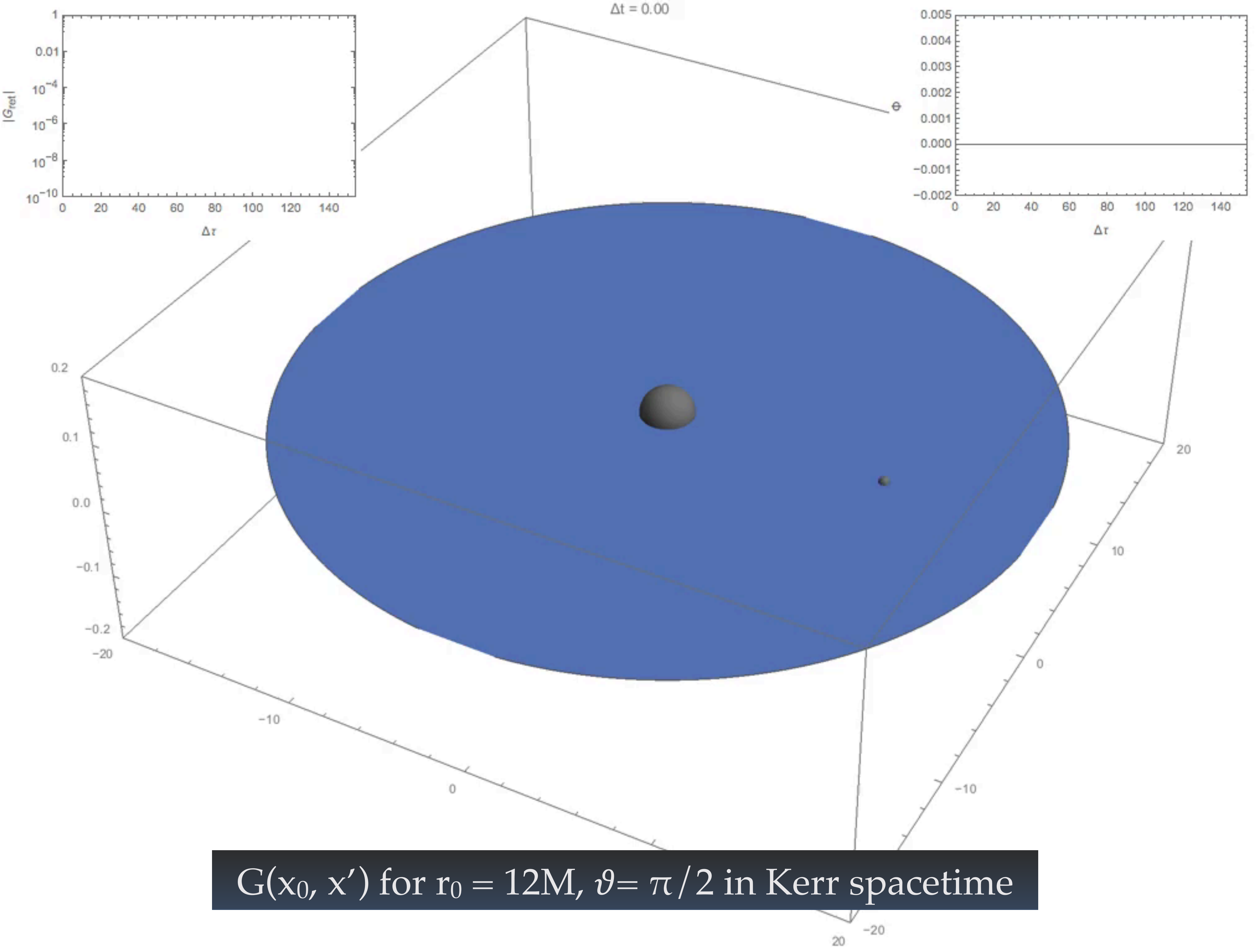
  light-crossings ('caustic echoes') at x_i



- "Propagation of singularities" theorems:

  $G_{ret}(x,x')$ diverges if x & x' are connected by a null geodesic

- But these theorems do not inform us about the form of the singularity



$G(x_0, x')$ for $r_0 = 12M$, $\vartheta = \pi/2$ in Kerr spacetime

DP Calculations

- Calculations of DP in weak-field: DeWitt&DeWitt'64, Capon PhD'98 (Schutz), Nakano&Sasaki'01, Anderson&Wiseman'05,…

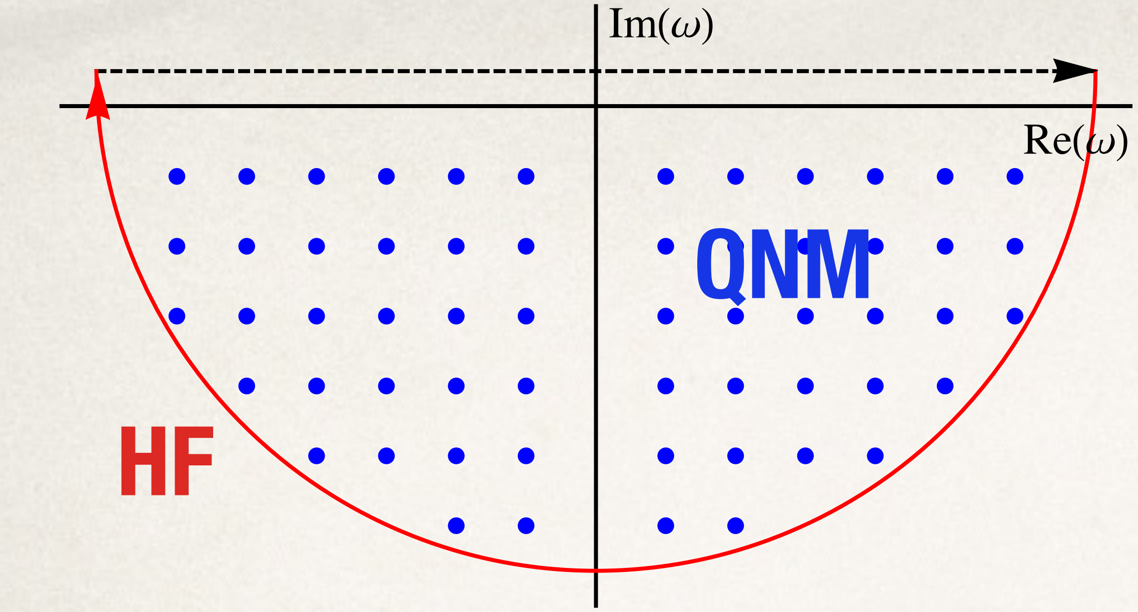
- Calculations of DP in strong-field performed so far:
  - Numerical: solve a PDE (Barry's part of the talk)
  - Semi-analytical: Fourier mode decomposition

DP - Fourier series

- Axisymmetric & stationary spacetime
- Fourier transform in time and harmonic mode decomposition:
- Need to calculate the spheroidal harmonics and 2 lin. indep. slns. of the homogenous radial ODE

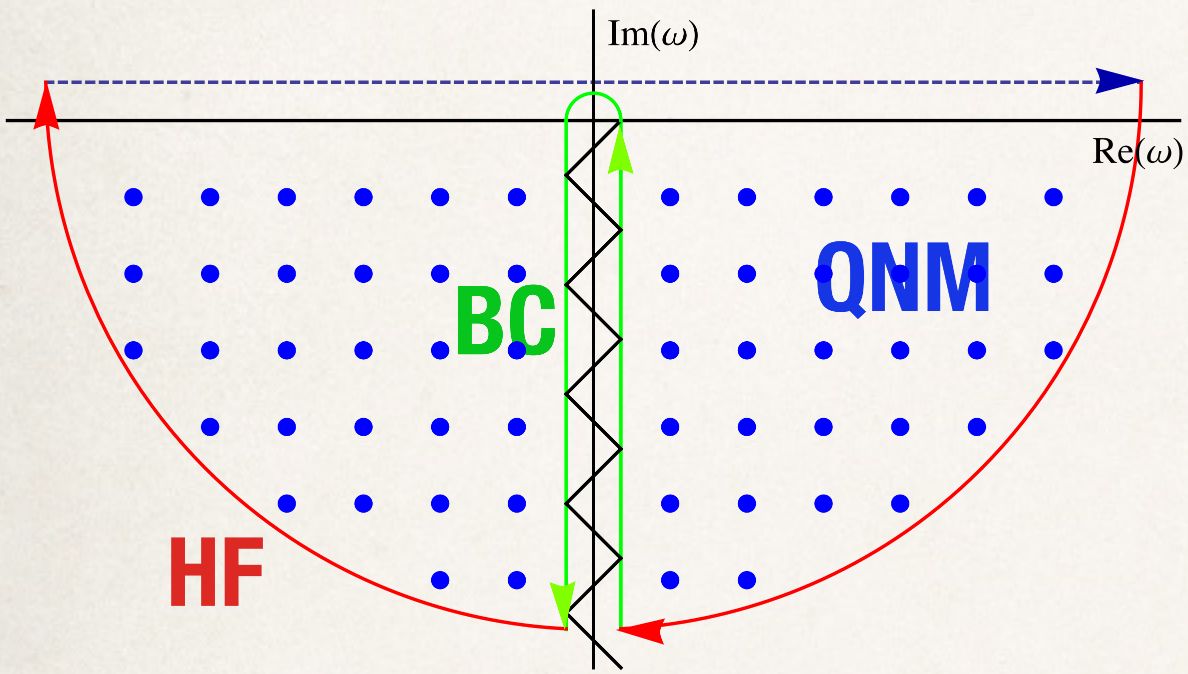
DP - Complex-ω plane

- Deform contour of integration into complex-ω plane.
- Apply residue th. to account for the singularities of the Fourier modes G_m



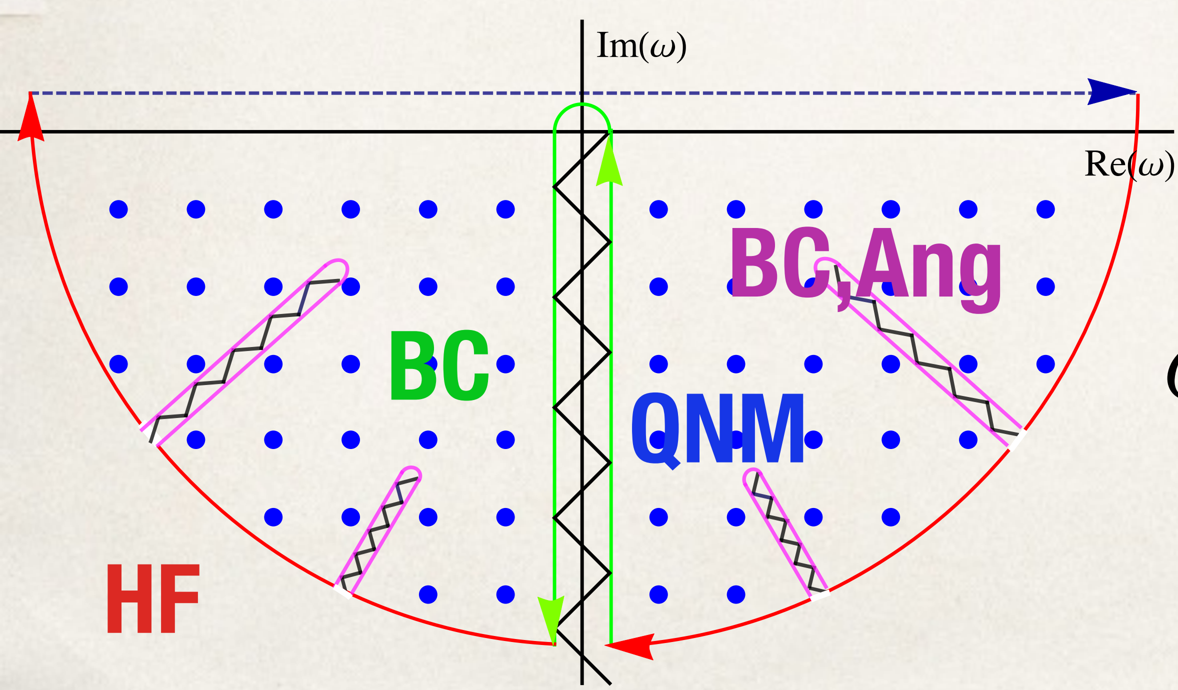
Nariai
(Casals,Dolan,Ottewill&Wardell'09)

$$G_m^{ret} = \cancel{G_m^{HF}} + G_m^{QNM}$$



Schwarzschild
(Casals,Dolan,Ottewill&Wardell'13)

$$G_m^{ret} = \cancel{G_m^{HF}} + G_m^{QNM} + G_m^{BC}$$



Kerr (In progress)

