A discontinuous method for time-domain gravitational self-force computation

Charalampos Markakis[1,2]
Leor Barack[1]

[1]Mathematical Sciences, University of Southampton
[2]Theoretical Physics Institute, University of Jena

Contents

Teukolsky equation

$$(\nabla^\mu \nabla_\mu + A^\mu \nabla_\mu + V)\psi_0 = 4\pi T$$

- Strongly hyperbolic PDE describing linear perturbations in Kerr
- Usually solved as ODE in frequency domain
- Used to compute gravitational self-force in a radiation gauge (Shah, Friedman, Whiting; Pound, Merlin, Barack; et al.)
- For highly eccentric or hyperbolic orbits, *time-domain self-consistent evolution* is ideal
- Time-domain Teukolsky solvers have been increasingly used (Burko, Khanna, Pullin, Hughes, Poisson, Lousto, Zenginoğlu, Harms, Bernuzzi, Brügmann et al.)

Method of lines

- First order reduction

$$\left.\begin{aligned} \nabla^{\mu}\pi_{\mu} + A^{\mu}\pi_{\mu} + V\psi_{0} &= 4\pi T \\ \nabla_{\mu}\pi_{\nu} - \nabla_{\nu}\pi_{\mu} &= 0 \\ \nabla_{\mu}\psi_{0} &= \pi_{\mu} \end{aligned}\right\} \quad \Rightarrow \quad \partial_{t}U_{A} = L^{AB}U_{B} + S_{A}, \;\; U_{A} = \begin{pmatrix} \pi_{t} \\ \pi_{i} \\ \psi_{0} \end{pmatrix}$$

- Discretize *U*, *L* in space, evolve as coupled ODEs in time

$$\frac{d\mathbf{U}}{dt} = \mathbf{L}\,\mathbf{U} + \mathbf{S}$$

- *L* contains spatial derivative matrices, which are typically valid only for smooth methods.

Teukolsky equation with particle source

$$(\nabla^\mu \nabla_\mu + A^\mu_{lm} \nabla_\mu + V_{lm})\psi_0^{lm}(r) = G_{lm}\delta(r-\xi) + F_{lm}\delta'(r-\xi)$$

Some methods of incorporating a point particle source

1. Approximate δ as Gaussian pulse
2. Construct finite-difference representation of δ
3. ‘Particle without particle’ (domain decomposition + coordinate mapping + jump conditions across particle)

$$J_0 = \psi(\xi^+) - \psi(\xi^-) = F(\xi)$$
$$J_1 = \psi'(\xi^+) - \psi'(\xi^-) = \tfrac{1}{2}[G(\xi) - A^r(\xi)F(\xi)]$$

...

4. *Systematic approach: generalize finite-difference & pseudospectral methods to include (<u>known</u>) jump discontinuities in approximating function*

Discontinuous method

- Jumps in solution *ψ* and its derivatives across particle known *a priori* from field equation

$$\psi^{(k)}(\xi^{+}) - \psi^{(k)}(\xi^{-}) = J_k, \quad k = 1, 2, \ldots, \infty$$

- FD & PS methods are based on Lagrange interpolation
- Construct discontinuous generalization to Lagrange interpolation that uses $\{J_k\}$ as input

Lagrange interpolation

- N-th order polynomial

$$p(x) = \sum_{j=0}^{N} c_j x^j$$

- Collocation conditions

$$p(x_i) = f_i, \quad i = 0,1,\ldots,N$$

- Solution: Lagrange's interpolating polynomial

$$p(x) = \sum_{j=0}^{N} f_j \pi_j(x), \quad \pi_j(x) = \prod_{\substack{k=0 \\ k \neq j}}^{N} \frac{x - x_k}{x_j - x_k}$$

Discontinuous Lagrange interpolation

- N-th order polynomials

$$p_-(x) = \sum_{j=0}^{N} c_j^- x^j, \; x < \xi; \quad p_+(x) = \sum_{j=0}^{N} c_j^+ x^j, \; x > \xi$$

