Gravitational self-force from radiation gauge reconstructed metric perturbations.

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Outline

- Strategy for computing the gravitational SF from reconstructed MP. Adam Pound, Cesar Merlin and Leor Barack.[Phys. Rev. D 89, 024009]

- The completion of the solution. Cesar Merlin, Leor Barack, Amos Ori and Adam Pound.

- Numerical Implementation. Cesar Merlin and Abhay Shah.

- The original formulation of the SF was given in the Lorenz gauge [Poisson *et al.* (2011)]. For Kerr the field equations in the LG are not separable.

- The treatment of black-hole perturbations for Kerr is much simpler in a radiation gauge, where it is possible to reconstruct the perturbations from the perturbed Weyl scalars. [Chrzanowski-Cohen-Kegeles (1975), Wald (1978), Ori (2008), Friedman *et al.* (2010)]

- Gauge invariant quantities have been successfully computed using this idea. [Friedman *et al.* (2011-12)]

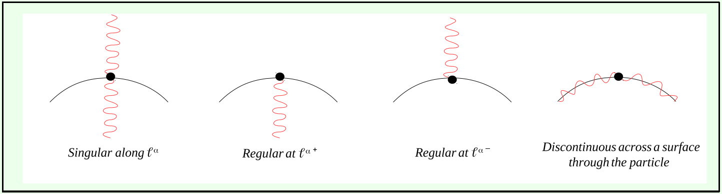
- The solution in the RG needs to be completed to include the non-radiative modes.[Wald, (1973)]

$$h_{\alpha\beta} = h^{\mathrm{Rec}}_{\alpha\beta} + h^{\mathrm{Comp}}_{\alpha\beta}.$$

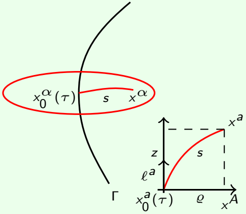
- We want to include $h^{\mathrm{Comp}}_{\alpha\beta}$ with a rigorous method for SF calculations.

Singular structure of the Radiation gauges

Pathological singularity in the radiation gauge



Half-string solutions	Full-string solution	No-string solution
$h^{\pm}_{\tau A} = \mp \dfrac{2\mu x_A}{s(s \pm z)}$	$h_{\tau A} = \dfrac{2\mu z x_A}{s\varrho^2}$	$h_{\tau A} = h^{+}_{\tau A}\theta(z) + h^{-}_{\tau A}\theta(-z)$
$h^{\pm}_{zA} = \pm \dfrac{2\mu x_A}{s(s \pm z)}$	$h_{zA} = -\dfrac{2\mu z x_A}{s\varrho^2}$	$h_{zA} = h^{+}_{zA}\theta(z) + h^{-}_{zA}\theta(-z)$



Method 1: Mode-sum formula for Locally-Lorenz Gauge

In a LL gauge the perturbation near the particle has the same leading-order singular form as the Lorenz gauge,

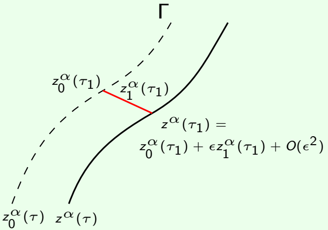
$$\boxed{h^{\rm LL}_{\alpha\beta} = \frac{2\mu\delta_{\alpha\beta}}{s} + O(1).}$$

and we can calculate the self-force using the same LG mode-sum formula

$$F^{\rm LL}_{\alpha} = \sum_{\ell=0}^{\infty}\left[F^{\rm LL\,\ell}_{\alpha} - A_\alpha L - B_\alpha - C_\alpha/L\right] - D_\alpha, \quad \text{with} \quad L = \ell + 1/2.$$

And the first order correction to geodesic motion

$$\mu\frac{D^2 z_1^{\alpha}{}_{\rm LL}}{D\tau^2} = -\mu R^{\alpha}_{\mu\beta\nu} u^{\mu} z_1^{\beta}{}_{\rm LL} u^{\nu} + F^{\alpha}{}_{\rm LL}$$



