Taming the post-Newtonian expansion: How to simplify (and understand) high-order post-Newtonian expressions for EMRIs

Nathan K. Johnson-McDaniel
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arXiv:1405.1572 [gr-qc]

Theoretisch-Physikalisches Institut,
Friedrich-Schiller-Universität Jena

Motivation: High-order PN expansions of quantities for EMRIs

- There has recently been a fair amount of work in obtaining high-order post-Newtonian (PN) expansions of various quantities for EMRIs.

- These calculations all use the black hole perturbation theory formalism developed by Mano, Suzuki, and Takasugi (based on Leaver's work on computing BH quasinormal modes, itself based on relativity early work in quantum mechanics) [Prog. Theor. Phys. **95**, 1079 (1996); **96**, 549 (1996)] allows one to calculate to arbitrarily high PN order, in principle (either analytically or numerically).

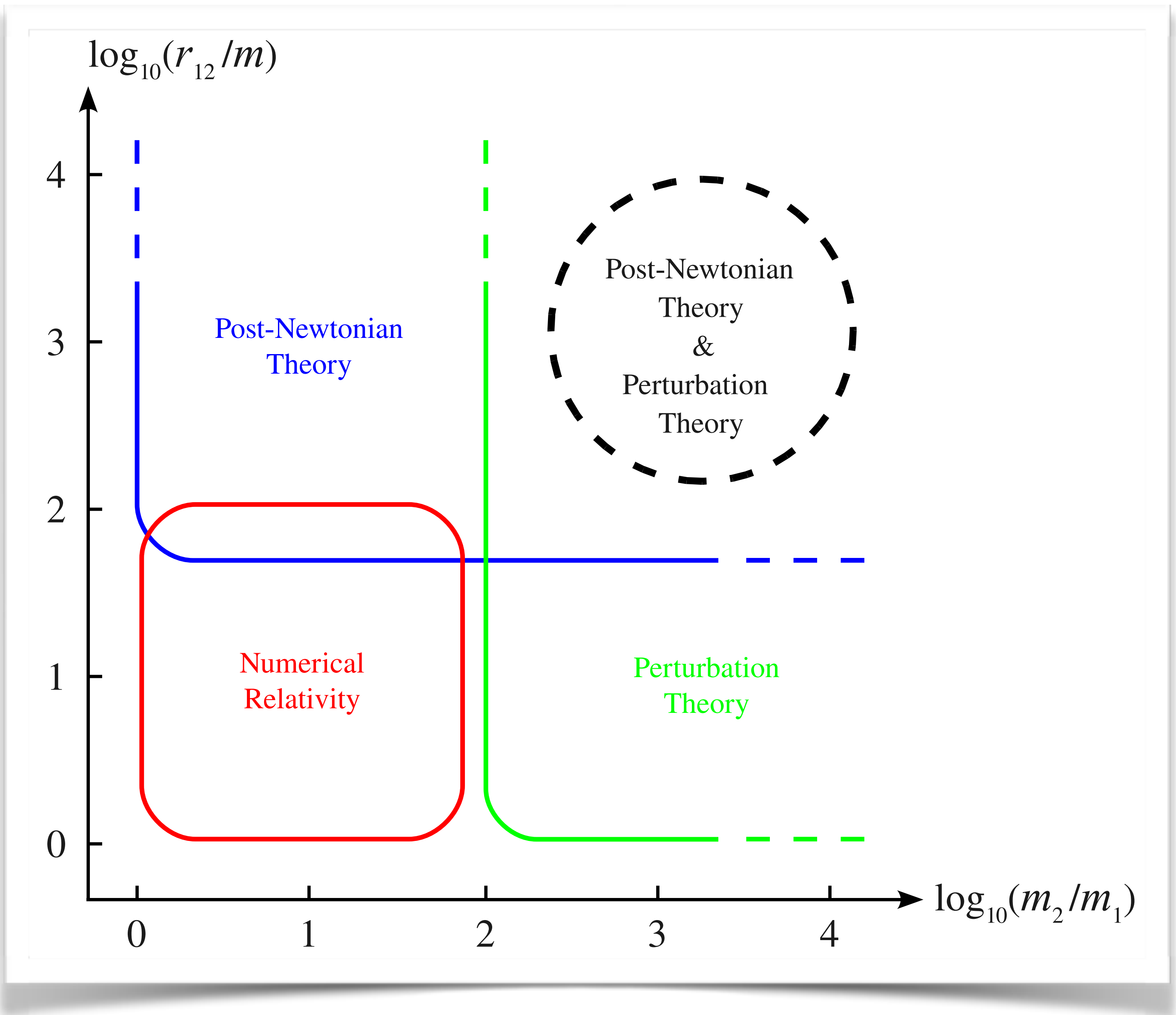
- This should be contrasted with the situation for comparable mass binaries, where one has to solve PDEs instead of just ODEs, and the current state-of-the-art is only 3.5PN [$O(v^7)$, i.e., $O(G^{3.5})$, past Newtonian predictions] for many quantities, with some quantities known to higher orders.

- Reaching 3.5PN in the comparable mass case involved decades of work by many researchers, and often new conceptual insights. The recent determination of the 4PN Hamiltonian by Damour, Jaranowski, and Schäfer was a major tour de force!

Motivation: High-order PN expansions of quantities for EMRIs

- The currently state-of-the-art for high-order EMRI computations is the 22PN calculation of the energy flux from a point particle in a circular orbit around a Schwarzschild black hole, by Ryuichi Fujita [Prog. Theor. Phys. **128**, 971 (2012)]. (Going to even higher orders is limited by ever-increasing demands on computer time and memory.)
- Fujita also has unpublished 10PN and 8PN calculations of the flux at infinity and the horizon for a circular equatorial orbit in Kerr, respectively, and there are other notable high-order calculations, both analytic and numerical:
- The 8.5PN analytic calculations of the redshift observable and spin precession frequency by Bini and Damour [Phys. Rev. D. **89**, 104047 (2014) and arXiv:1404.2747 [gr-qc]]
- Numerical calculations of the redshift observable to 10.5PN by Shah, Friedman, and Whiting [Phys. Rev. D **89**, 064042 (2014)] and the fluxes at infinity and the horizon in Kerr (for circular, equatorial orbits) to 20PN by Shah [arXiv:1403.2697 [gr-qc]], with some coefficients determined analytically in both cases.
- The analytic results for eccentric orbits are not yet known to high order (4PN for Schwarzschild and 2.5PN for Kerr and equatorial orbits and only to quadratic order in the eccentricity), but there is progress in obtaining higher-order terms numerically—see the talks by Evans and Forseth.