$$G_m^{ret} = \cancel{G_m^{HF}} + G_m^{QNM} + G_m^{BC} + G_m^{BC,Ang} \, ?$$

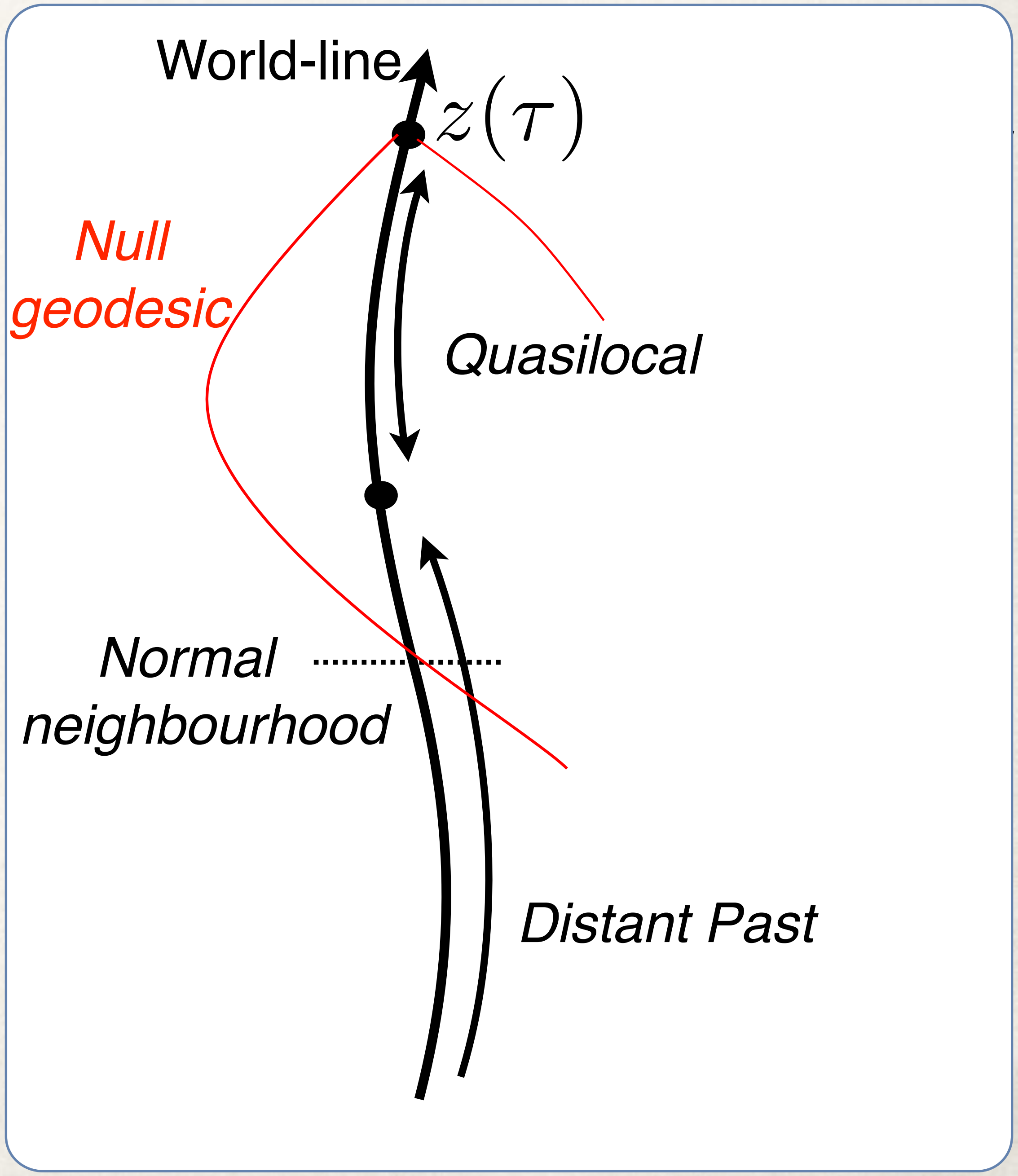
HF: expected to be zero in DP

DP - Matched Expansions - Kerr

- Matched expansions in Kerr:

$$\int_{-\infty}^{\tau^-} G_{ret}\ d\tau' =$$

$$\underbrace{\int_{\tau_m}^{\tau^-} V\ d\tau'}_{\text{QL}} + \underbrace{\int_{-\infty}^{\tau_m} (G_{QNM} + G_{BC})\ d\tau'}_{\text{DP}}$$



Radial solutions

- Use MST method (Sasaki&Tagoshi'03) for QNM radial coefficients: series of hypergeometric functions (typically, a couple hundred terms -> 'renormalized ang.mom. parameter' ν
- Use Jaffé series for QNM frequencies & radial functions: series about the horizon

BC in Kerr

- It may be seen from using a Leaver'86 series:

$$R^{up}_{\ell m} = \text{“}\sum \log(r\omega)\text{”}$$

- Small-frequency BC yields late-time GF:

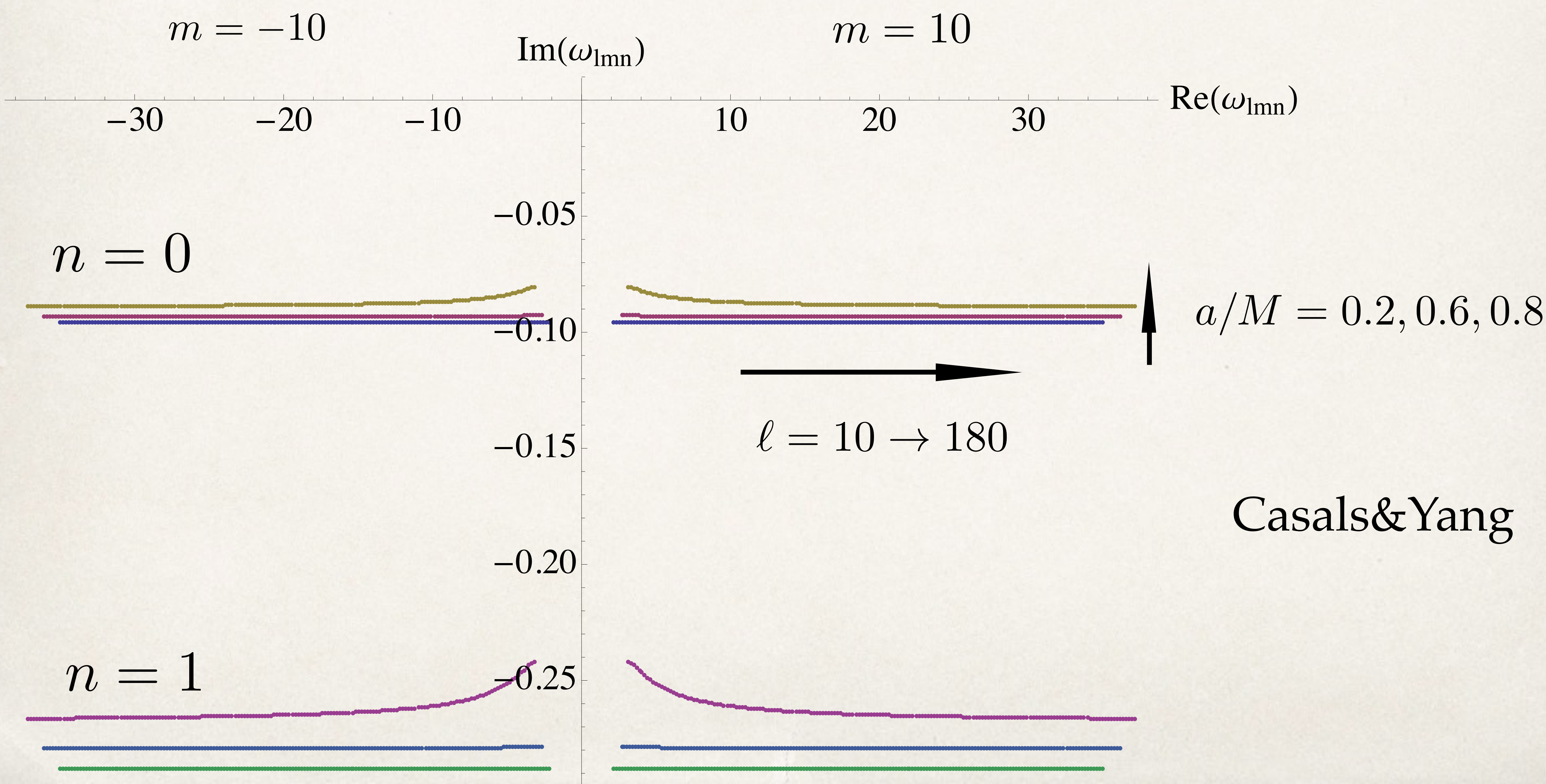
$$t^{-3},\ t^{-4},\ t^{-5}\log t, \ldots$$

- Not so small-frequency BC is important for SF accuracy
- Schwarzschild: Casals&Ottewill'12

  Kerr: see Chris Kavanagh's talk

QNMs in Kerr: Frequencies

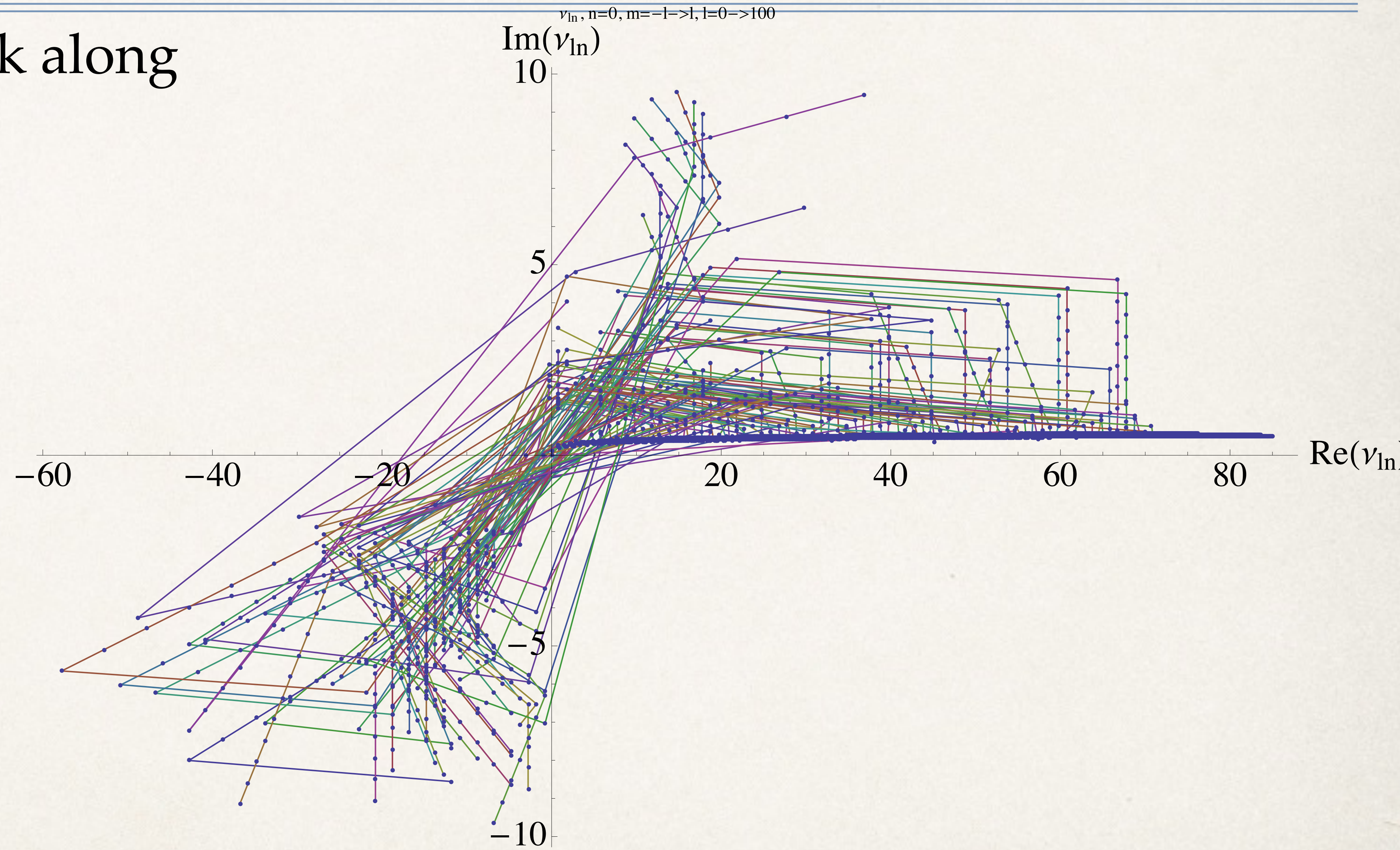
- QNMs: frequencies $\omega_{\ell m n}$ that are simple poles of the GF Fourier modes