- Collocation conditions

$$p(x) = \theta(x-\xi)p_+(x) + \theta(\xi - x)p_-(x)$$

$$p(x_i) = f_i \quad (i = 0, 1, \ldots, N)$$

- Jump conditions

$$p_+^{(k)}(\xi^+) - p_-^{(k)}(\xi^-) = J_k \quad (k = 0, 1, \ldots, M)$$

- Solution: *piecewise*-polynomial interpolant

$$s_j(x;\xi) = [\theta(x-\xi)\theta(\xi - x_j) - \theta(\xi - x)\theta(x_j - \xi)] \sum_{m=0}^{M} \frac{J_m}{m!}(x_j - \xi)^m$$

$$p(x) = \sum_{j=0}^{N} [f_j + s_j(x;\xi)]\pi_j(x)$$

Discontinuous method

- Interpolation

$$p(x) = \sum_{j=0}^{N} [f_j + s_j(x;\xi)]\pi_j(x)$$

- Differentiation

$$p^{(n)}(x_i) = \sum_{j=0}^{N} D_{ij}^{(n)}[f_j + s_j(x;\xi)], \quad D_{ij}^{(n)} = \pi_j^{(n)}(x_i)$$

- Integration

$$\int_a^b p(x)dx = \sum_{j=0}^{N} [w_j f_j + q_j(\xi)]$$

$$w_j = \int_a^b \pi_j(x)dx, \;\; q_j(\xi) = \int_a^b s_j(x;\xi)\pi_j(x)dx$$

Discontinuous method

- Interpolation

$$p(x) = \sum_{j=0}^{N} [f_j + s_j(x;\xi)]\pi_j(x)$$

- Finite differencing (equidistant nodes)

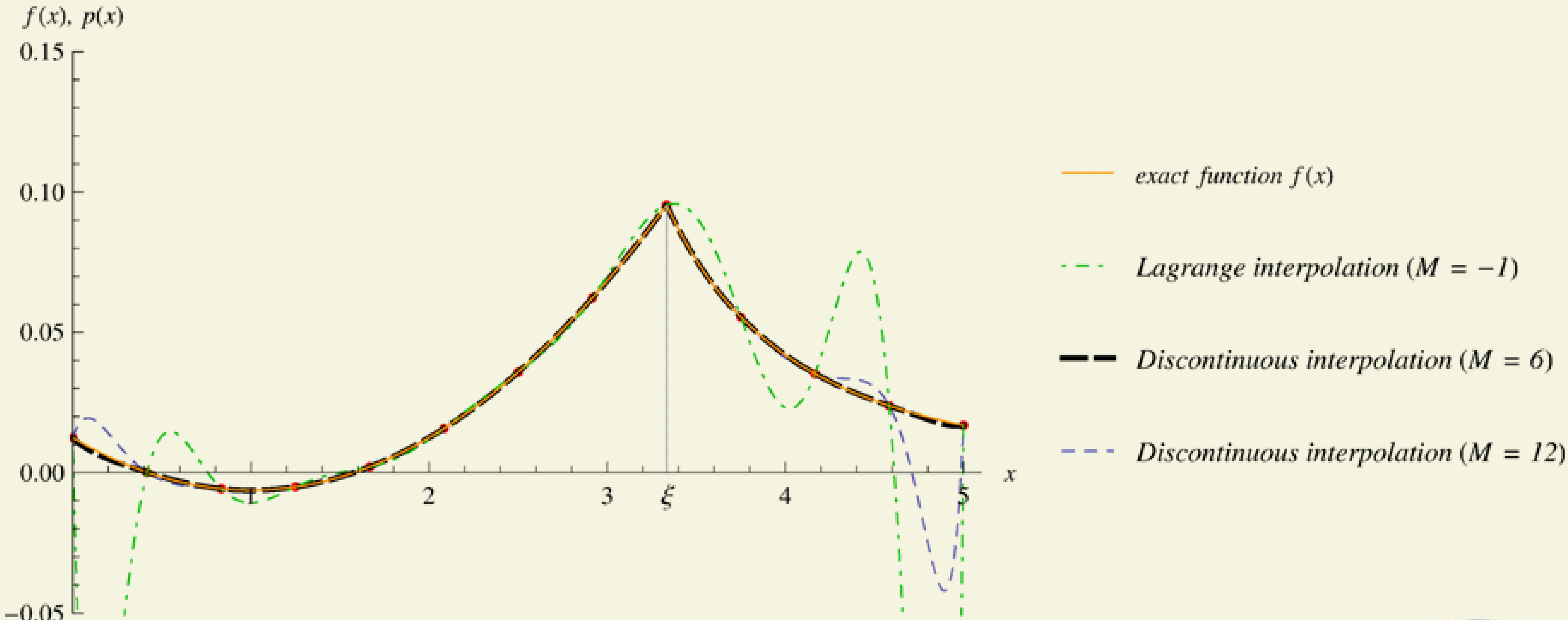
$$x_i = a + i\frac{b-a}{N} \quad (i = 0,1,\ldots,N)$$