Why are these high-order expansions interesting?



From Blanchet, Detweiler, Le Tiec, and Whiting, Phys, Rev. D **81**, 064004 (2010)

- High-order PN results for EMRI fluxes can be used to obtain adiabatic templates comparable in accuracy to numerical adiabatic templates.
- Perhaps more importantly, though, these higher-order perturbative results can give important insights even in the comparable mass case (cf. recent work on EOB; also, the 4PN determination of Detweiler's redshift observable was necessary to complete the 4PN comparable mass Hamiltonian).
- They are also interesting for a study of the structure of the PN expansion, in general.

Complexity of high-order PN results

- High-order perturbation theory results generally display a combinatorial increase in complexity, and PN results are no exception.
- We will consider the GW energy flux at infinity in the following discussion, since it is the quantity currently known to the highest order.

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$$
\left\langle \frac{dE}{dt} \right\rangle_{\infty} \bigg/ \left\langle \frac{dE}{dt} \right\rangle_{\text{Newt}} =
$$

$$
\begin{aligned}
&1 - \frac{1247}{336} v^2 + 4\pi v^3 - \frac{44711}{9072} v^4 - \frac{8191}{672}\pi v^5 + \left[\frac{6643739519}{69854400} + \frac{16}{3}\pi^2 - \frac{1712}{105}\gamma - \frac{3424}{105}\log 2 - \frac{1712}{105}\log v\right] v^6 - \frac{16285}{504}\pi v^7 \\
&+ \left[-\frac{323105549467}{3178375200} - \frac{1369}{126}\pi^2 + \frac{232597}{4410}\gamma + \frac{39931}{294}\log 2 - \frac{47385}{1568}\log 3 + \frac{232597}{4410}\log v\right] v^8 \\
&+ \left[\frac{265978667519}{745113600} - \frac{6848}{105}\gamma - \frac{13696}{105}\log 2 - \frac{6848}{105}\log(v)\right]\pi v^9 \\
&+ \left[-\frac{2500861660823683}{2831932303200} - \frac{424223}{6804}\pi^2 + \frac{916628467}{7858620}\gamma - \frac{83217611}{1122660}\log 2 + \frac{47385}{196}\log 3 + \frac{916628467}{7858620}\log v\right] v^{10} \\
&+ \left[\frac{8399309750401}{101708006400} + \frac{177293}{1176}\gamma + \frac{8521283}{17640}\log 2 - \frac{142155}{784}\log 3 + \frac{177293\log v}{1176}\right]\pi v^{11} \\
&+ \left[\frac{2067586193789233570693}{602387400044430000} + \frac{3803225263}{10478160}\pi^2 - \frac{256}{45}\pi^4 - \frac{27392}{315}\pi^2\gamma - \frac{54784}{315}\pi^2\log 2 - \frac{27392}{105}\zeta(3)\right. \\
&- \frac{246137536815857}{157329572400}\gamma + \frac{1465472}{11025}\gamma^2 + \frac{5861888}{11025}\gamma\log 2 - \frac{271272899815409}{157329572400}\log 2 + \frac{5861888}{11025}\log^2 2 - \frac{437114506833}{789268480}\log 3 \\
&\left.- \frac{37744140625}{260941824}\log 5 + \left(-\frac{246137536815857}{157329572400} - \frac{27392}{315}\pi^2 + \frac{2930944}{11025}\gamma + \frac{5861888}{11025}\log 2\right)\log v + \frac{1465472}{11025}\log^2 v\right] v^{12} \\
&+ \left[-\frac{81605095538444363}{20138185267200} + \frac{300277177}{436590}\gamma - \frac{42817273}{71442}\log 2 + \frac{142155}{98}\log 3 + \frac{300277177}{436590}\log v\right]\pi v^{13} \\
&+ \left[\frac{58327313257446476199371189}{8332222517414555760000} + \frac{2621359845833}{2383781400}\pi^2 - \frac{9523}{945}\pi^4 + \frac{531077}{6615}\pi^2\gamma + \frac{128223}{245}\pi^2\log 2 - \frac{142155}{392}\pi^2\log 3\right. \\
&+ \frac{531077}{2205}\zeta(3) + \frac{9640384387033067}{17896238860500}\gamma - \frac{52525903}{154350}\gamma^2 - \frac{471188717}{231525}\gamma\log 2 + \frac{1848015}{2744}\gamma\log 3 + \frac{1940223255075 1339}{17896238860500}\log 2 \\
&- \frac{5811697}{2450}\log^2 2 + \frac{1848015}{2744}\log 2\log 3 - \frac{6136997968378863}{1256910054400}\log 3 + \frac{1848015}{5488}\log^2 3 + \frac{9926708984375}{5088365568}\log 5 \\
&\left.+ \left(\frac{9640384387033067}{17896238860500} - \frac{52525903}{77175}\gamma + \frac{531077}{6615}\pi^2 - \frac{471188717}{231525}\log 2 + \frac{1848015}{2744}\log 3\right)\log v - \frac{52525903}{154350}\log^2 v\right] v^{14} \\
&+ \cdots \\
&+ \Big\{\mathcal{R}_{168,155} + \cdots + \left[\mathcal{R}_{128,115} + \cdots + \mathcal{R}_{31,20}\zeta(3)\gamma\log 2\log 3 + \cdots + \mathcal{R}_{35,22}\zeta(3)\zeta(5)\log 2 + \cdots\right]\pi^2 + \cdots + \mathcal{R}_{13,9}\pi^{14} + \cdots \\
&+ \mathcal{R}_{15,4}\zeta(13) + \cdots + \mathcal{R}_{24,18}\gamma^7 + \cdots + \mathcal{R}_{69,59}\log 19 + \cdots + \left[\mathcal{R}_{146,134} + \cdots + \mathcal{R}_{19,9}\zeta(3)\zeta(5)\log 2 + \cdots + \mathcal{R}_{24,18}\gamma^6 + \cdots\right]\log v \\
&+ \mathcal{R}_{24,18}\log^7 v\Big\} v^{44} + O\left(v^{45}\right)
\end{aligned}
$$

where $\mathcal{R}_{A,B}$ denotes a rational number with A digits in the numerator and B digits in the denominator.