The metric perturbation transforms (from Rad→ LL) according to

$$h^{\rm LL}_{\alpha\beta} = h^{\rm Rad}_{\alpha\beta} + \xi_{\alpha;\beta} + \xi_{\beta;\alpha}.$$

where ξ induces a change in the force given by

$$\delta F^{\alpha\,{\rm Rad}\to{\rm LL}}_{grav} = -\mu\left[(g^{\alpha\gamma} + u^\alpha u^\gamma)\frac{D^2\xi_\gamma}{D\tau^2} + R^\alpha{}_{\beta\gamma\nu}u^\beta\xi^\gamma u^\nu\right].$$

Using the half-string radiation gauge modes we write

$$\begin{aligned} F^{\rm LL}_\alpha &= \sum_{\ell=0}^{\infty}\left[F^{{\rm Rad}\,\ell}_{\alpha\,\pm} + \delta F^{{\rm Rad}\to{\rm LL}\,\ell}_{\alpha\,\pm} - A^\pm_\alpha L - B_\alpha - C_\alpha/L\right] - D_\alpha \\ &= \sum_{\ell=0}^{\infty}\left[(F^{\rm Rad}_\alpha)^\ell_\pm - (A^\pm_\alpha - \delta A^\pm_\alpha)L - (B_\alpha - \delta B_\alpha) - (C_\alpha - \delta C_\alpha)/L\right] - (D_\alpha - \delta D^\pm_\alpha). \end{aligned}$$

with

$$\delta D^\pm_\alpha = \sum_{\ell=0}^{\infty}\left[\left(\delta F^{{\rm Rad}\to{\rm LL}}_\alpha\right)^\ell_\pm - \delta A^\pm_\alpha L - \delta B_\alpha - \delta C_\alpha/L\right].$$

Circular equatorial orbits in Kerr

For a fixed extension of u^α and the covariant derivatives we get:

$$\delta D^{\pm}_{\alpha} = \pm \mathcal{Q}_\alpha \frac{2\mu^2 c}{r_0(b-c^2)}\left(1-\frac{1}{\sqrt{1+b-c^2}}\right),$$

where

$$\begin{aligned} b =& r_0^{-3}\left[\mathcal{L}^2 r_0 + a^2(2M+r_0)\right], \\ c =& \frac{a^2\mathcal{E}\mathcal{L}+\mathcal{E}\mathcal{L}r_0^2 - a\mathcal{L}^2 - a\Delta}{r_0\left(a^2\mathcal{E}-a\mathcal{L}+\mathcal{E}r_0^2\right)}, \quad \textit{with} \quad \Delta \equiv r_0^2 - 2Mr_0 + a^2, \end{aligned}$$

and

$$\begin{aligned} \mathcal{Q}_t =& \mathcal{Q}_\theta = \mathcal{Q}_\varphi = 0, \\ \mathcal{Q}_r =& \frac{3M}{r_0^3}\,\frac{vr_0^2 - a(r_0-M) - a^2 v}{r_0 - 3M + 2av} \quad \textit{with} \quad v \equiv \sqrt{M/r_0}. \end{aligned}$$

Method 2: Mode-sum formula for a no-string Radiation Gauge

The correction to the motion in between two parity regular gauges:

$$\mu \frac{D^2}{D\tau^2}\left(\Delta z_1^\alpha\right) = -\mu R^\alpha_{\ \mu\beta\nu} u^\mu \Delta z_1^\beta u^\nu + \Delta F^\alpha$$

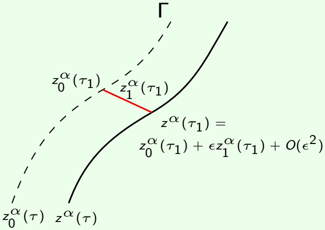
with

$$\Delta z_1^\alpha = \langle \xi \rangle_{s\to 0} \quad \text{and} \quad \Delta F^\alpha = \left\langle \ddot{\xi} \right\rangle_{s\to 0}$$

if ξ_α satisfies

- $\xi_\tau = f_1(\tau)\ln s + o(\ln s)$,
- $\xi_a = f_2(\tau, n^a) + o(1)$,
- temporal derivatives don't increase the degree of singularity,
- each spatial derivative increases the degree by one at most,

with f_1, f_2 at least twice differentiable and f_2 parity regular.