- Pseudospectral (Chebyshev-Gauss-Lobatto nodes)

$$x_i = \frac{a+b}{2} + \frac{a-b}{2}\cos\left(\frac{i\pi}{N}\right)$$

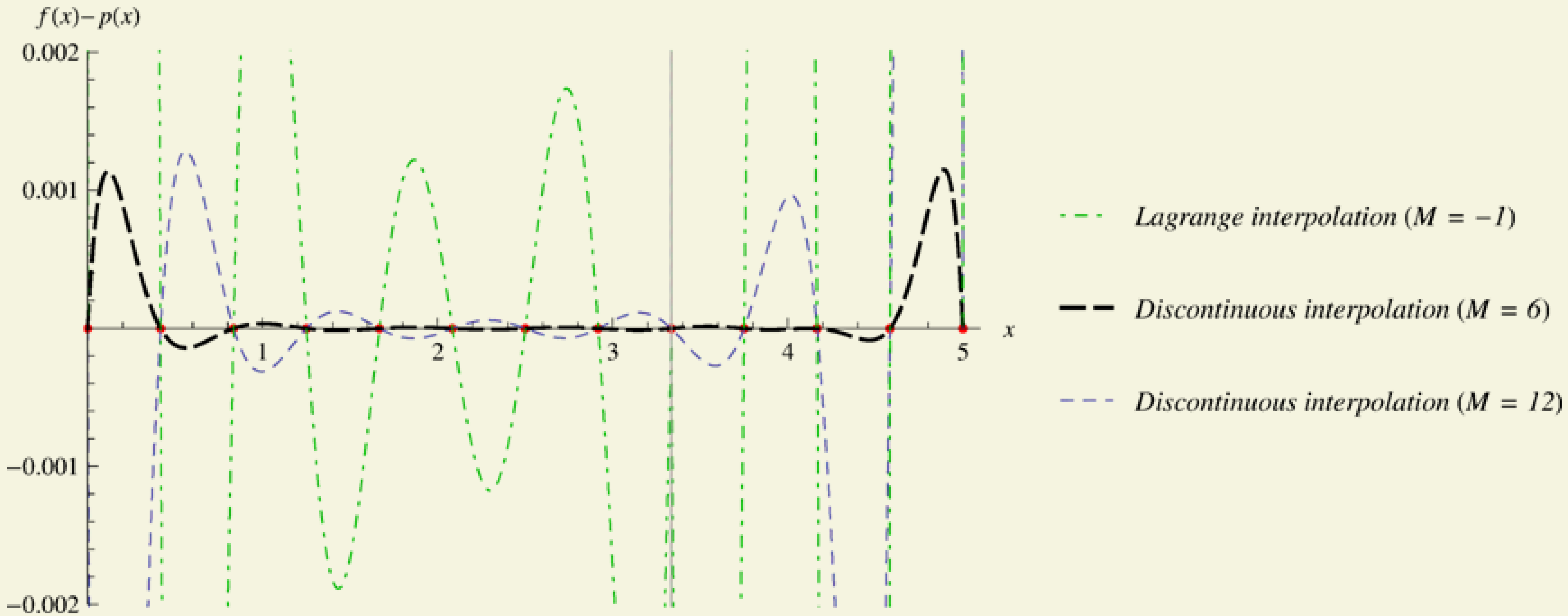
Static scalar particle in Schwarzschild

Equidistant nodes



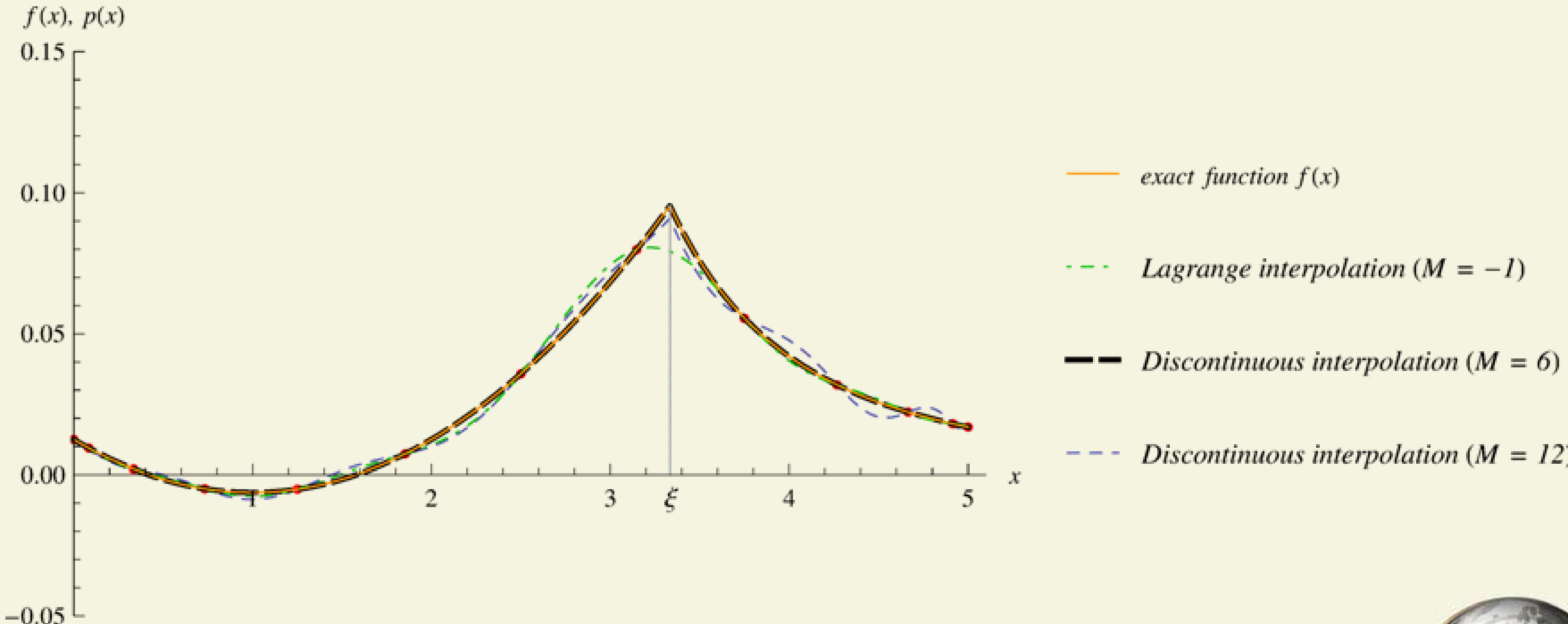
Static scalar particle in Schwarzschild

Equidistant nodes



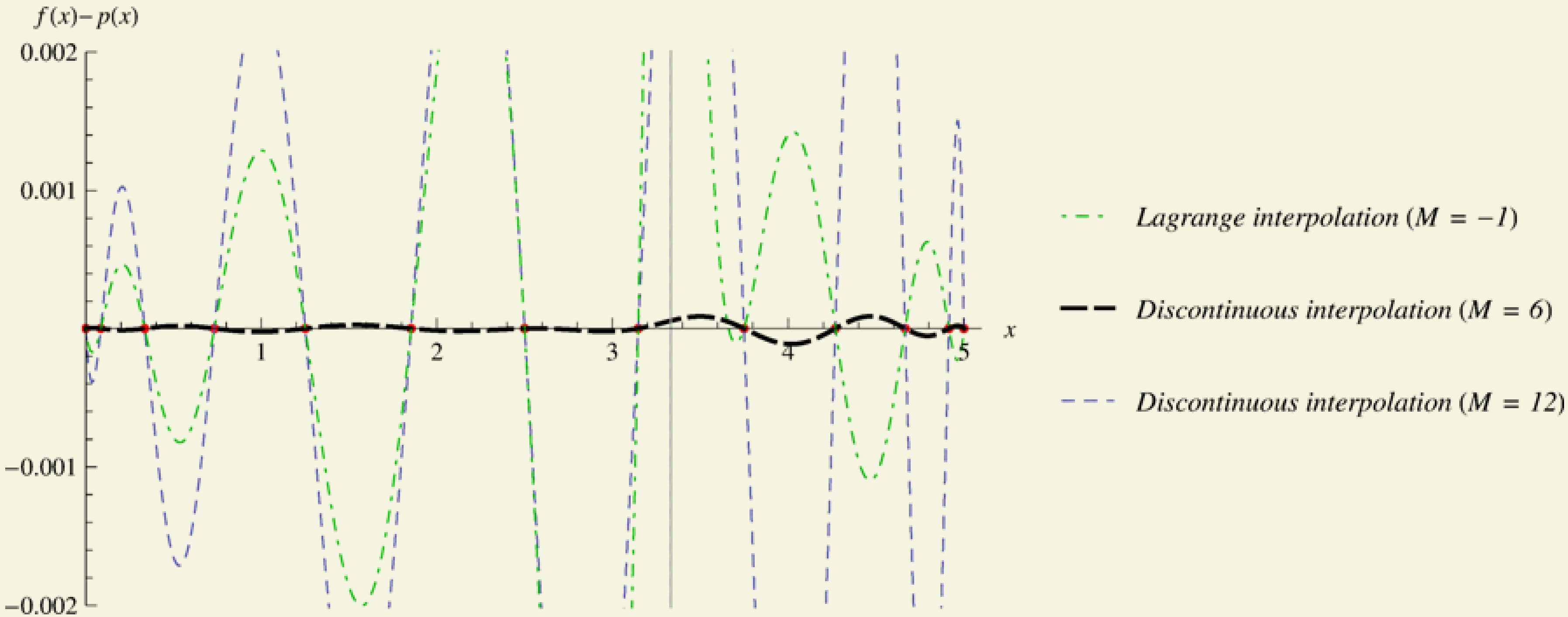