QNMs in Kerr: Parameter ν

- Let ν run amok along

  $\nu + k, \forall k \in \mathbb{Z}$

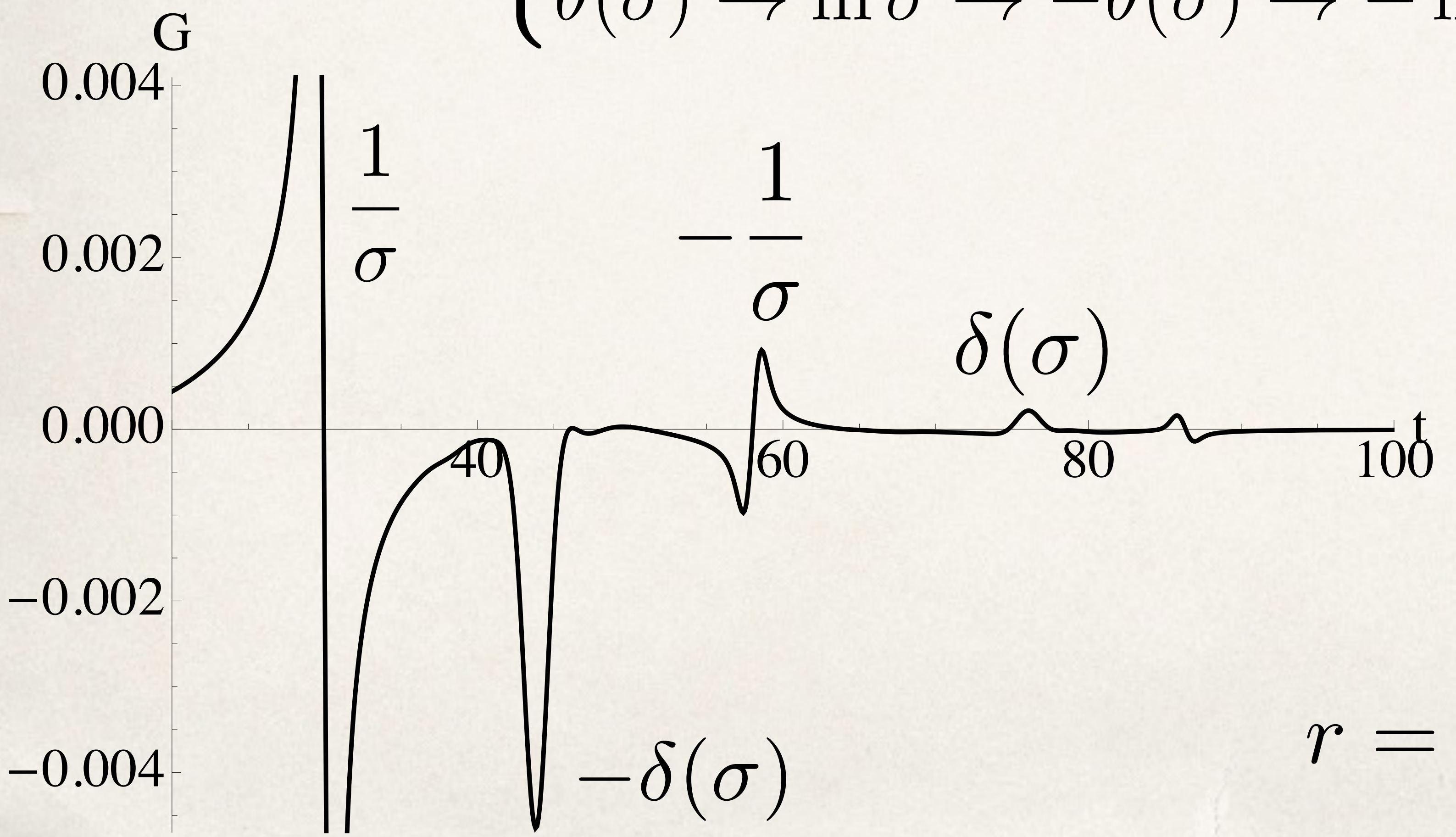


- Give correct value for radial coefficients but this means that MST series for $R^{in}_{\ell m}$ cannot be used -> use Jaffé series instead

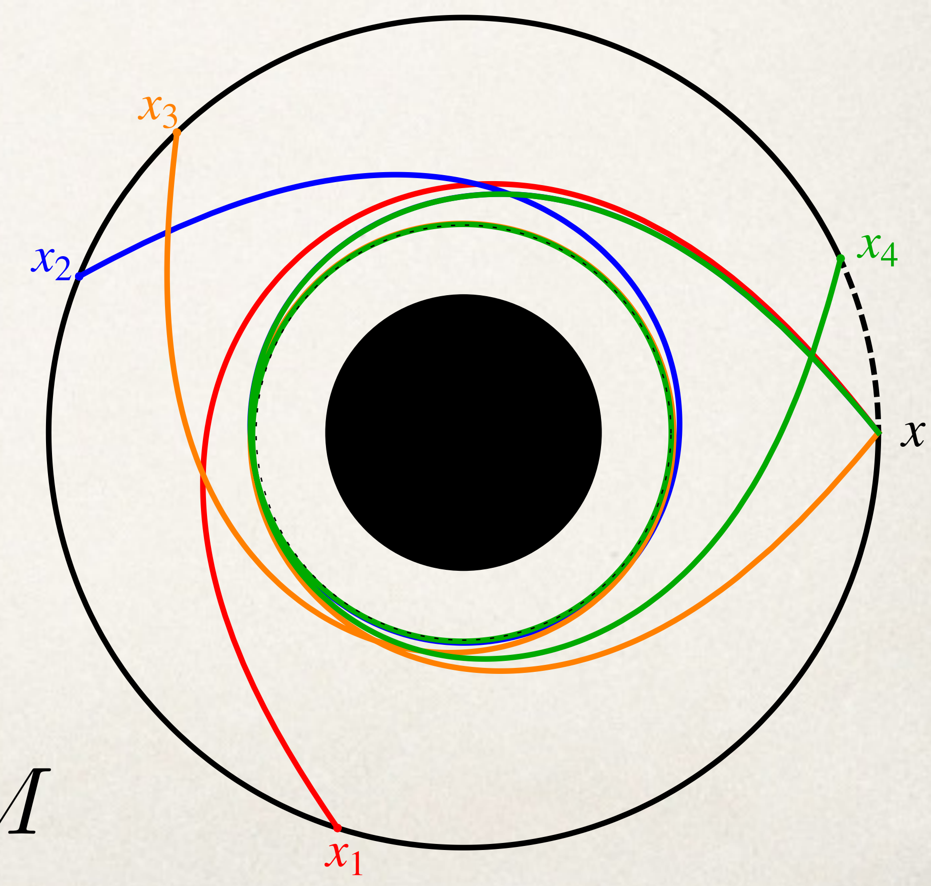
QNMs: Singularity structure of GF

- Divergence of GF has a 4-fold structure in Schwarzschild, Kerr & others:

$$G_{ret} \sim \begin{cases} \delta(\sigma) \to \dfrac{1}{\sigma} \to -\delta(\sigma) \to -\dfrac{1}{\sigma} \\ \\ \theta(\sigma) \to \ln\sigma \to -\theta(\sigma) \to -\ln\sigma \end{cases}$$



$r = 6M$



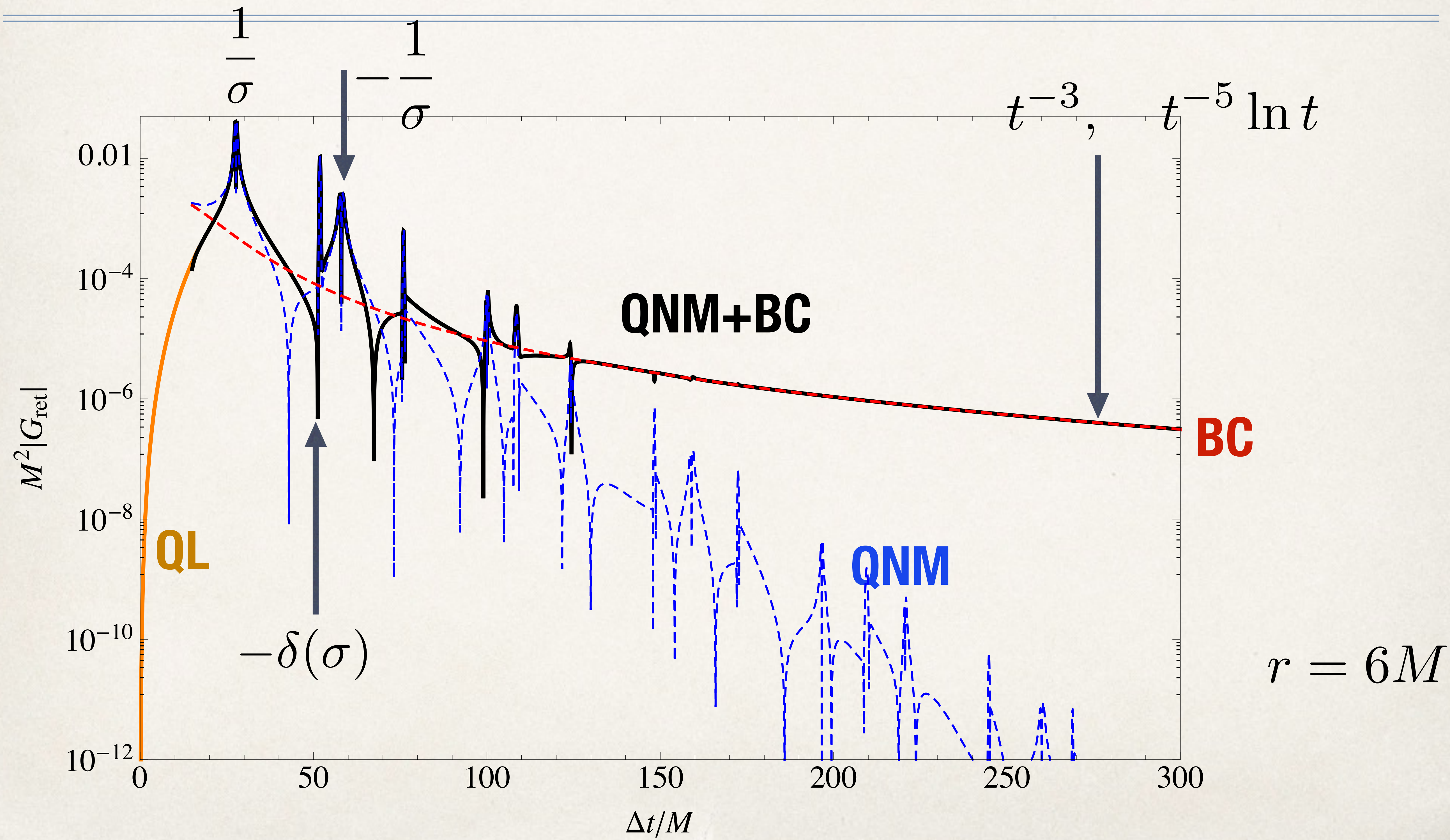
Singularity structure of GF

- Along $\Delta\varphi = 0, \pi$ in Schwarzschild it is 2-fold:

$$\pm\delta(\sigma),\ \Delta\varphi = \pi, \quad \text{or} \quad \pm 1/\sigma,\ \Delta\varphi = 0$$

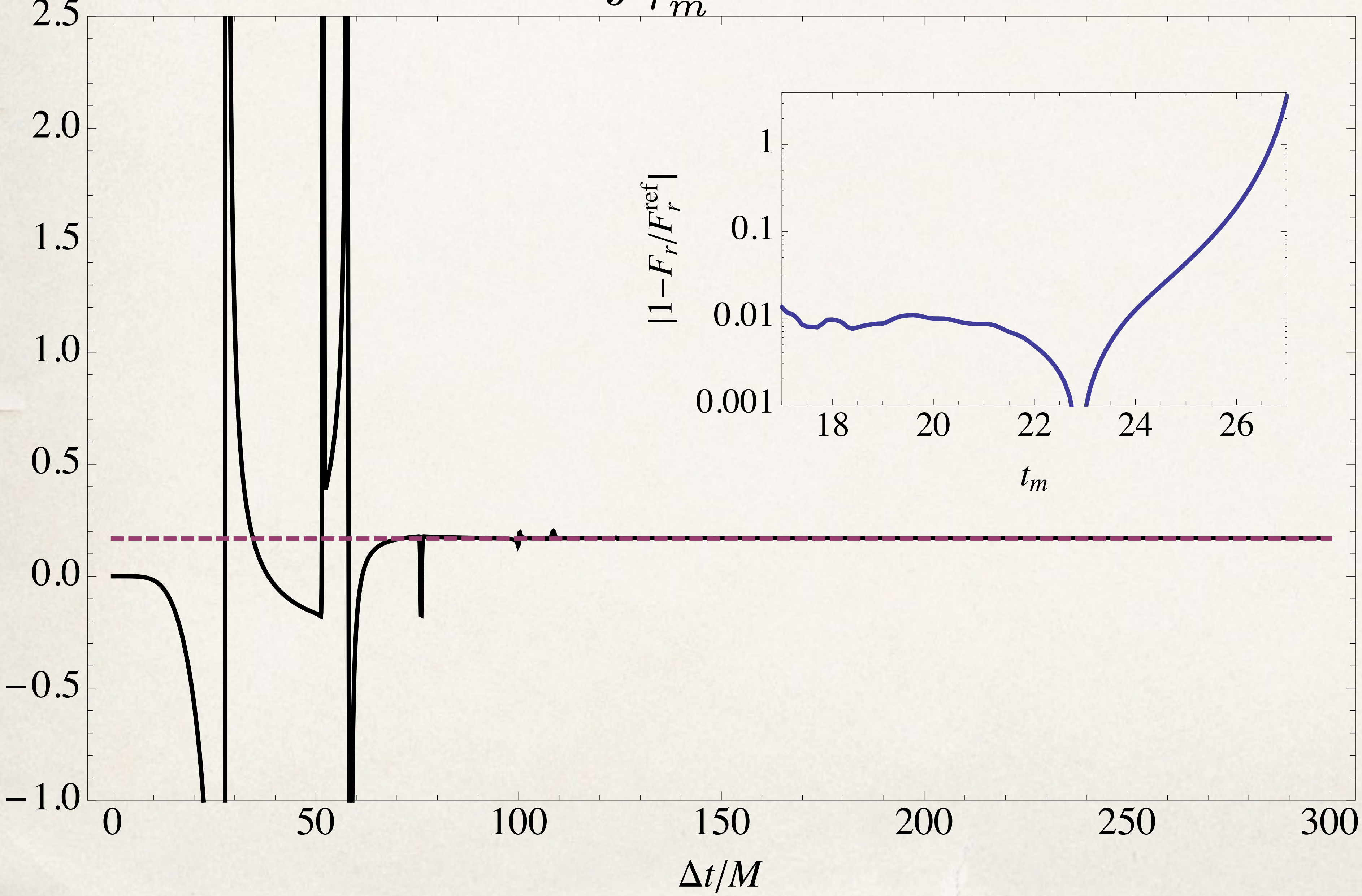
- Obtained via large-ℓ of QNMs, large-ℓ of sln. to (1+1)-PDE, geometrical optics, Penrose limit,… by Ori'09; Casals,Dolan,Ottewill&Wardell'09; Dolan&Ottewill'11; Harte&Drivas'12; Casals&Nolan'12; Zenginoglu&Galley'12; Yang, Zhang,Zimmerman&Chen'14