Mode-sum formula

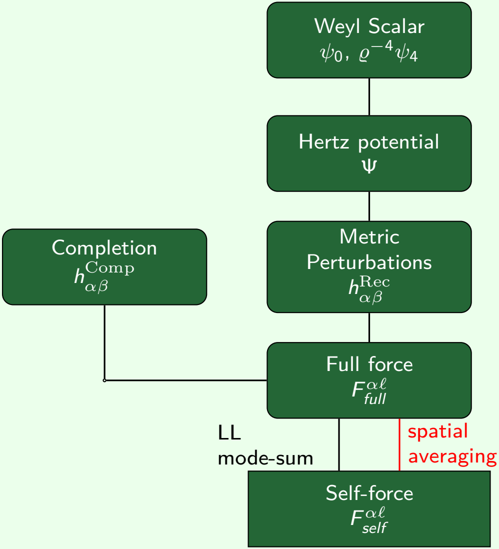
The gauge transformation between Lorenz to a half-string radiation gauge is not parity regular unlike the one between Lorenz to LL.

Corrections to D^α would include terms coming from supertranslations.

On the no-string RG we can use the two sided-averaged version of the mode-sum by virtue of $\delta D^{\alpha +} = -\delta D^{\alpha -}$:

$$\boxed{F^\alpha = \sum_\ell \left[\frac{1}{2} (F^\alpha)^\ell_+ + \frac{1}{2} (F^\alpha)^\ell_- - B^\alpha \right].}$$

Implementation



Completion of the solution

Wald showed (1973) that the only things we can add to the metric reconstruction are:

- Mass and angular momentum perturbations (δM and δJ).
- Perturbations to other algebraically special solutions (C-metric and Kerr-NUT).
- Gauge perturbations.

The Weyl scalars ψ_0 and ψ_4 are gauge invariant, but they only carry information from the radiative part of the solution.

- Obtain δM and δJ in "Boyer–Lindquist" gauge:

$$h^{\delta M}_{\alpha\beta} \equiv \delta M \frac{\partial}{\partial M} g_{\alpha\beta}(M,J),$$
$$h^{\delta J}_{\alpha\beta} \equiv \delta J \frac{\partial}{\partial J} g_{\alpha\beta}(M,J).$$

- Amplitudes at $r > r_0$ can be obtained from the limit $r \to \infty$.
- Amplitudes at $r < r_0$ determined by continuity condition off the particle for certain invariants.
- The invariants are constructed from $m = 0$ sector of the full (completed) MP.
- In eccentric/inclined case we will impose continuity at the level of the Green's function.

Metric perturbations in the preferred gauge

Under a gauge transformation ψ_2 transforms

$$\tilde{\psi}_2 \to \psi_2 - \xi^\alpha \psi_{2,\alpha}$$

We move to a "preferred gauge" $\tilde{h}$, in which $\delta\psi_2$ vanishes. The relevant components of the gauge vector are

$$\boxed{\tilde{\xi}^r = \mathrm{Re}\left(\frac{\varrho^{-4}}{3M}\delta\psi_2\right), \qquad \tilde{\xi}^\theta = \frac{\mathrm{Im}\left(\frac{\varrho^{-4}}{3M}\delta\psi_2\right)}{a\sin\theta}.}$$

The new components of the MP are given by

$$\tilde{h}_{\alpha\beta} = h_{\alpha\beta} - \left(\tilde{\xi}_{\alpha,\beta} + \tilde{\xi}_{\beta,\alpha}\right) + 2\Gamma^\mu_{\alpha\beta}\tilde{\xi}_\mu.$$