Only 7PN…

1949 terms at $O(v^{44})$; 9227 terms total

Simplifying the PN expansion of the energy flux

- How can we simplify this expression?

- Perhaps we should first look at the individual modes, starting with the dominant quadrupolar (2,2) mode.

Simplifying the PN expansion of the energy flux

- How
- Perh… odes, starti… de

$$
\begin{aligned}
\eta_{22} = 1 &- \frac{107}{21}v^2 + 4\pi v^3 + \frac{4784}{1323}v^4 - \frac{428}{21}\pi v^5 + \left[\frac{99210071}{1091475} + \frac{16}{3}\pi^2 - \frac{1712}{105}\gamma - \frac{3424}{105}\log 2 - \frac{1712}{105}\log v\right] v^6 + \frac{19136}{1323}\pi v^7 \\
&+ \left[-\frac{27956920577}{81265275} - \frac{1712}{63}\pi^2 + \frac{183184}{2205}\gamma + \frac{366368}{2205}\log 2 + \frac{183184}{2205}\log v\right] v^8 \\
&+ \left[\frac{396840284}{1091475} - \frac{6848}{105}\gamma - \frac{13696}{105}\log 2 - \frac{6848}{105}\log v\right] \pi v^9 \\
&+ \left[\frac{187037845924}{6257426175} + \frac{76544}{3969}\pi^2 - \frac{8190208}{138915}\gamma - \frac{16380416}{138915}\log 2 - \frac{8190208}{138915}\log v\right] v^{10} \\
&+ \left[-\frac{111827682308}{81265275} + \frac{732736}{2205}\gamma + \frac{1465472}{2205}\log 2 + \frac{732736}{2205}\log v\right] \pi v^{11} \\
&+ \Bigg[\frac{139638221186546204}{29253467368125} + \frac{295709968\pi^2}{654885} - \frac{256}{45}\pi^4 - \frac{27392}{315}\pi^2\gamma - \frac{54784}{315}\pi^2\log 2 - \frac{27392}{105}\zeta(3) - \frac{36117727568}{22920975}\gamma \\
&+ \frac{1465472}{11025}\gamma^2 + \frac{5861888}{11025}\gamma\log 2 - \frac{72235455136}{22920975}\log 2 + \frac{5861888}{11025}\log^2 2 \\
&+ \left(-\frac{36117727568}{22920975} - \frac{27392}{315}\pi^2 + \frac{2930944}{11025}\gamma + \frac{5861888}{11025}\log 2\right)\log v + \frac{1465472}{11025}\log^2 v\Bigg] v^{12} \\
&+ \left[\frac{748151383696}{6257426175} - \frac{32760832}{138915}\gamma - \frac{65521664}{138915}\log 2 - \frac{32760832}{138915}\log v\right] \pi v^{13} \\
&+ \Bigg[-\frac{1922289287156615370 8684}{13656396171461 79375} - \frac{406031680304}{243795825}\pi^2 + \frac{27392}{945}\pi^4 + \frac{2930944}{6615}\pi^2\gamma + \frac{5861888}{6615}\pi^2\log 2 + \frac{2930944}{2205}\zeta(3) \\
&+ \frac{51936991437808}{8532853875}\gamma - \frac{156805504}{231525}\gamma^2 - \frac{627222016}{231525}\gamma\log 2 + \frac{103873982875616}{8532853875}\log 2 - \frac{627222016}{231525}\log^2 2 \\
&+ \left(\frac{51936991437808}{8532853875} + \frac{2930944}{6615}\pi^2 - \frac{313611008}{231525}\gamma - \frac{627222016}{231525}\log 2\right)\log v - \frac{156805504}{231525}\log^2 v\Bigg] v^{14} + \cdots \\
&+ \Big\{\mathcal{R}_{111,100} + \cdots + \left[\mathcal{R}_{89,79} + \cdots + \mathcal{R}_{37,26}\zeta(3)\gamma\log^2 2 + \cdots + \mathcal{R}_{17,8}\zeta(3)\zeta(5)\log 2 + \cdots\right]\pi^2 + \cdots + \mathcal{R}_{12,9}\pi^{14} + \cdots \\
&+ \mathcal{R}_{13,4}\zeta(13) + \cdots + \mathcal{R}_{24,18}\gamma^7 + \cdots + \left[\mathcal{R}_{92,80} + \cdots + \mathcal{R}_{19,10}\zeta(3)\zeta(5)\log 2 + \cdots + \mathcal{R}_{24,18}\gamma^6 + \cdots\right]\log v \\
&+ \mathcal{R}_{24,18}\log^7 v\Big\} v^{44} + O\left(v^{45}\right)
\end{aligned}
$$

Still only 7PN…

650 terms at $O(v^{44})$; 4947 terms total

Simplifying the 2,2 mode of the energy flux

- Well, unfortunately the 2,2 mode wasn't much simpler than the full energy flux (and other modes share about the same complexity)—the primary simplification came about because we now only have logarithms of 2 and v.