GF results in Schwarzschild



GF results in Schwarzschild

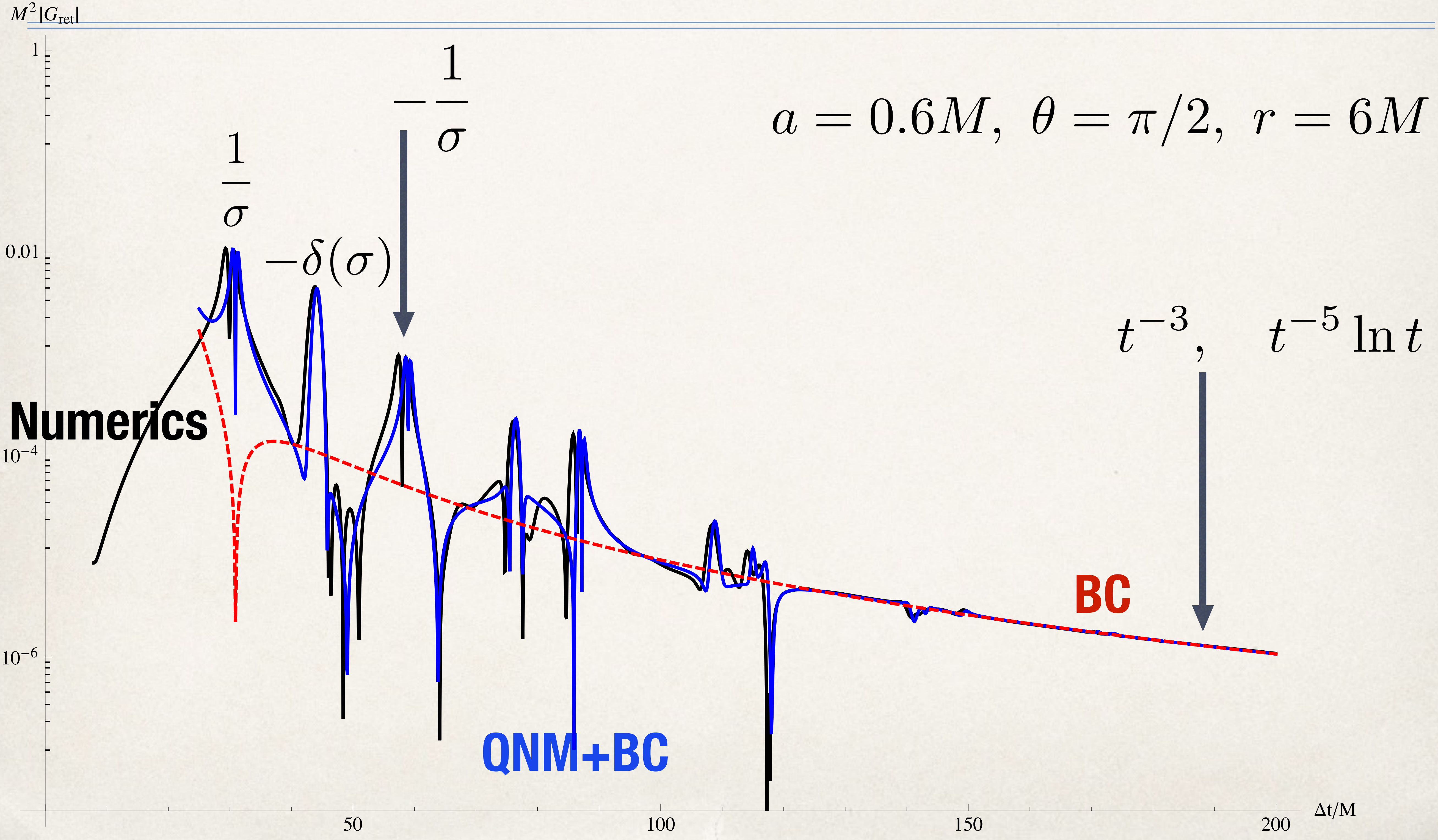
- Partial SF $q^2\nabla_\mu \int_{\tau_m}^{\tau^-} d\tau' G_{ret}$



- Value settles after 3rd light-crossing
- SF rel.err. $\approx 1\%$

$r = 6M$

GF results in Kerr-in Progress



Some features of Matched Exp.

- Matched expansions:

  -Regularization is trivial

  - It gives physical insight (wave propagation, how much ‘memory’ does the SF has, may explain sign, Thornburg’s SF oscillations,…)

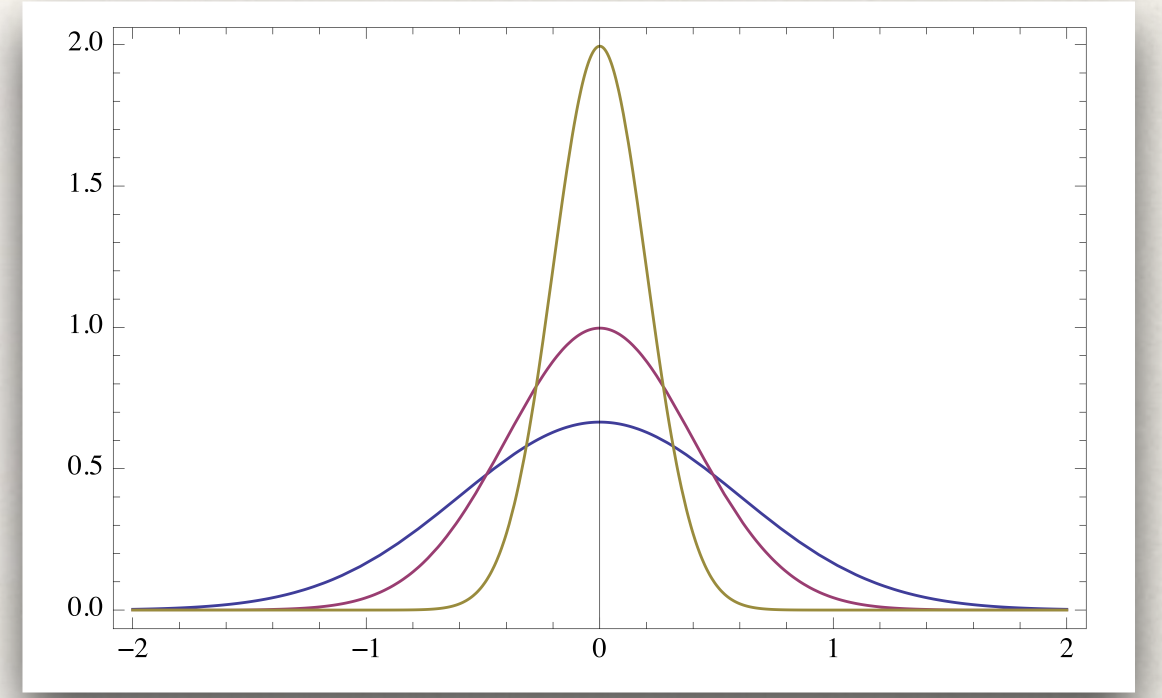
  - Once GF is known for all pairs, the SF can be easily calculated for any orbit (geodesic, accelerated, highly eccentric,…)

- DP via QNM+BC: only requires solving ODEs; one or two QNM overtones might suffice

Numerical Calculation of the Green Function

Mollified Green function

- Don't need exact Green function to compute the self-force accurately using worldline convolutions.
- It is sufficient to have a smeared, or *mollified* Green function.
- One way to do so is using a finite, smoothed sum over QNMs along with a branch cut integral.



- Alternative, analogous approach: (almost) fully numerical calculation using smeared Gaussians in place of δ-functions.

$$\delta_4^{\varepsilon}(x - x') = \frac{1}{(2\pi\varepsilon^2)^2} \exp\left[-\sum_{\alpha=0}^{3} \frac{(x^\alpha - x'^\alpha)^2}{2\varepsilon^2} \right]$$

Numerical time-domain evolution

- Two closely related numerical schemes for computing a mollified Green function using Gaussian approximations to δ-functions.
- One options is to solve the sourced wave equation for the retarded Green function (Zenginoğlu & Galley, 2012):

$$\Box_x G_{\text{ret}}(x, x') = -4\pi \delta_4(x, x')$$

$$\Downarrow$$

$$\Box_x G^{\varepsilon}_{\text{ret}}(x, x') = -4\pi \frac{1}{(2\pi\varepsilon^2)^2} \exp\left[-\sum_{\alpha=0}^{3} \frac{(x^\alpha - x'^\alpha)^2}{2\varepsilon^2}\right]$$

Numerical time-domain evolution

- Alternatively, reformulate as an initial value problem (Wardell, Galley, Zenginoğlu, Casals, Dolan & Ottewill 2013).
- Given initial data on a spatial hyper-surface Σ and the full Green function, one can determine the solution at an arbitrary point x' in the future of Σ (Kirchhoff theorem)

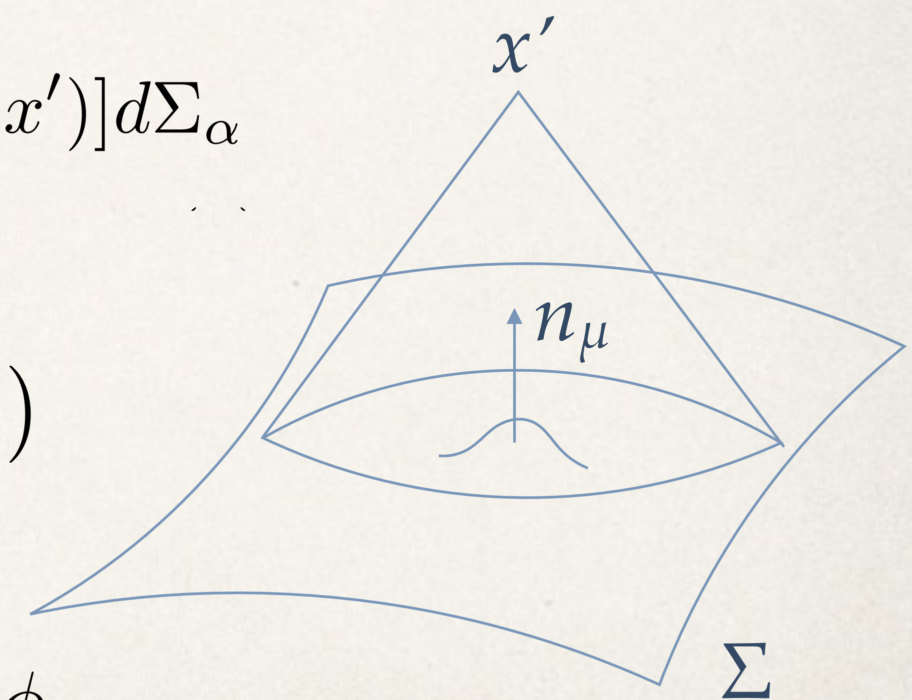
$$\Phi(x') = -\frac{1}{4\pi}\int_\Sigma [G(x,x')\nabla^\alpha\Phi(x) - \Phi(x)\nabla^\alpha G(x,x')]d\Sigma_\alpha$$

- Basic idea: choose as initial data

$$\Phi(x)|_\Sigma = 0 \quad n_\mu\nabla^\mu\Phi(x)|_\Sigma = -4\pi\delta_3^\epsilon(\mathbf{x},\mathbf{x}_0)$$

then in the limit $\varepsilon \to 0$

$$\Phi(x') = \int_\Sigma G(x,x')\delta_3(x-x_0)r^2\sin\theta\, dr\, d\theta\, d\phi$$
$$= G(x_0,x').$$



Numerical time-domain evolution

- So, we evolve the homogeneous wave equation with initial data

$$\Phi(x)|_\Sigma = 0 \quad n_\mu \nabla^\mu \Phi(x)|_\Sigma = -4\pi\delta_3^\epsilon(\mathbf{x}, \mathbf{x}_0)$$