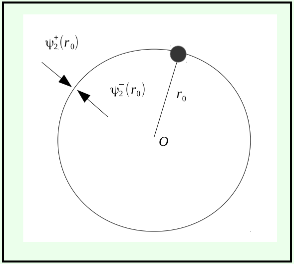
We shall focus now on three metric components (where $a, b, c ... = r, \theta$):

$$\boxed{\tilde{h}_{ab} = h_{ab} - \left(\tilde{\xi}_{a,b} + \tilde{\xi}_{b,a}\right) + 2\Gamma^c_{ab}\tilde{\xi}_c,}$$

which are gauge-invariant.

Elementary example: Invariants in Flat-space

For a static particle sitting in flat space ψ_2 is trivially invariant



- Perform the MP reconstruction mode by mode on both limits $r \to r_0^{\pm}$.
- Calculate ψ_2 on the corresponding sides:

$$\psi_2^-(r) = -\sum_{\ell=2}^{\infty} \frac{2\mu(\ell-1)\ell\pi r^{\ell-2} r_0^{-\ell-1}}{2\ell+1} Y_{\ell 0}(\theta,\varphi) Y'_{\ell 0}(\theta_0,\varphi_0),$$

$$\psi_2^+(r) = -\sum_{\ell=2}^{\infty} \frac{2\mu(\ell+1)(\ell+2)\pi r^{-\ell-3} r_0^{\ell}}{2\ell+1} Y_{\ell 0}(\theta,\varphi) Y'_{\ell 0}(\theta_0,\varphi_0).$$

- Evaluate the jump off the particle at $r = r_0$:

$$[\psi_2]^{\mathrm{Rec}}_{r=r_0} = -\frac{4\pi\mu}{r_0^3}\sum_{\ell=2}^{\infty} Y_{\ell 0}(\theta,\varphi) Y'_{\ell 0}(\theta_0,\varphi_0)$$

$$= -\frac{\mu}{r_0^3}\delta(\Omega-\Omega_0) + \frac{\mu}{r_0^3} + \frac{3\mu\cos\theta\cos\theta_0}{r_0^3}.$$

- Determine analytically the perturbation in the exterior solution from limit $r \to \infty$:

$$\psi_2^{\mathrm{Comp}+} = -\frac{\mu}{R^3} \quad \text{with} \quad R = \sqrt{r^2 + r_0^2 - 2rr_0\cos\gamma}.$$

- We define $[\psi_2]^{\mathrm{Comp}} \equiv \psi_2^{\mathrm{Comp}+} - \psi_2^{\mathrm{Comp}-}$ and impose continuity off the particle

$$[\psi_2]_{r=r_0} = [\psi_2]^{\mathrm{Rec}}_{r=r_0} + [\psi_2]^{\mathrm{Comp}}_{r=r_0} = 0 \Longrightarrow \psi_2^{\mathrm{Comp}-} = 0.$$

Second Example: Circular equatorial orbits in Schwarzschild

In Schwarzschild we have two invariants $\mathrm{Im}(\delta\psi_2)$ and $\tilde{h}_{rr}$.

For $\mathrm{Im}(\delta\psi_2)$ the analytical solutions are

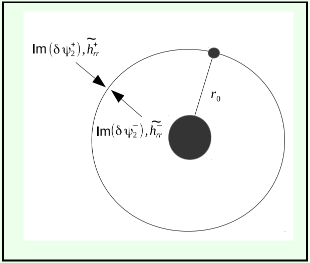
$$\mathrm{Im}(\delta\psi_2^-) = C_-(\mathrm{Q}_\ell^2(x_0), \mathrm{Q}_\ell^{2\prime}(x_0)) R_{2-}(x) Y_{\ell 0}(\theta,\varphi) Y'_{\ell 0}(\theta_0,\varphi_0),$$
$$\mathrm{Im}(\delta\psi_2^+) = C_+(\mathrm{P}_\ell^2(x_0), \mathrm{P}_\ell^{2\prime}(x_0)) R_{2+}(x) Y_{\ell 0}(\theta,\varphi) Y'_{\ell 0}(\theta_0,\varphi_0).$$