- We can first try a substitution, where we introduce the standard PN (tail-related) $\mathrm{eulerlog}_m(v)$ function,

  $\mathrm{eulerlog}_m(v) = \gamma + \log(2mv)$

  and write the remaining logarithms in terms of $\log(2v^2)$.

N.B.: This form is independent of m

Simplifying the 2,2 mode of the energy flux

- Well, unfortunately the 2,2 mode wasn't much simpler than t… about the sa… came about … and v.

$$
\begin{aligned}
\eta_{22} = 1 &- \frac{107}{21}v^2 + 4\pi v^3 + \frac{4784}{1323}v^4 - \frac{428}{21}\pi v^5 + \left[\frac{99210071}{1091475} + \frac{16}{3}\pi^2 - \frac{1712}{105}\,\mathrm{eulerlog}_2(v)\right] v^6 + \frac{19136}{1323}\pi v^7 \\
&+ \left[-\frac{27956920577}{81265275} - \frac{1712}{63}\pi^2 + \frac{183184}{2205}\,\mathrm{eulerlog}_2(v)\right] v^8 + \left[\frac{396840284}{1091475} - \frac{6848}{105}\,\mathrm{eulerlog}_2(v)\right] \pi v^9 + \\
&+ \left[\frac{187037845924}{6257426175} + \frac{76544}{3969}\pi^2 - \frac{8190208}{138915}\,\mathrm{eulerlog}_2(v)\right] v^{10} + \left[-\frac{111827682308}{81265275} + \frac{732736}{2205}\,\mathrm{eulerlog}_2(v)\right] \pi v^{11} \\
&+ \left[\frac{1396382211865462 04}{29253467368125} + \frac{295709968}{654885}\pi^2 - \frac{256}{45}\pi^4 - \frac{27392}{105}\zeta(3) - \left(\frac{36117727568}{22920975} + \frac{27392}{315}\pi^2\right)\mathrm{eulerlog}_2(v)\right. \\
&\left. + \frac{1465472}{11025}\,\mathrm{eulerlog}_2^2(v)\right] v^{12} + \left[\frac{748151383696}{6257426175} - \frac{32760832}{138915}\,\mathrm{eulerlog}_2(v)\right] \pi v^{13} + \left[-\frac{1922289287156615370 8684}{13656396171461 79375}\right. \\
&- \frac{406031680304}{243795825}\pi^2 + \frac{27392}{945}\pi^4 + \frac{2930944}{2205}\zeta(3) + \left(\frac{51936991437808}{8532853875} + \frac{2930944}{6615}\pi^2\right)\mathrm{eulerlog}_2(v) \\
&\left. - \frac{156805504}{231525}\,\mathrm{eulerlog}_2^2(v)\right] v^{14} + \cdots \\
&+ \Big\{\mathcal{R}_{111,100} + \cdots + [\mathcal{R}_{89,79} + \cdots + \mathcal{R}_{26,17}\zeta(3)\,\mathrm{eulerlog}_2^3(v) + \cdots + \mathcal{R}_{14,8}\zeta(3)\log^3(2v^2) + \cdots + \mathcal{R}_{17,8}\zeta(3)\zeta(5)\,\mathrm{eulerlog}_2(v) \\
&+ \cdots]\pi^2 + \cdots + \mathcal{R}_{12,9}\pi^{14} + \cdots + \mathcal{R}_{13,4}\zeta(13) + \cdots + [\mathcal{R}_{92,80} + \cdots + \mathcal{R}_{19,10}\zeta(3)\zeta(5) + \cdots]\,\mathrm{eulerlog}_2(v) + \cdots \\
&+ \mathcal{R}_{24,18}\,\mathrm{eulerlog}_2^7(v) + [\mathcal{R}_{52,43} + \cdots + \mathcal{R}_{10,3}\zeta(7) + \cdots]\log(2v^2) + \cdots + \mathcal{R}_{17,12}\log^5(2v^2) + \cdots + \mathcal{R}_{17,12}\log^5(2v^2)\Big\} v^{44} \\
&+ O\left(v^{45}\right)
\end{aligned}
$$

Still just showing things to 7PN…

- We c… e the stand…

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and write the remaining logarithms in terms of $\log(2v^2)$.

171 terms at $O(v^{44})$; 1533 terms total

N.B.: This form is independent of m

The Damour-Nagar tail factorization (T_{lm})

- Well, the substitution was effective in producing some simplification, but there are still many terms left, and considerable complexity in the transcendentals at high orders [e.g., $\zeta(13)$, $\pi^2\zeta(3)\zeta(5)\mathrm{eulerlog}_2(v)$, and powers of $\mathrm{eulerlog}_2(v)$ up to the seventh…]

- Let us consider the Damour-Nagar tail factorization $\eta_{22}/|T_{22}|^2$, where

$$|T_{\ell m}| = e^{\pi m v^3} \frac{|\Gamma(1+\ell-2\mathrm{i}mv^3)|}{\Gamma(1+\ell)}$$

The Damour-Nagar tail factorization (T_{lm})