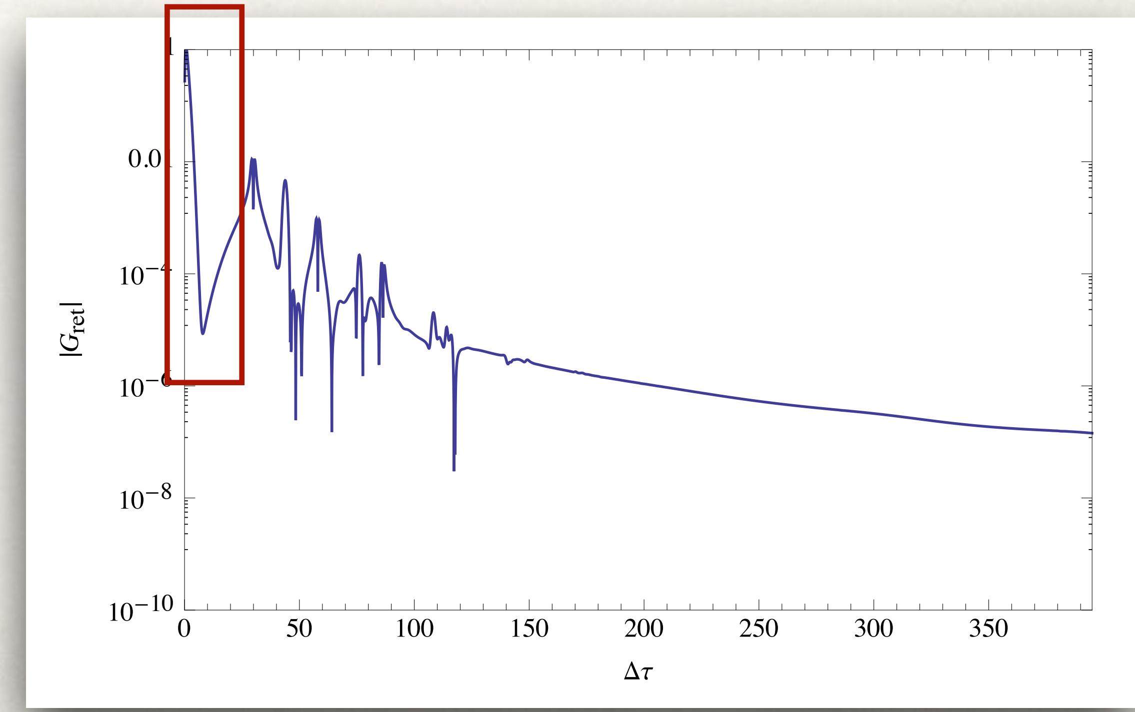
  for a sequence of values of ε, then extrapolate to $\varepsilon \to 0$ to get the Green function.
- Similarly for derivatives of the Green function (i.e. self-force).
- Somewhat surprisingly, this works very well for computing the self-force, even for quite large ε/M ~ 0.1 - 1.0.
- Narrower Gaussian improves resolution of small-scale features at null-geodesic crossings. Between crossings, even a large ε is sufficient.

Practical issues: regularization

- MiSaTaQuWa equation only includes *tail* part of the integral of the retarded Green function over the past world-line, excluding coincidence-limit ($t = 0$) δ-function part

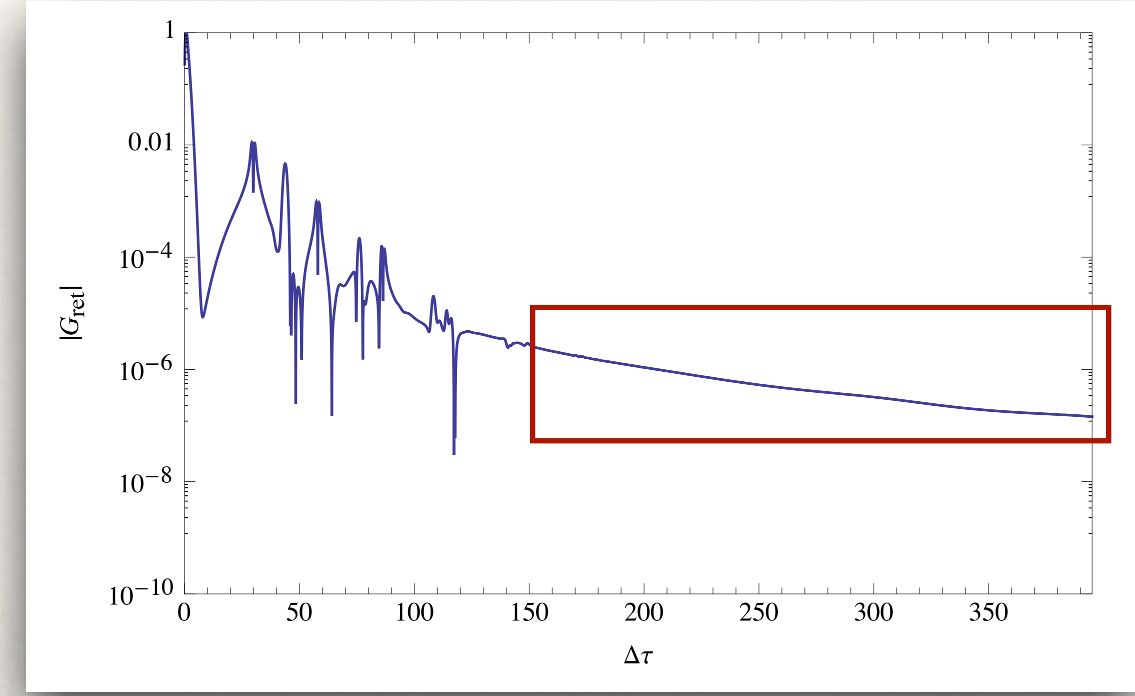
$$f^a = (\text{local terms}) + \lim_{\epsilon \to 0} q^2 \int_{-\infty}^{\tau - \epsilon} \nabla^a G_{\text{ret}}(x, x') d\tau'$$

- But the numerical solution has smeared out this δ-function to have Gaussian support up to $t \approx \varepsilon$.
- Have to supplement numerical solution with approximation at early times → quasilocal series.



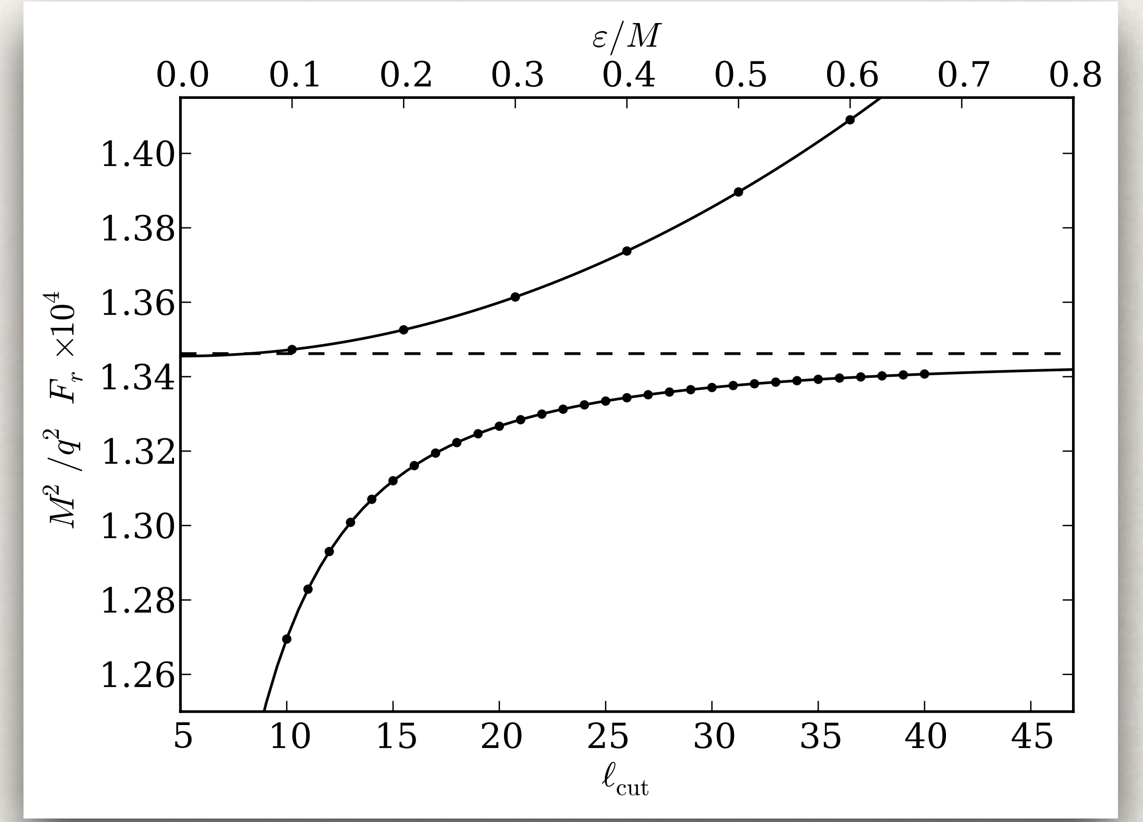
Practical issues: late times

- Numerical integration can only be done up to some finite time $t_{\max}$.
- At late times the solution is well approximated by branch cut.
- Once the the solution has settled down to this regime, switch over to analytical branch-cut expression.
- This can be very significant for computing the regularized self-field, less so for the self-force.
- See talk by Chris Kavanagh later today.



Practical issues: Gaussian width

- Dominant source of error comes from Gaussian smearing.
- Fortunately, this is still a relatively small error, and converges quite rapidly as ε is decreased.
- Can achieve relative errors of ~10^{-4} with a Gaussian of width $\varepsilon = 0.1M$.



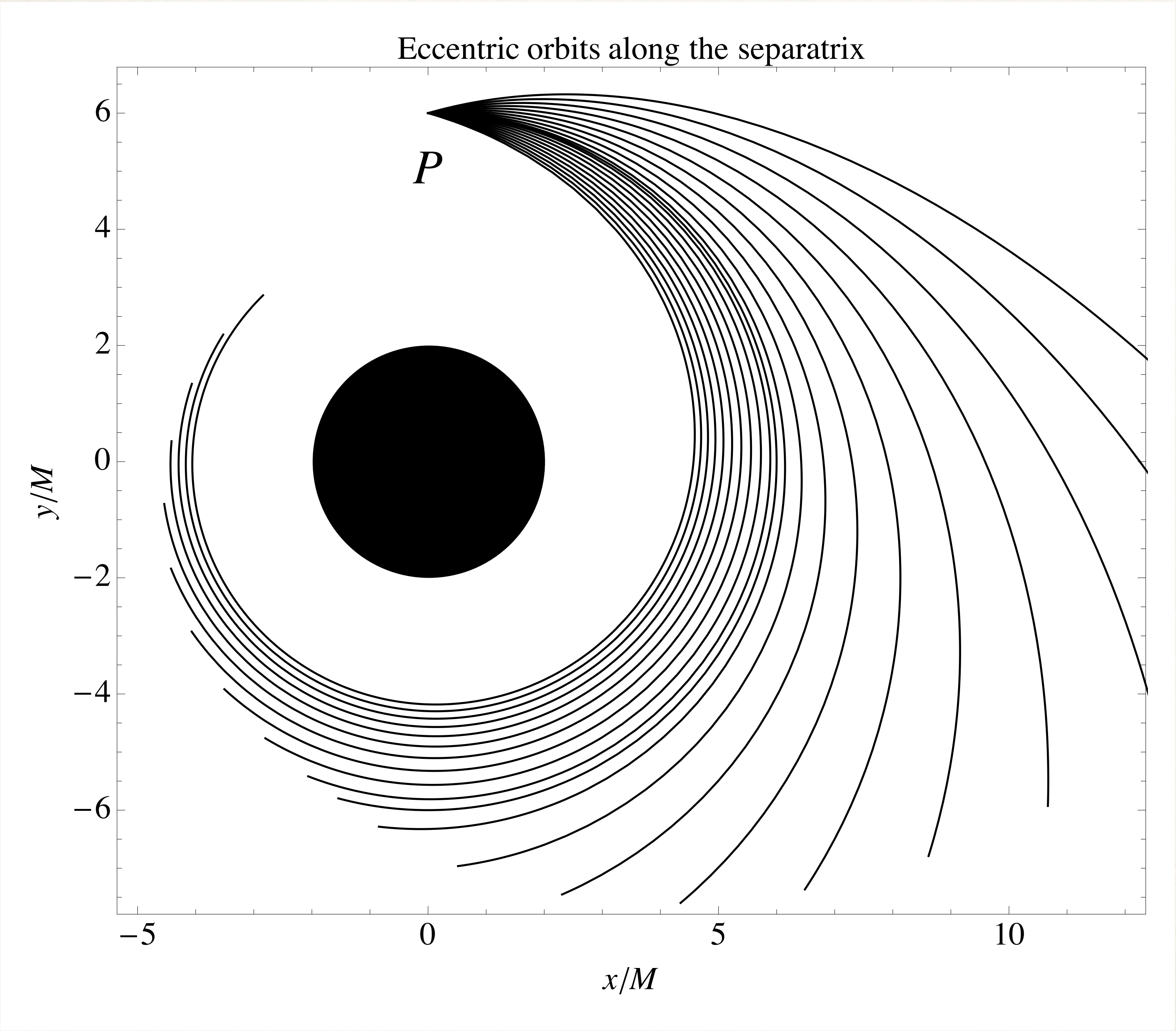
Practical issues: accuracy

- Can achieve relative errors of ~10^{-4} with a Gaussian of width $\varepsilon = 0.1M$.
- Run time on the order of 1 hour on 3 compute nodes. Could easily be optimized significantly.