Static scalar particle in Schwarzschild

Chebyshev-Gauss-Lobbato nodes



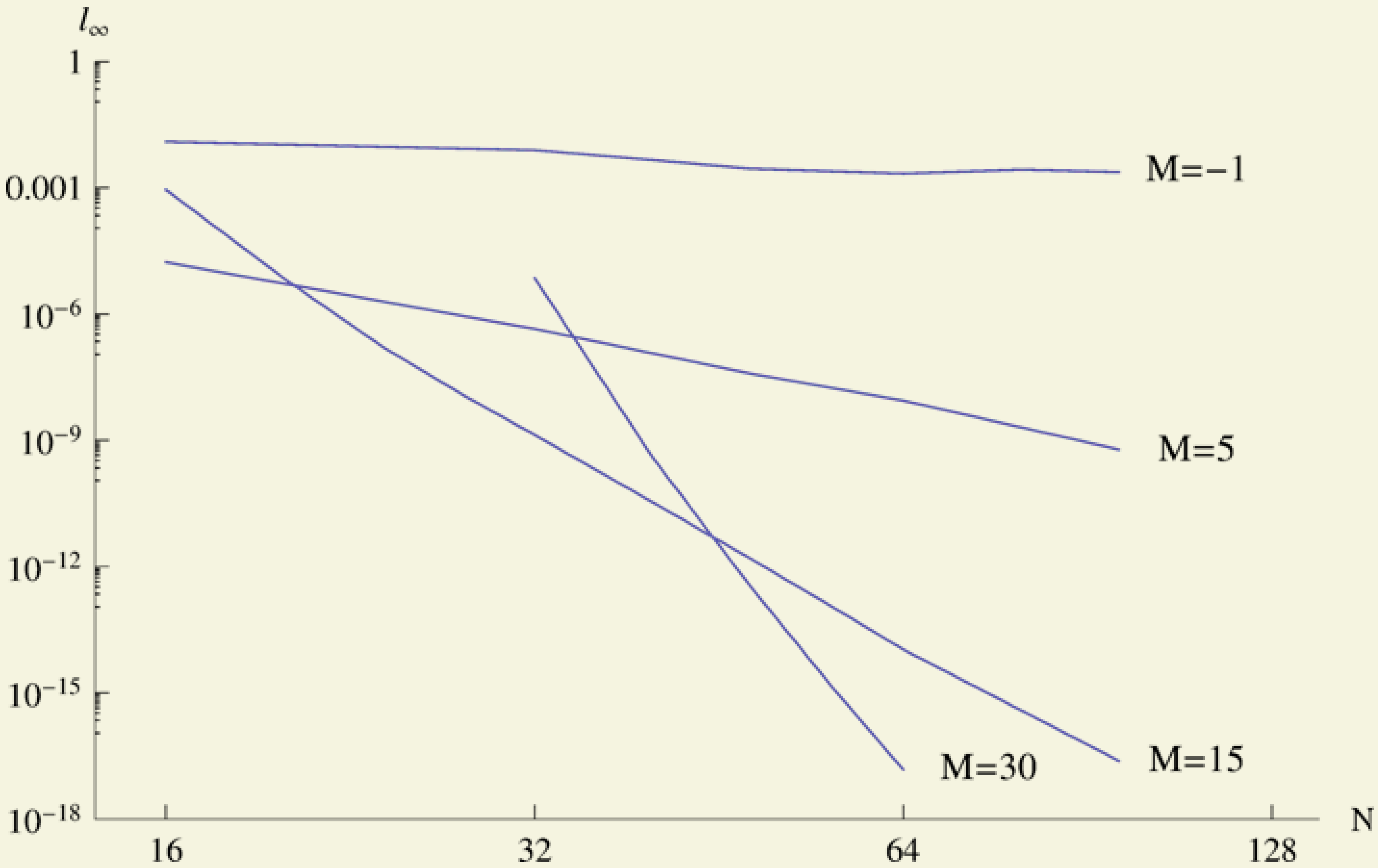
Static scalar particle in Schwarzschild

Chebyshev-Gauss-Lobbato nodes



Static scalar particle in Schwarzschild

sup error norm (Chebyshev-Gauss-Lobbato nodes)



Discontinuous method

Merits:

- High accuracy for pseudospectral nodes
- Particle can move freely inside domain
- No domain decomposition, coordinate mapping, etc needed
- Very easy to implement
- Efficient

Cost:

- Must use large number of jumps (roughly half the # of nodes)
- Must update $s_j(x;\xi)$ as particle moves

From Weyl curvature to the Hertz potential

- If solution ψ_0 (or ψ_4) is known, one reconstructs the Hertz potential Ψ via

$$\psi_0 = D^4 \Psi^* = (l^\mu \partial_\mu)^4 \Psi^*$$

- Invert (with 4 integrations) to recover Ψ
- Differentiate (twice) to recover metric perturbation $h_{\alpha\beta}$ (see e.g. Lousto & Whiting)

- Recall, however, that Ψ also satisfies homogenous Teukolsky equation off the particle
- Can one skip ψ_0 and solve directly for Ψ?