The previous expressions only went to 7PN…

$$
\begin{aligned}
\frac{\eta_{22}}{|T_{22}|^2} = {} & 1 - \frac{107}{21}v^2 + \frac{4784}{1323}v^4 + \left[\frac{77380571}{1091475} - \frac{1712}{105}\,\mathrm{eulerlog}_2(v)\right]v^6 + \left[-\frac{19675602077}{81265275} + \frac{183184}{2205}\,\mathrm{eulerlog}_2(v)\right]v^8 \\
& + \left[-\frac{265502242076}{6257426175} - \frac{8190208}{138915}\,\mathrm{eulerlog}_2(v)\right]v^{10} + \left[\frac{96287266208715704}{29253467368125} - \frac{366368}{11025}\pi^2 - \frac{27392}{105}\zeta(3)\right. \\
& \left. - \frac{28643306768}{22920975}\,\mathrm{eulerlog}_2(v) + \frac{1465472}{11025}\,\mathrm{eulerlog}_2^2(v)\right]v^{12} + \left[-\frac{1216470740485020564 4184}{1365639617146179375} + \frac{39201376}{231525}\pi^2\right. \\
& \left. + \frac{2930944}{2205}\zeta(3) + \frac{37759374165808}{8532853875}\,\mathrm{eulerlog}_2(v) - \frac{156805504}{231525}\,\mathrm{eulerlog}_2^2(v)\right]v^{14} + \left[-\frac{4754978280237051 8469284}{955947732002325 5625}\right. \\
& \left. - \frac{1752704512}{14586075}\pi^2 - \frac{131043328}{138915}\zeta(3) + \frac{46374364943936}{131405949675}\,\mathrm{eulerlog}_2(v) + \frac{7010818048}{14586075}\,\mathrm{eulerlog}_2^2(v) - \frac{512}{15}\log(2v^2)\right]v^{16} \\
& + \left[\frac{3737433184837217266106 48630768}{30836483965065016832 34375} - \frac{32946911447392}{12033511875}\pi^2 - \frac{5861888}{165375}\pi^4 - \frac{2581241112832}{114604875}\zeta(3) + \frac{438272}{105}\zeta(5)\right. \\
& + \left(-\frac{189488492579157761216}{3071614073653125} + \frac{627222016}{1157625}\pi^2 + \frac{46895104}{11025}\zeta(3)\right)\mathrm{eulerlog}_2(v) + \frac{131787645789568}{12033511875}\,\mathrm{eulerlog}_2^2(v) \\
& \left. - \frac{2508888064}{3472875}\,\mathrm{eulerlog}_2^3(v) + \frac{11264}{315}\log(2v^2)\right]v^{18} + \cdots - \frac{16384}{75}\pi v^{21} + \cdots \\
& + \Big\{\mathcal{R}_{111,100} + \cdots + \left[\mathcal{R}_{89,79} + \cdots + \mathcal{R}_{22,18}\zeta(3)\,\mathrm{eulerlog}_2^3(v) + \cdots + \mathcal{R}_{14,8}\zeta(3)\log^3(2v^2) + \cdots + \mathcal{R}_{18,20}\zeta(3)\zeta(5) + \cdots\right]\pi^2 \\
& + \cdots + \mathcal{R}_{17,14}\pi^{12} + \cdots + \mathcal{R}_{13,4}\zeta(13) + \cdots + \left[\mathcal{R}_{92,80} + \cdots + \mathcal{R}_{24,14}\zeta(3)\zeta(5) + \cdots\right]\mathrm{eulerlog}_2(v) + \cdots + \mathcal{R}_{24,18}\,\mathrm{eulerlog}_2^7(v) \\
& + \left[\mathcal{R}_{51,43} + \cdots + \mathcal{R}_{10,3}\zeta(7) + \cdots\right]\log(2v^2) + \cdots + \mathcal{R}_{17,12}\log^5(2v^2)\Big\}v^{44} + O\left(v^{45}\right)
\end{aligned}
$$

First odd power of v that is not removed

Now 9PN

127 terms at $O(v^{44})$; 881 terms total

The Damour-Nagar tail factorization (T_{lm})

The previous expressions only went to 7PN…

$$
\begin{aligned}
\frac{\eta_{22}}{|T_{22}|^2} = {} & 1 - \frac{107}{21}v^2 + \frac{4784}{1323}v^4 + \left[\frac{77380571}{1091475} - \frac{1712}{105}\,\mathrm{eulerlog}_2(v)\right]v^6 + \left[-\frac{19675602077}{81265275} + \frac{183184}{2205}\,\mathrm{eulerlog}_2(v)\right]v^8 \\
& + \left[-\frac{265502242076}{6257426175} - \frac{8190208}{138915}\,\mathrm{eulerlog}_2(v)\right]v^{10} + \left[\frac{96287266208715704}{29253467368125} - \frac{366368}{11025}\pi^2 - \frac{27392}{105}\zeta(3)\right. \\
& \left. - \frac{28643306768}{22920975}\,\mathrm{eulerlog}_2(v) + \frac{1465472}{11025}\,\mathrm{eulerlog}_2^2(v)\right]v^{12} + \left[-\frac{1216470740485020564418 4}{136563961714617937 5} + \frac{39201376}{231525}\pi^2\right. \\
& \left. + \frac{2930944}{2205}\zeta(3) + \frac{37759374165808}{8532853875}\,\mathrm{eulerlog}_2(v) - \frac{156805504}{231525}\,\mathrm{eulerlog}_2^2(v)\right]v^{14} + \left[-\frac{475497828023705184692 84}{955947732002325562 5}\right. \\
& \left. - \frac{1752704512}{14586075}\pi^2 - \frac{131043328}{138915}\zeta(3) + \frac{46374364943936}{131405949675}\,\mathrm{eulerlog}_2(v) + \frac{7010818048}{14586075}\,\mathrm{eulerlog}_2^2(v) - \frac{512}{15}\log(2v^2)\right]v^{16} \\
& + \left[\frac{37374331848372172661064863076 8}{3083648396506501683234375} - \frac{32946911447392}{12033511875}\pi^2 - \frac{5861888}{165375}\pi^4 - \frac{2581241112832}{114604875}\zeta(3) + \frac{438272}{105}\zeta(5)\right. \\
& + \left(-\frac{1894884925791577612 16}{3071614073653125} + \frac{627222016}{1157625}\pi^2 + \frac{46895104}{11025}\zeta(3)\right)\mathrm{eulerlog}_2(v) + \frac{131787645789568}{12033511875}\,\mathrm{eulerlog}_2^2(v) \\
& \left. - \frac{2508888064}{3472875}\,\mathrm{eulerlog}_2^3(v) + \frac{11264}{315}\log(2v^2)\right]v^{18} + \cdots - \frac{16384}{75}\pi v^{21} + \cdots \\
& + \Big\{\mathcal{R}_{111,100} + \cdots + \left[\mathcal{R}_{89,79} + \cdots + \mathcal{R}_{22,18}\zeta(3)\,\mathrm{eulerlog}_2^3(v) + \cdots + \mathcal{R}_{14,8}\zeta(3)\log^3(2v^2) + \cdots + \mathcal{R}_{18,20}\zeta(3)\zeta(5) + \cdots\right]\pi^2 \\
& + \cdots + \mathcal{R}_{17,14}\pi^{12} + \cdots + \mathcal{R}_{13,4}\zeta(13) + \cdots + \left[\mathcal{R}_{92,80} + \cdots + \mathcal{R}_{24,14}\zeta(3)\zeta(5) + \cdots\right]\mathrm{eulerlog}_2(v) + \cdots + \mathcal{R}_{24,18}\,\mathrm{eulerlog}_2^7(v) \\
& + \left[\mathcal{R}_{51,43} + \cdots + \mathcal{R}_{10,3}\zeta(7) + \cdots\right]\log(2v^2) + \cdots + \mathcal{R}_{17,12}\log^5(2v^2)\Big\}v^{44} + O\left(v^{45}\right)
\end{aligned}
$$