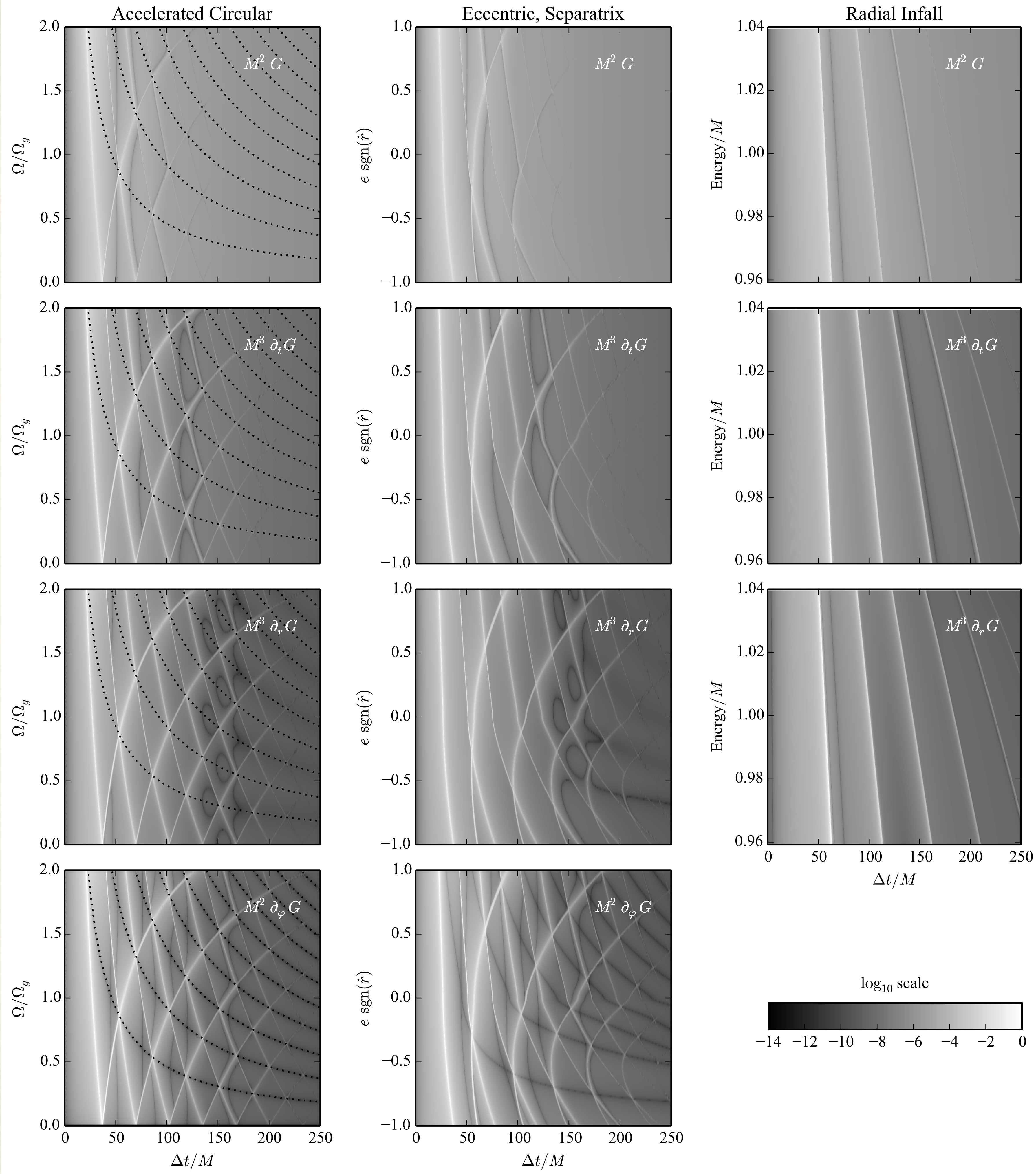
		Computed value	Rel. Err.	Est. Err.
Circular	$M/q\,\Phi$	-5.45517×10^{-3}	6×10^{-5}	3×10^{-3}
	$M^2/q^2\,F_t$	3.60779×10^{-4}	4×10^{-4}	2×10^{-3}
	$M^2/q^2\,F_r$	1.67861×10^{-4}	8×10^{-4}	2×10^{-3}
	$M^2/q^2\,F_\varphi$	-5.30452×10^{-3}	5×10^{-5}	5×10^{-4}
Eccentric	$M/q\,\Phi$	-7.70939×10^{-3}	1×10^{-3}	1×10^{-3}
	$M^2/q^2\,F_t$	6.65241×10^{-4}	2×10^{-4}	1×10^{-3}
	$M^2/q^2\,F_r$	1.3473×10^{-4}	8×10^{-4}	4×10^{-3}
	$M^2/q^2\,F_\varphi$	-7.28088×10^{-3}	4×10^{-5}	5×10^{-4}

Physical applications & results

Computing many orbits at once



- Using a single Green function we can quickly compute the self-force for many orbits.
- (But need a separate Green function for each point on the orbit)

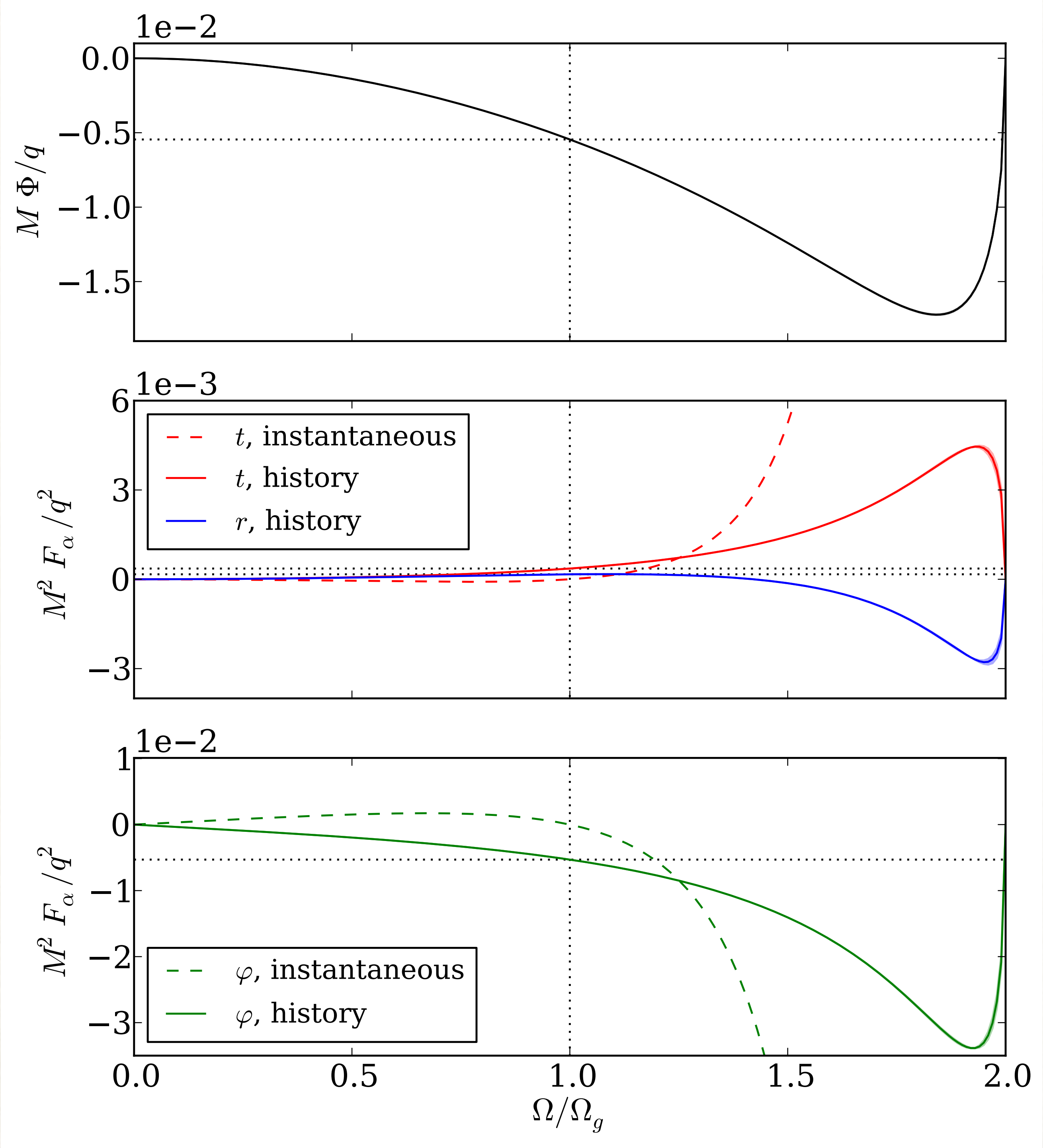

Accelerated Circular
Eccentric, Separatrix
Radial Infall
$M^2\,G$
$M^3\,\partial_t G$
$M^3\,\partial_r G$
$M^2\,\partial_\varphi G$
Ω/Ω_g
$e\ \mathrm{sgn}(\dot r)$
Energy/M
$\Delta t/M$
$\log_{10}$ scale

Interesting physical cases

- World line integration method applies equally to any world line.
- No extra difficulties in dealing with orbits which cause difficulties for other methods.
- No problems with “junk” initial data. No issues with non-discrete frequency spectrum. No problems with accelerated worldlines.
- Easily handle aperiodic (or nearly aperiodic) trajectories, such as unbound orbits, highly-eccentric or zoom-whirl orbits, and ultra-relativistic trajectories.
- Three examples: Accelerated orbits, high eccentricities, unbound.

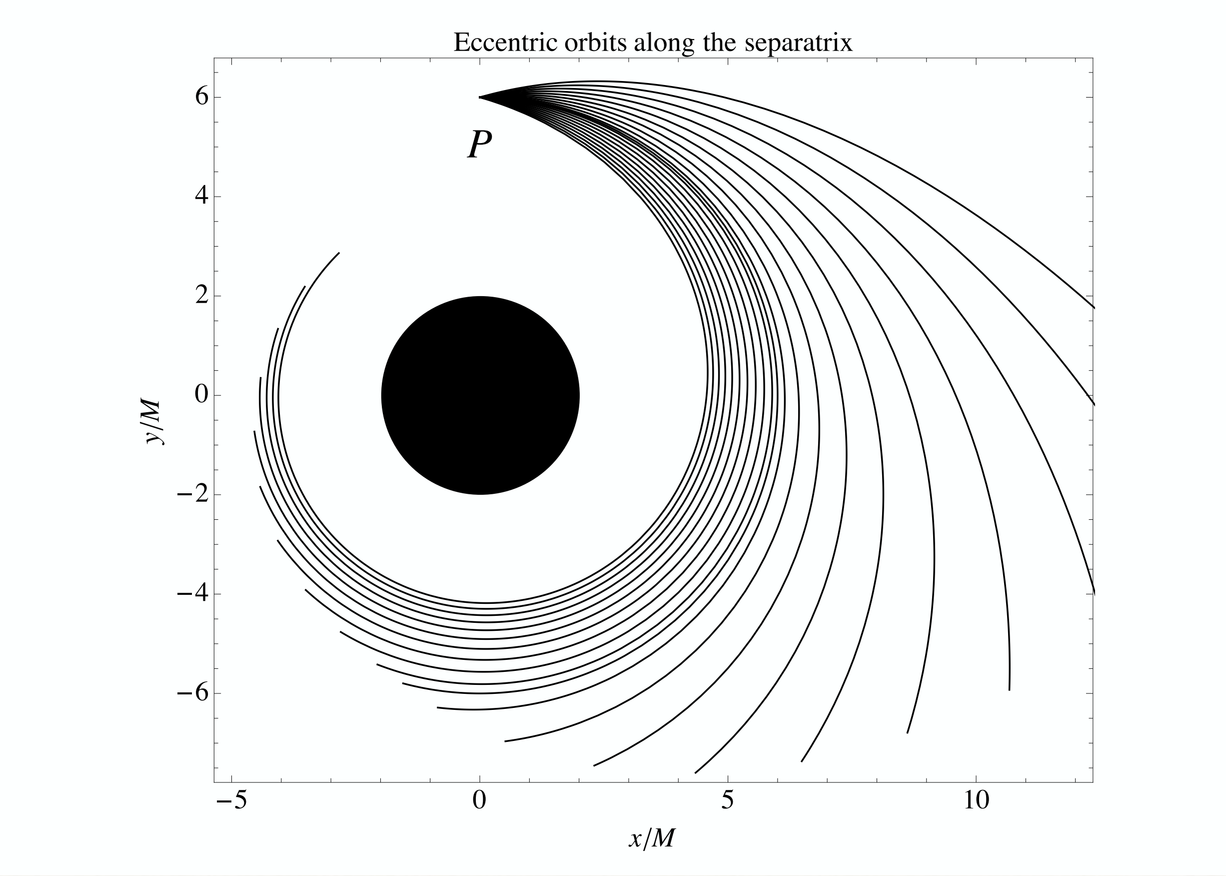
Accelerated Circular Orbits (including ultra-relativistic)