From Weyl curvature to the Hertz potential

- To solve Teukolsky equation for Ψ, one must impose BCs (no incoming at infinity, no outgoing on horizon)
- One must also compute (Dirac δ, δ') source terms due to particle
- Equivalent to determining the jumps $[\Psi]$, $[\Psi']$ in Ψ, Ψ' across the particle location
- In turn, these can be determined from the jumps $[\psi_0]$, $[\psi_0']$ in ψ_0, ψ_0' as follows…

From Weyl curvature to the Hertz potential

Example: static particle in Minkowski

$$\partial_r{}^2\Psi^* = -\tfrac{6}{r}\partial_r\Psi^* + \frac{(l+3)(l-2)}{r^2}\Psi^*$$

$$\partial_r{}^3\Psi^* = \frac{l^2+l+36}{r^2}\partial_r\Psi^* - \frac{8(l+3)(l-2)}{r^3}\Psi^*$$

$$\partial_r{}^4\Psi^* = -\frac{16(l^2+l+15)}{r^3}\partial_r\Psi^* + \frac{(l+3)(l-2)(l^2+l+60)}{r^4}\Psi^*$$

$$\partial_r{}^5\Psi^* = \frac{1800+l(l+1)(l^2+l+198)}{r^4}\partial_r\Psi^* - \frac{20(l+3)(l-2)(l^2+l+24)}{r^5}\Psi^*$$

From Weyl curvature to the Hertz potential

$$\psi_0 = -\frac{16(l^2+l+15)}{r^3}\partial_r\Psi^* + \frac{(l+3)(l-2)(l^2+l+60)}{r^4}\Psi^*$$

$$\partial_r\psi_0 = \frac{1800+l(l+1)(l^2+l+198)}{r^4}\partial_r\Psi^* - \frac{20(l+3)(l-2)(l^2+l+24)}{r^5}\Psi^*$$

$$\Psi^* = \frac{(l-6)!(l+2)!}{(l+6)!(l-2)!}\{16(l^2+l+15)r^5\partial_r\psi_0 + [1800+l(l+1)(l^2+l+198)]r^4\psi_0\}$$

$$\partial_r\Psi^* = \frac{(l-6)!(l+3)!}{(l+6)!(l-3)!}\{(l^2+l+60)r^4\partial_r\psi_0 + 20(l^2+l+24)r^3\psi_0\}$$

From Weyl curvature to the Hertz potential

- Knowledge of jumps in the Hertz potential determines source terms and allows one to write a (Teukolsky) field equation analogous to that satisfied by Weyl scalars:

$$(\nabla^\mu \nabla_\mu + A^\mu_{lm} \nabla_\mu + V_{lm}) \Psi^{*lm}(r) = G_{lm}\,\delta(r-\xi) + F_{lm}\,\delta'(r-\xi)$$

- Can be solved with time-domain methods such as those described earlier
- Two (one) derivatives on Ψ^* to reconstruct $h_{\alpha\beta}$

Conclusion

- Numerical methods exist to solve the Teukolsky equation efficiently & accurately in the time domain
- Solutions can be used to reconstruct metric perturbation and self-force in a radiation gauge
- Source to be determined by evolving geodesic equation in perturbed spacetime, in a self-consistent approach

References

C. Markakis and L. Barack, *High-order difference and pseudospectral methods for discontinuous problems* [arXiv:1406.4865]