First odd power of v that is not removed

Now 9PN

Slight simplifications of complexity…

127 terms at O(v^{44}); 881 terms total

The S_{lm} factorization

- The Damour-Nagar factorization successfully removed some terms (notably all the odd powers of v up to v^{21} and the largest power of π), but there is still plenty of complexity remaining, including lots of transcendentals and log v terms.

- However, it is possible to modify the Damour-Nagar factorization slightly and remove many more terms. Specifically, we use

$$S_{\ell m} := (2mv)^{\bar{\nu}_{\ell m}(v)} e^{\pi m v^3} \frac{\Gamma[1 + \bar{\nu}_{\ell m}(v) - 2\mathrm{i}mv^3]}{\Gamma[1 + 2\bar{\nu}_{\ell m}(v)]}$$

Fractional part of the renormalized angular momentum ν which is fundamental to the MST approach (series in v^6)

The S_{lm} factorization

The expression for the T_{lm} factorization only went to 9PN…

$$
\begin{aligned}
\frac{\eta_{22}}{|S_{22}|^2} = {} & 1 - \frac{107}{21}v^2 + \frac{4784}{1323}v^4 + \frac{99210071}{1091475}v^6 - \frac{27956920577}{81265275}v^8 + \frac{187037845924}{6257426175}v^{10} + \frac{1396382211865462004}{29253467368125}v^{12} \\
& - \frac{19222892871566153708684}{1365639617146179375}v^{14} + \left[-\frac{534496372687122603752 84}{9559477320023255625} - \frac{512}{15}\log(2v^2)\right]v^{16} \\
& + \left[\frac{59073025042448165511876 5186768}{30836483965065016832 34375} - \frac{114688}{1605}\,\mathrm{eulerlog}_2(v) + \frac{11264}{315}\log(2v^2)\right]v^{18} \\
& + \left[-\frac{32353692860249036036453613491 74816}{6854950385433953241830015625} + \frac{16384}{45}\,\mathrm{eulerlog}_2(v) - \frac{42752}{405}\log(2v^2)\right]v^{20} - \frac{16384}{75}\pi v^{21} \\
& + \left[-\frac{837797695839226310646716717859 1936}{22392837925750913923311384375} - \frac{876544}{4725}\pi^2 - \frac{16384}{15}\zeta(3) - \frac{78381056}{303345}\,\mathrm{eulerlog}_2(v) - \frac{146048512}{40425}\log(2v^2)\right. \\
& \left. + \frac{438272}{1575}\log^2(2v^2)\right]v^{22} + \frac{1753088}{1575}\pi v^{23} + \left[\frac{43157118871251878301723731731384960807 0192}{64125832996859366542516201416796875} - \frac{8667136}{19845}\pi^2\right. \\
& + \frac{32768}{33705}\zeta(3) - \frac{362080557826048}{26777779875}\,\mathrm{eulerlog}_2(v) + \frac{131072}{225}\,\mathrm{eulerlog}_2^2(v) + \frac{115721109504}{99324225}\,\mathrm{eulerlog}_2(v)\log(2v^2) \\
& \left. + \frac{3218927074816}{1489863375}\log(2v^2) - \frac{9641984}{33075}\log^2(2v^2)\right]v^{24} - \frac{78381056}{99225}\pi v^{25} + \cdots \\
& + \Big\{\mathcal{R}_{111,100} + \cdots + \left[\mathcal{R}_{89,79} + \cdots + \mathcal{R}_{15,7}\zeta(3)\,\mathrm{eulerlog}_2(v) + \cdots + \mathcal{R}_{16,8}\zeta(3)\log(2v^2) + \cdots\right]\pi^2 + \cdots + \mathcal{R}_{11,7}\pi^8 \\
& + \cdots + \mathcal{R}_{12,4}\zeta(3)\zeta(5) + \cdots + \mathcal{R}_{10,3}\zeta(9) + \cdots + \left[\mathcal{R}_{61,50} + \cdots + \mathcal{R}_{24,14}\zeta(3)\log^2(2v^2) + \cdots\right]\mathrm{eulerlog}_2(v) + \cdots \\
& + \mathcal{R}_{17,12}\,\mathrm{eulerlog}_2^5(v) + \left[\mathcal{R}_{52,43} + \cdots + \mathcal{R}_{12,5}\zeta(7) + \cdots\right]\log(2v^2) + \cdots + \mathcal{R}_{17,12}\log^5(2v^2)\Big\}v^{44} + O\left(v^{45}\right)
\end{aligned}
$$

Now 12.5PN

70 terms at $O(v^{44})$; 501 terms total

- Th… d
  sc
  ar
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- H
  fa
  S