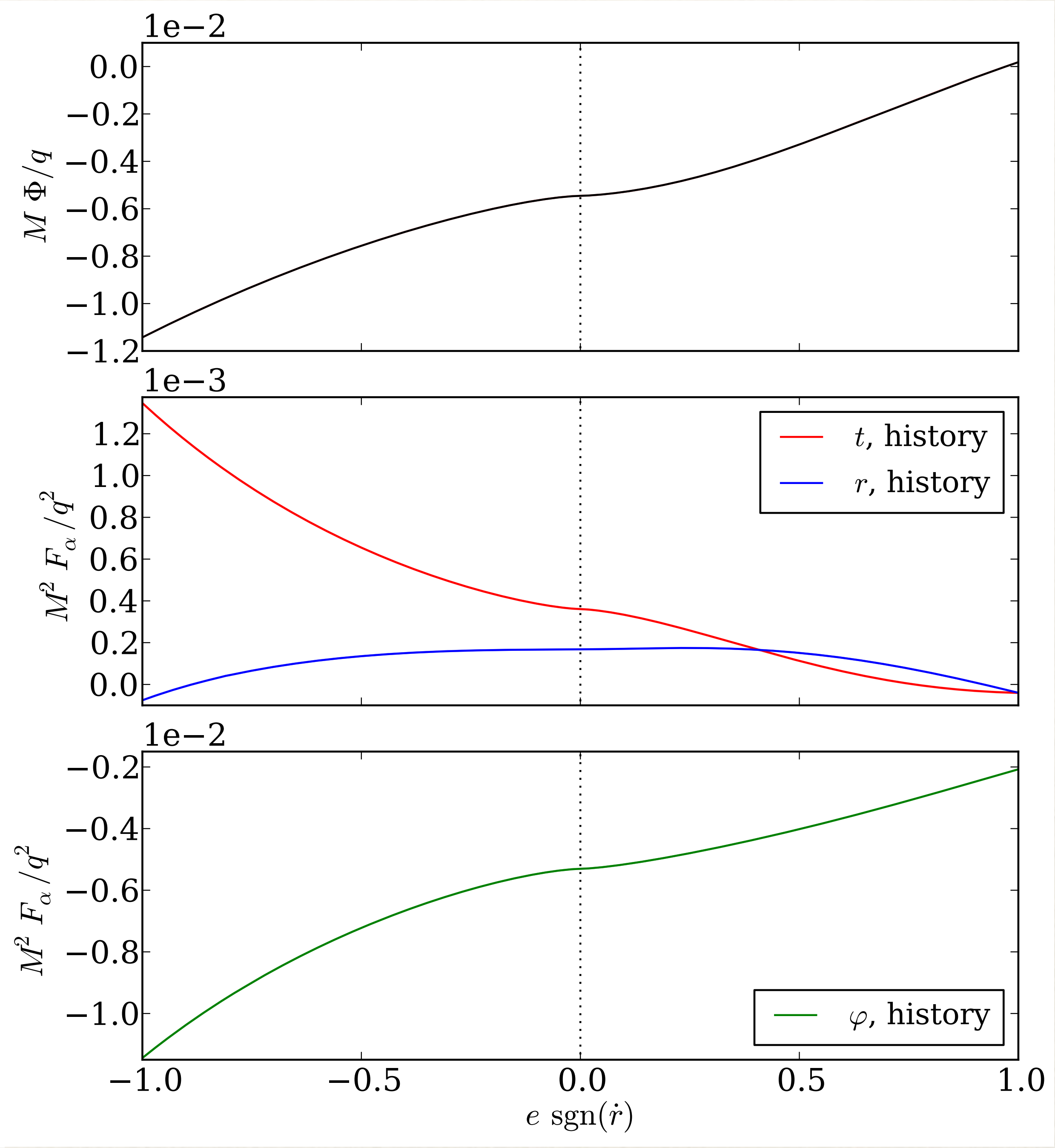
- Circular orbits of radius r_0=6M.
- Orbital frequency ranging from a static particle, $\Omega = 0$, to ultra-relativistic $\Omega = 2\Omega_g$.
- Self-force diverges and becomes purely local in the ultra-relativistic limit.
- Tail contribution vanishes in both static and ultra-relativistic limits.



Highly eccentric orbits

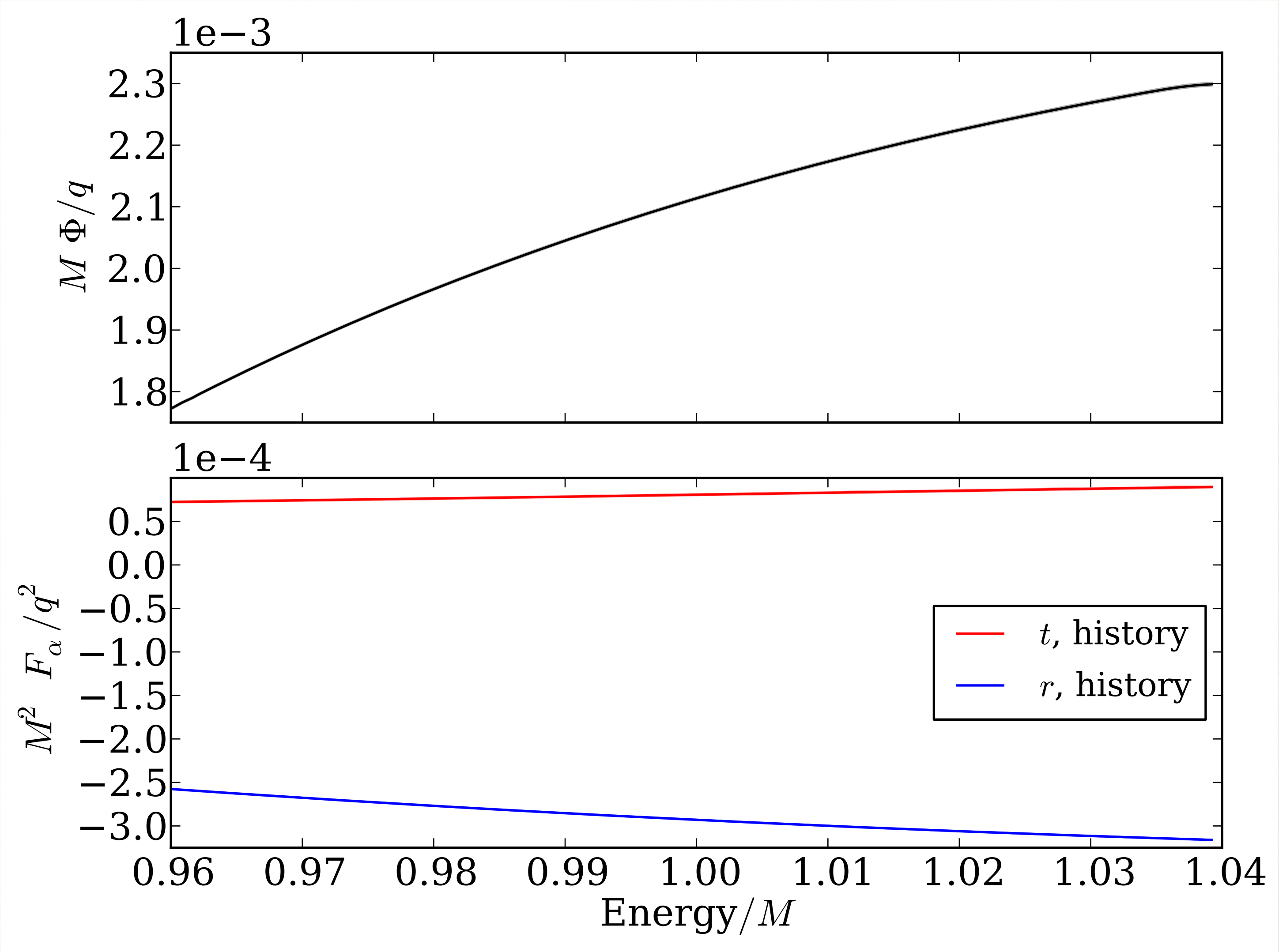
- Highly eccentric orbits near the separatrix between unstable and stable bound orbits.
- Possibly relevant to EOB?





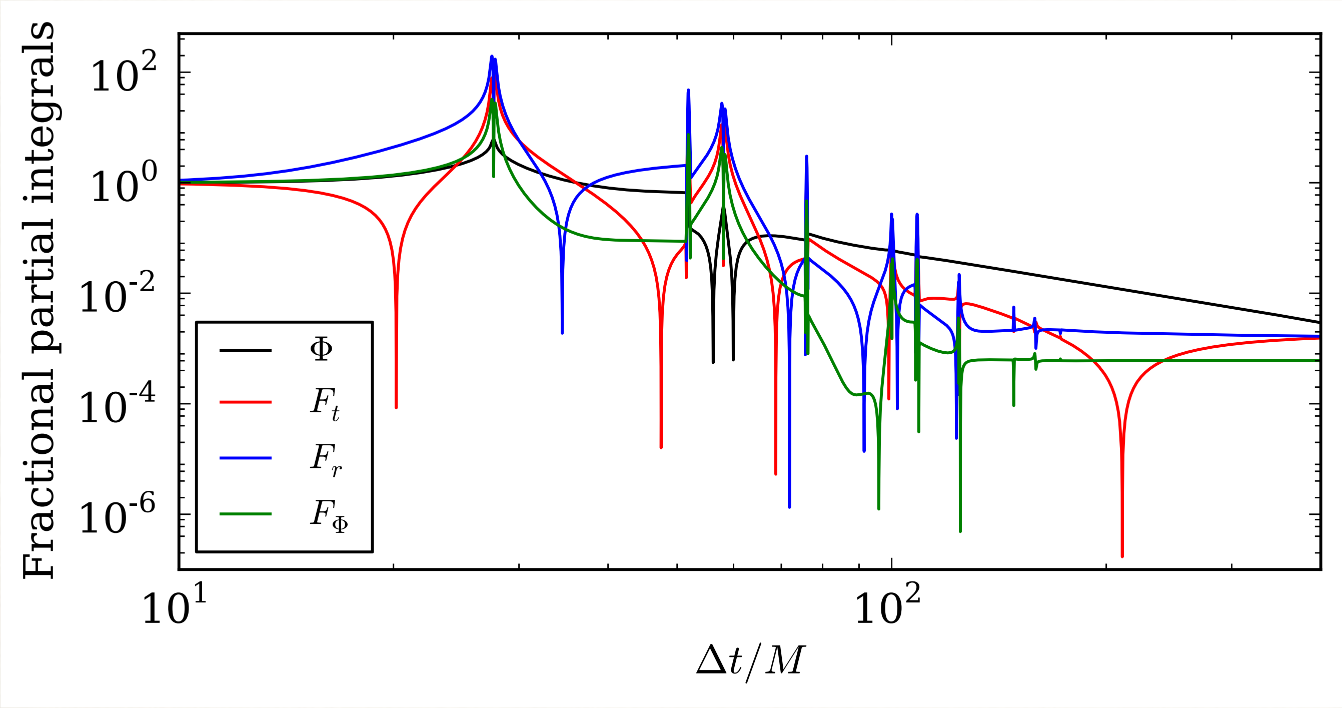
Unbound motion

- Radial plunge orbits
- Potentially interesting for cosmic censorship scenarios?