The jump at $r = r_0$ is given by

$$[\mathrm{Im}(\delta\psi_2)]^{\mathrm{Rec}}_{r=r_0} = \sum_{\ell=2}^{\infty} \frac{4\mu\mathcal{L}\pi}{r_0^4} Y_{\ell 0}(\theta,\varphi) Y'_{\ell 0}(\theta_0,\varphi_0),$$

which can be summed as

$$[\mathrm{Im}(\delta\psi_2)]^{\mathrm{Rec}}_{r=r_0} = -\frac{4\mu\mathcal{L}\pi}{r_0^4}\delta'(\Omega-\Omega_0) + \frac{3\mu\mathcal{L}}{r_0^4}\cos\theta.$$



In a similar way for $\tilde{h}_{rr}$ we find

$$[\tilde{h}_{rr}]^{\rm Rec}_{r=r_0} = \sum_{\ell=2}^{\infty} \frac{8\mu\mathcal{E}\pi}{3Mf_0^2} Y_{\ell 0}(\theta,\varphi) Y_{\ell 0}(\theta_0,\varphi_0) + \sum_{\ell=2}^{\infty} \frac{4\mu\mathcal{E}\pi(r_0-M)}{3Mr_0 f_0^3} Y_{\ell 0}(\theta,\varphi) Y''_{\ell 0}(\theta_0,\varphi_0),$$

which we sum analytically using the generating function:

$$[\tilde{h}_{rr}]^{\rm Rec}_{r=r_0} = \frac{8\mu\mathcal{E}\pi}{3Mf_0^2}\delta(\Omega-\Omega_0) - \frac{4\mu\mathcal{E}\pi(r_0-M)}{3Mr_0 f_0^3}\delta''(\Omega-\Omega_0) - \frac{2\mu\mathcal{E}}{3Mf_0^2}.$$

We find the mass and angular momentum perturbations at $r \to \infty$:

$$\tilde{h}^{\rm Comp+}_{rr} = \frac{2\mu\mathcal{E}}{3Mf^2},$$
$$\mathrm{Im}(\delta\psi_2^{\rm Comp+}) = -\frac{3\mu\mathcal{L}}{r^4}\cos\theta.$$

This way

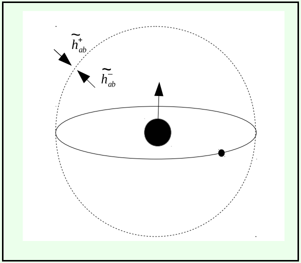
$$[\tilde{h}_{rr}]_{r=r_0} = [\tilde{h}_{rr}]^{\rm Rec}_{r=r_0} + [\tilde{h}_{rr}]^{\rm Comp}_{r=r_0} = 0, \quad \Longrightarrow \tilde{h}^{\rm Comp-}_{rr} = 0.$$
$$[\mathrm{Im}(\delta\psi_2)]_{r=r_0} = [\mathrm{Im}(\delta\psi_2)]^{\rm Rec}_{r=r_0} + [\mathrm{Im}(\delta\psi_2)]^{\rm Comp}_{r=r_0} = 0 \quad \Longrightarrow \mathrm{Im}(\delta\psi_2^{\rm Comp-}) = 0.$$

Third example: Circular equatorial orbits in Kerr

In Kerr $\mathrm{Im}(\psi_2) \neq 0$ and $\mathrm{Im}(\delta\psi_2)$ is no longer invariant. The components $\tilde{h}_{ab}$ are still invariant.