The S_{lm} factorization

The expression for the T_{lm} factorization only went to 9PN…

$$
\begin{aligned}
\frac{\eta_{22}}{|S_{22}|^2} = 1 &- \frac{107}{21}v^2 + \frac{4784}{1323}v^4 + \frac{99210071}{1091475}v^6 - \frac{27956920577}{81265275}v^8 + \frac{187037845924}{6257426175}v^{10} + \frac{1396382211865462 04}{29253467368125}v^{12} \\
&- \frac{1922289287156615370868 4}{13656396171461793 75}v^{14} + \left[-\frac{534496372687122603752 84}{955947732002325562 5} - \frac{512}{15}\log(2v^2)\right]v^{16} \\
&+ \left[\frac{5907302504244816551187651867 68}{308364839650650168323437 5} - \frac{114688}{1605}\,\mathrm{eulerlog}_2(v) + \frac{11264}{315}\log(2v^2)\right]v^{18} \\
&+ \left[-\frac{3235369286024903603645361349174816}{685495038543395324183001562 5} + \frac{16384}{45}\,\mathrm{eulerlog}_2(v) - \frac{42752}{405}\log(2v^2)\right]v^{20} - \frac{16384}{75}\pi v^{21} \\
&+ \left[-\frac{8377976958392263106467167178591936}{223928379257509139233113843 75} - \frac{876544}{4725}\pi^2 - \frac{16384}{15}\zeta(3) - \frac{78381056}{303345}\,\mathrm{eulerlog}_2(v) - \frac{146048512}{40425}\log(2v^2)\right. \\
&\left. + \frac{438272}{1575}\log^2(2v^2)\right]v^{22} + \frac{1753088}{1575}\pi v^{23} + \left[\frac{431571188712518783017237317313849608070192}{6412583299685936654251620141679687 5} - \frac{8667136}{19845}\pi^2\right. \\
&+ \frac{32768}{33705}\zeta(3) - \frac{362080557826048}{26777779875}\,\mathrm{eulerlog}_2(v) + \frac{131072}{225}\,\mathrm{eulerlog}_2^2(v) + \frac{115721109504}{99324225}\,\mathrm{eulerlog}_2(v)\log(2v^2) \\
&\left. + \frac{3218927074816}{1489863375}\log(2v^2) - \frac{9641984}{33075}\log^2(2v^2)\right]v^{24} - \frac{78381056}{99225}\pi v^{25} + \cdots \\
&+ \Big\{\mathcal{R}_{111,100} + \cdots + \left[\mathcal{R}_{89,79} + \cdots + \mathcal{R}_{15,7}\zeta(3)\,\mathrm{eulerlog}_2(v) + \cdots + \mathcal{R}_{16,8}\zeta(3)\log(2v^2) + \cdots\right]\pi^2 + \cdots + \mathcal{R}_{11,7}\pi^8 \\
&+ \cdots + \mathcal{R}_{12,4}\zeta(3)\zeta(5) + \cdots + \mathcal{R}_{10,3}\zeta(9) + \cdots + \left[\mathcal{R}_{61,50} + \cdots + \mathcal{R}_{24,14}\zeta(3)\log^2(2v^2) + \cdots\right]\mathrm{eulerlog}_2(v) + \cdots \\
&+ \mathcal{R}_{17,12}\,\mathrm{eulerlog}_2^5(v) + \left[\mathcal{R}_{52,43} + \cdots + \mathcal{R}_{12,5}\zeta(7) + \cdots\right]\log(2v^2) + \cdots + \mathcal{R}_{17,12}\log^5(2v^2)\Big\}v^{44} + O(v^{45})
\end{aligned}
$$

Now 12.5PN

Substantial decreases in complexity…

70 terms at O(v^{44}); 501 terms total

The V_{lm} and V'_{lm} factorizations

- While the S_{lm} factorization produces the maximum simplification one can likely hope (a simple integer PN series with rational coefficients), it only does so up to a certain order, due to the structure of the expansion. [However, as we shall see, the order to which the Slm factorization produces this complete simplification increases as ℓ increases.]

- Can we simplify further?

- Yes, though here the simplification is not nearly as dramatic, and the factorization is rather more involved.

The V_{lm} and V'_{lm} factorizations (cont.)

$$V_{\ell m} := \frac{V_{\ell m}^{\mathrm{num}}}{V_{\ell m}^{\mathrm{denom}}} \qquad V_{\ell m}^{\mathrm{num}} := 1 + q_{\ell m}(2v^2)^{1+2\ell+2\bar{\nu}_{\ell m}(v)} \frac{\Gamma[1-2\bar{\nu}_{\ell m}(v)]}{\Gamma[1+2\bar{\nu}_{\ell m}(v)]} \left\{ \frac{\Gamma[1+\bar{\nu}_{\ell m}(v) - 2\mathrm{i}mv^3]}{\Gamma[1-\bar{\nu}_{\ell m}(v) - 2\mathrm{i}mv^3]} \right\}^2$$

Fixed by requiring that the factorization remove certain terms

$$V_{\ell m}^{\mathrm{denom}} := 1 + \mathrm{i}s_\ell(4mv^3)^{1+2\ell+2\bar{\nu}_{\ell m}(v)} e^{-\mathrm{i}\pi\bar{\nu}_{\ell m}(v)} \frac{\bar{\nu}_{\ell m}(v) + 2\mathrm{i}mv^3}{\bar{\nu}_{\ell m}(v) - 2\mathrm{i}mv^3} \left\{ \frac{\Gamma[1-2\bar{\nu}_{\ell m}(v)]}{\Gamma[1+2\bar{\nu}_{\ell m}(v)]} \right\}^2 \left\{ \frac{\Gamma[1+\bar{\nu}_{\ell m}(v) - 2\mathrm{i}mv^3]}{\Gamma[1-\bar{\nu}_{\ell m}(v) - 2\mathrm{i}mv^3]} \right\}^3$$

$$s_\ell \to s_\ell \left[1 + \sum_{k=1}^{\infty} [\bar{s}_\ell]_k (2mv^3)^{2k} \right]$$