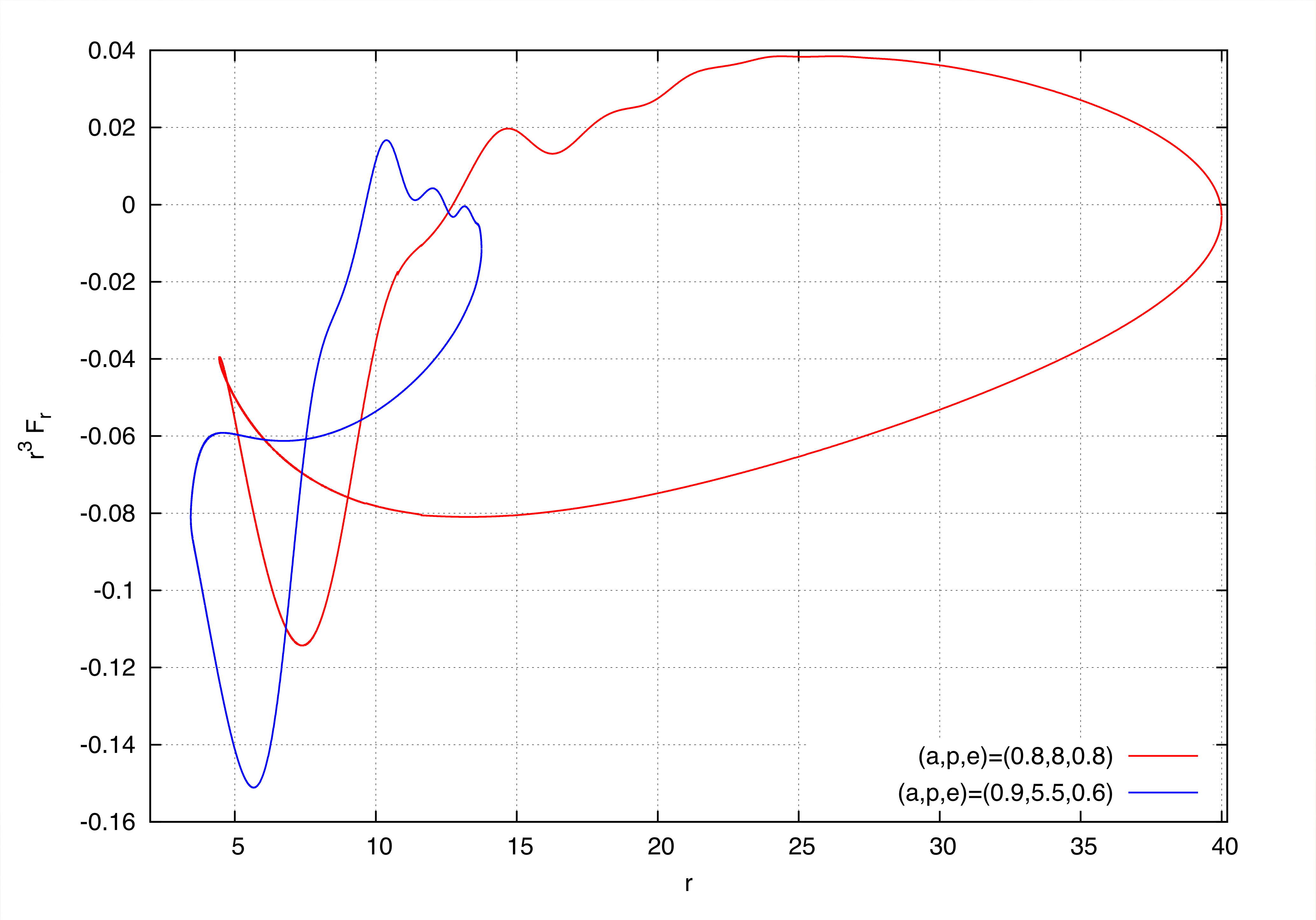
Physical insight

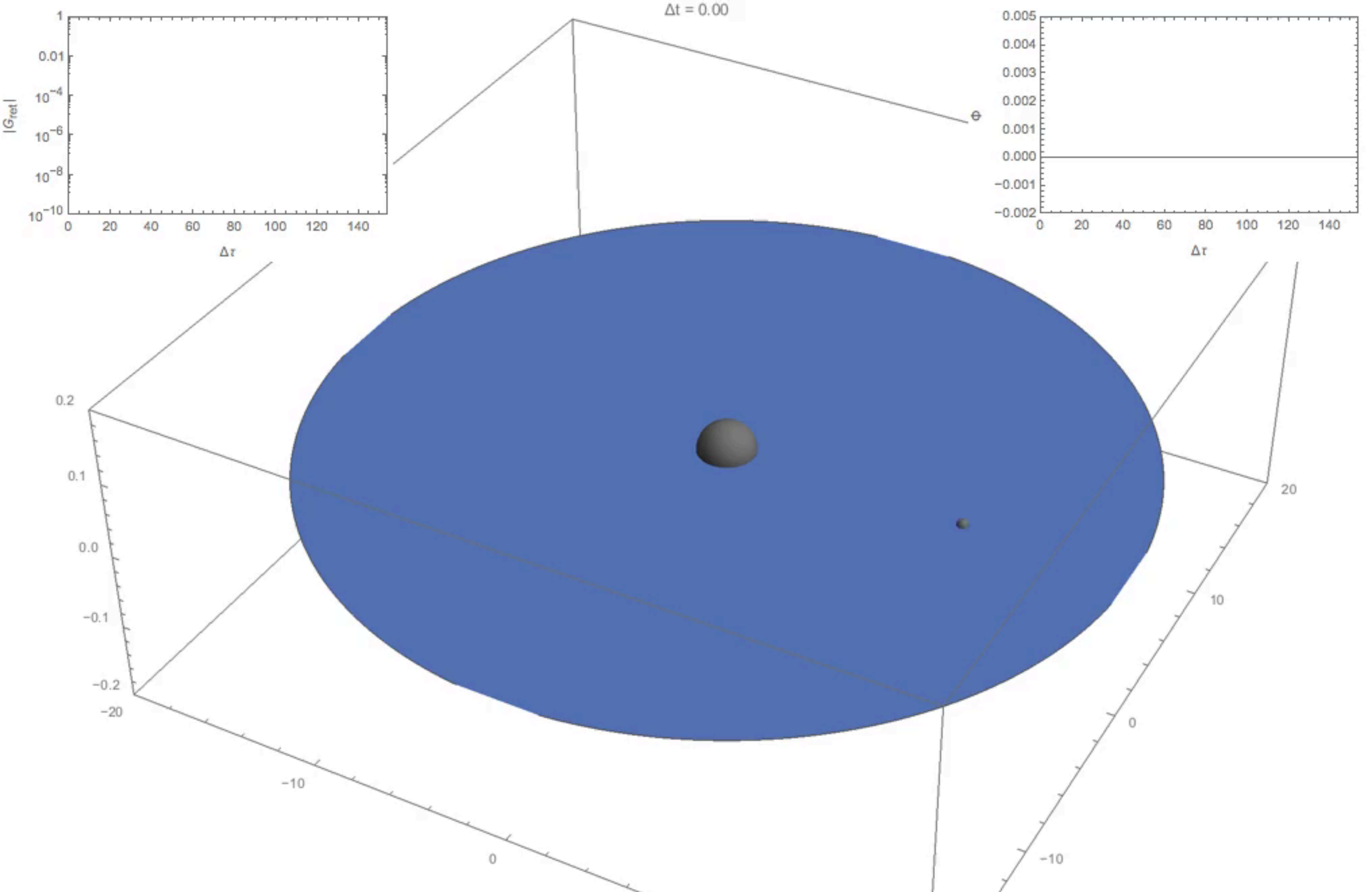
History dependence



- Circular orbit, $r_0 = 6M$, orbital period $T \approx 100M$.
- Self-force "remembers" ~1-2 orbits.
- Field has a longer memory ~10s of orbits.
- Self-force gets a kick near each null-geodesic intersection.

History dependence

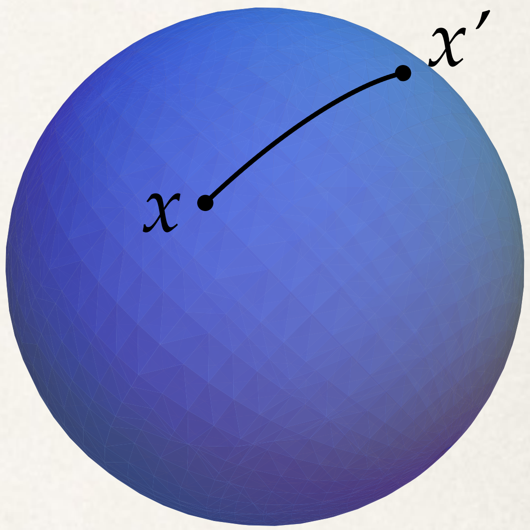




$G(x_0, x')$ for $r_0 = 12M$, $\vartheta = \pi/2$ in Kerr spacetime.
Geodesic with for $(a,p,e)=(0.9,5.5,0.6)$.

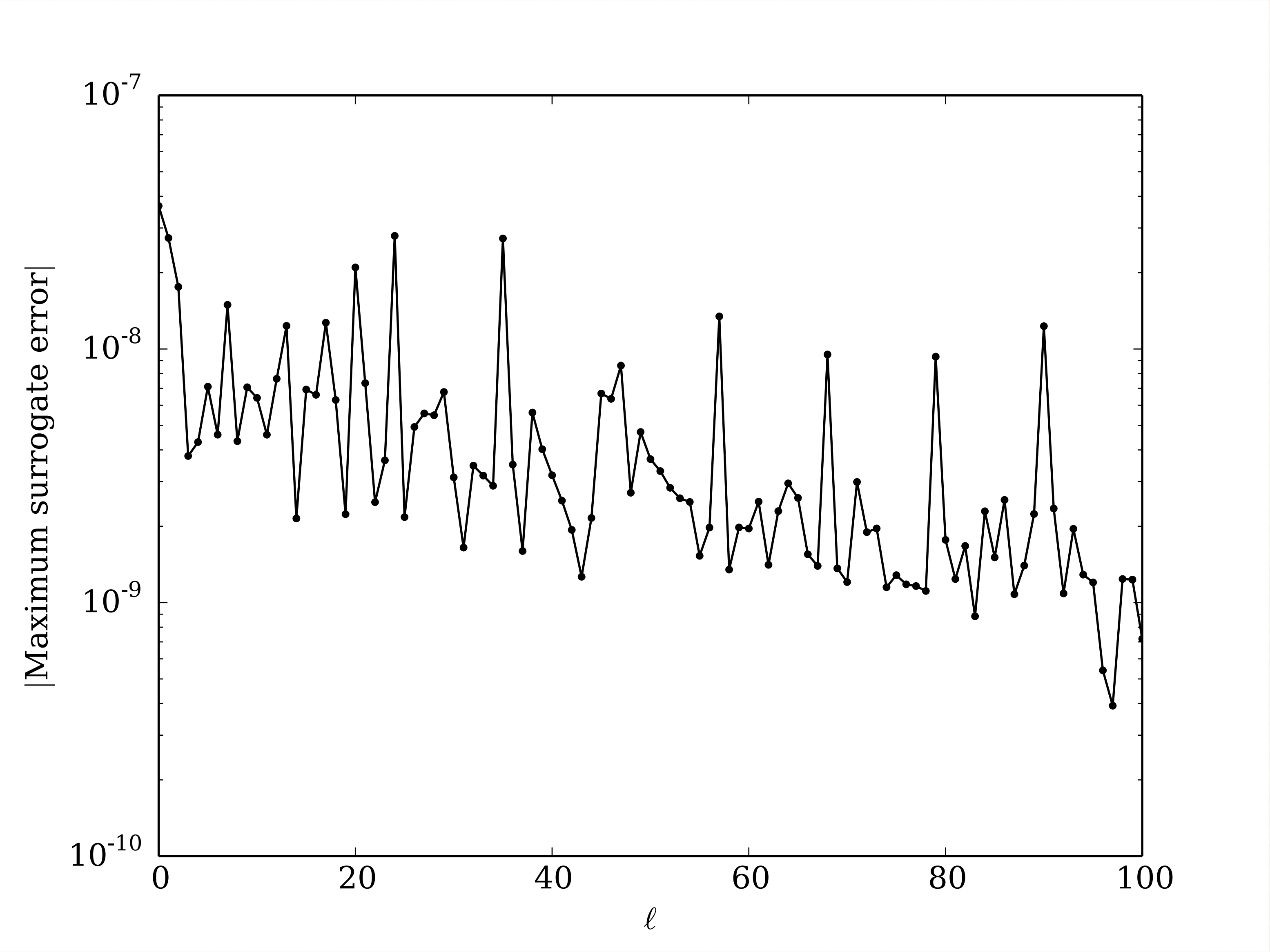
Other applications

Other interesting applications: Surrogate models



- Need the Green function for all pairs of points x and x'. In Schwarzschild, this is a four-dimensional parameter space. In Kerr, six-dimensional.
- Reduced order model methods have been shown to work very well for gravitational wave templates from binary black hole systems.
- Construct a surrogate model using reduced order methods which have been very successful with waveform templates.
- Proof of principal done for Green function, works very well. Generating data for the model took ~1 day running on a few nodes of a cluster.

Other interesting applications: Surrogate models



Once the surrogate is constructed, each evaluation of the Green function for the pair of points x and x' takes ~0.06s on a laptop and is very accurate.

Other interesting applications: Second order

- Second order scalar self-force (Galley 2012) can be written in terms of convolutions of the retarded Green function.

$$F^\mu(\tau) = (a^\mu + P^{\mu\nu}\nabla_\nu)\left\{\frac{m^2c_1^2}{m_{pl}^2}I_R(z^\mu) - \frac{m^3c_1^2c_2}{m_{pl}^4}\left(\frac{1}{2}I_R^2(z^\mu) + \int d\tau'\, D_R(z^\mu, z^{\mu'})I_R(z^{\mu'})\right)\right.$$
$$+\frac{m^4c_1^2c_2^2}{m_{pl}^6}\left(I_R(z^\mu)\int d\tau'\, D_R(z^\mu, z^{\mu'})I_R(z^{\mu'}) + \int d\tau' d\tau''\, D_R(z^\mu, z^{\mu'})D_R(z^{\mu'}, z^{\mu''})I_R(z^{\mu''})\right)$$
$$\left.+\frac{m^4c_1^3c_3}{2m_{pl}^6}\left(\frac{1}{3}I_R^3(z^\mu) + \int d\tau'\, D_R(z^\mu, z^{\mu'})I_R^2(z^{\mu'})\right) + O(\varepsilon^4)\right\},$$

- Here, D_R is just the retarded Green function which has already been computed (and I_R is constructed from D_R).

Other interesting applications: Self-consistent evolution

- Green function can be used to self-consistently solve the coupled delay-differential equation for the field and the self-forced worldline.

$$\Box\Phi = \rho \qquad f_a = \nabla_a \Phi^{\mathrm{R}} \qquad a^\alpha = (g^{\alpha\beta} + u^\alpha u^\beta) f_\beta$$

- Original (Quinn) equation of motion including fully self-consistent evolved orbit.
- Analytic version possible for plane-wave spacetimes (Harte). Numerical version possible using surrogate model for Schwarzschild/Kerr.
- Can also use the Green function to assess difference between osculating geodesic and self-consistent orbits. So far find that they agree well, to within error bars.

Conclusions and prospects

- Schwarzschild case now complete (Phys. Rev. D 89, 084021), Kerr in advanced stages.
- Green functions are a flexible approach to self-force calculations.
- Gives insight into history dependence of the self-force.
- Compute Green function once, get all orbits through that base point.
- Need a separate calculation for each point on the orbit - Reduced Order Models useful.
- Interesting orbits not accessible by other means
- Second and higher order
- Extension to gravitational case.
- Self-force as a test of alternative theories of gravity
- Other applications beyond self-force.