We can still obtain the mass and angular momentum perturbations at $r \to \infty$ for $\tilde{h}_{rr}$ corresponds to

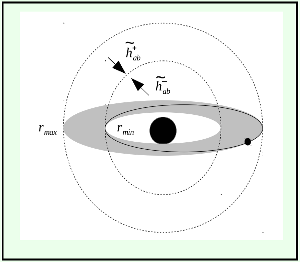
$$\tilde{h}_{rr}^{\mathrm{Comp}+} = -\frac{2a\delta J\Sigma}{M\Delta^2} + \delta M \Big[9a^2M^5r + 7a^2M^4r^2 + 2M^4r^4 + 10a^4M^4\cos^2\theta - 18a^2M^5r\cos^2\theta + 8a^2M^4r^2\cos^2\theta + 9a^2M^5r\cos(2\theta) - 3a^2M^4r^2\cos(2\theta)\Big] / 3M^5\Delta^2.$$

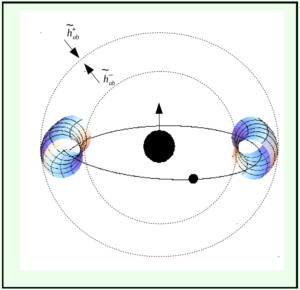


More general orbits in Kerr

- At each point in the region $r_{\rm min} \leq r_0 \leq r_{\rm max}$, $\theta_{\rm min} \leq \theta_0 \leq \theta_{\rm max}$ we find the jump on the gauge invariants corresponding to a infinitesimal ring.

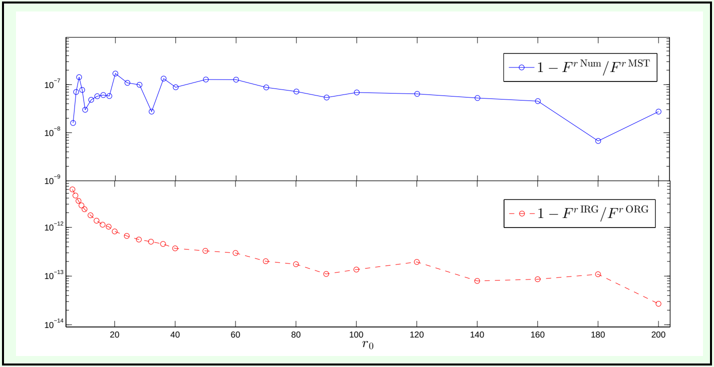
- We impose continuity for each ring and integrate all the throughout the source region.





Circular equatorial orbits in Schwarzschild

$r_0\,/\,M$	$F^{r\,\mathrm{Num}}(r_0)\times\frac{M^2}{\mu^2}$	$F^{r\,\mathrm{MST}}(r_0)\times\frac{M^2}{\mu^2}$	$\frac{\lvert\Delta F^r(r_0)\rvert}{F^{r\,\mathrm{MST}}(r_0)}$
6	-0.03686748(1)	-0.036867480588628(1)	2e-08
10	-0.0127062190(9)	-0.012706219381419343525(4)	3e-08
20	-0.0033944359(9)	-0.0033944353218027041758919(1)	2e-07
50	-0.0005752097(2)	-0.000575209706623811834752961743(1)	1e-07
100	-0.00014682425(2)	-0.000146824239894828699361863278978(1)	7e-08
200	-0.000037098354(3)	-0.00003709835297838419521139616578857(2)	3e-08



Summary and future work

- We have identified three types of RG gauges and discussed the motion in them. We presented two methods to calculate the SF starting from RG MP and using new versions of the mode-sum method.
- To include the non-radiative modes that are not recovered by the CCK MP reconstruction, we want to construct gauge invariant quantities. By imposing continuity off the particle we can extract amplitude of the mass and angular momentum perturbations inside the orbit.
- Finish the calculation of the invariants required to complete the CCK metric reconstruction in Kerr. Extend our analysis to complete the low multipoles for more general orbits (eccentric, inclined) using extended homogeneous solutions.
- We have obtained SF numerical values using the averaged version of the mode-sum and RG metric perturbations. Two methods to calculate the homogeneous solutions of Teukolsky equation were used and the agreement between them validates our results.
- An extension to eccentric orbits in Kerr is in progress.