ℓ	$-[\nu_\ell]_1$	$\bar{q}_\ell$	$-(\bar{q}_\ell [\nu_\ell]_1)^{-1}$	s_ℓ	$-(s_\ell [\nu_\ell]_1)^{-1}$
2	$\frac{107}{210} = \frac{107^1}{2^1 3^1 5^1 7^1}$	$\frac{7}{214}$	$2^2 3^1 5^1$	$\frac{7}{17120}$	$2^6 3^1 5^2$
3	$\frac{13}{42} = \frac{13^1}{2^1 3^1 7^1}$	$\frac{1}{520}$	$2^4 3^1 5^1 7^1$	$\frac{1}{10483200}$	$2^{10} 3^3 5^2 7^2$
4	$\frac{1571}{6930} = \frac{1571^1}{2^1 3^2 5^1 7^1 11^1}$	$\frac{11}{87976}$	$2^4 3^2 5^1 7^2$	$\frac{11}{595928309760}$	$2^{14} 3^5 5^2 7^4$
5	$\frac{773}{4290} = \frac{773^1}{2^1 3^1 5^1 11^1 13^1}$	$\frac{13}{1558368}$	$2^6 3^3 5^1 7^1 11^1$	–	–
6	$\frac{901}{6006} = \frac{17^1 53^1}{2^1 3^1 7^1 11^1 13^1}$	$\frac{1}{1783980}$	$2^3 3^3 5^1 7^1 11^2 13^1$	–	–

This substitution can be used to remove all the odd powers of v, and gives V'_{lm}
One can also remove terms with a similar series for q_{lm} (in v^2), but this is not quite as effective, so we do not consider it further.

The q_{lm} and s_l coefficients are simply related to the lowest-order PN correction to ν.

The V_{lm} and V'_{lm} factorizations (cont.)

$$
\begin{aligned}
\frac{\eta_{22}}{|S_{22}V'_{22}|^2} = {} & 1 - \frac{107}{21}v^2 + \frac{4784}{1323}v^4 + \frac{99210071}{1091475}v^6 - \frac{27956920577}{81265275}v^8 + \frac{18611386050668}{669544600725}v^{10} + \frac{139950258171806204}{29253467368125}v^{12} \\
& - \frac{205795569089625397226918 8}{146123439034641193125}v^{14} - \frac{591374761909136017627298 8}{102286407324248835187 5}v^{16} \\
& + \left[\frac{634457645667045016842203214441 76}{329950378426195680106078125} - \frac{114688}{1605}\,\mathrm{eulerlog}_2(v) - \frac{8704}{63}\log(2v^2)\right] v^{18} \\
& + \left[-\frac{370463959894752174010310496698958283 84}{784823269628333306657118488906 25} + \frac{16384}{45}\,\mathrm{eulerlog}_2(v) + \frac{70912}{3969}\log(2v^2)\right] v^{20} \\
& + \left[-\frac{919480615890647231616774790102430912}{23960336580553477897943181281 25} - \frac{876544}{4725}\pi^2 - \frac{78381056}{303345}\,\mathrm{eulerlog}_2(v) - \frac{110035171328}{350363475}\log(2v^2)\right] v^{22} \\
& + \left[\frac{4970118699958639953829221250019069029006428208}{734176661981042887545267990020907421875} + \frac{43368448}{99225}\pi^2 - \frac{187547648}{33705}\zeta(3)\right. \\
& - \frac{81953900429312}{5355555975}\,\mathrm{eulerlog}_2(v) + \frac{131072}{225}\,\mathrm{eulerlog}_2^2(v) + \frac{262144}{225}\,\mathrm{eulerlog}_2(v)\log(2v^2) - \frac{165767673909248}{13408770375}\log(2v^2) \\
& \left. + \frac{7450624}{6615}\log^2(2v^2)\right] v^{24} + \cdots \\
& + \Big\{\mathcal{R}_{111,100} + \cdots + \left[\mathcal{R}_{89,79} + \cdots + \mathcal{R}_{15,7}\zeta(3)\,\mathrm{eulerlog}_2(v) + \cdots + \mathcal{R}_{16,8}\zeta(3)\log(2v^2) + \cdots\right]\pi^2 + \cdots + \mathcal{R}_{11,7}\pi^8 \\
& + \cdots + \mathcal{R}_{12,4}\zeta(3)\zeta(5) + \cdots + \mathcal{R}_{10,3}\zeta(9) + \cdots + \left[\mathcal{R}_{61,50} + \cdots + \mathcal{R}_{24,14}\zeta(3)\log^2(2v^2) + \cdots\right]\mathrm{eulerlog}_2(v) + \cdots \\
& + \mathcal{R}_{17,12}\,\mathrm{eulerlog}_2^5(v) + \left[\mathcal{R}_{52,43} + \cdots + \mathcal{R}_{12,5}\zeta(7) + \cdots\right]\log(2v^2) + \cdots + \mathcal{R}_{17,12}\log^5(2v^2)\Big\} v^{44} + O\left(v^{45}\right)
\end{aligned}
$$

Purely rational coefficients to one order higher than with S_{lm}

Still to 12.5PN (12.5PN term zero)

Integer PN series to all orders known!

70 terms at $O(v^{44})$ [same as S_{lm} factorization] but only 360 terms total

Same complexity of v^{44} term as with S_{lm} factorization, though other terms are simpler…

Taking the logarithm

- The final method of simplification we consider consists of consistently expanding the logarithm of a given PN order.

- This is part of the exponential resummation introduced by Isoyama *et al.* [Phys Rev. D **87**, 024010 (2013)] as a way to improve the convergence of the full energy flux [and ensure its positivity near the horizon in the Kerr case].

- This does not produce the drastic simplification of the lowest orders and complete removal of the odd powers of v provided by the $S_{lm}\, V'_{lm}$ factorization, but still manages to remove terms that this factorization does not.

- Of course, it is possible to take expand the logarithm of the $S_{lm}\, V'_{lm}$ factorization and obtain the most significant simplification we have found. (This gives the maximum simplification of both cases, with a total of only 263 terms for the 2,2 mode.)

Taking the logarithm

Only odd powers of v of the form $v^{9+6(\ell+n)}$ at higher orders

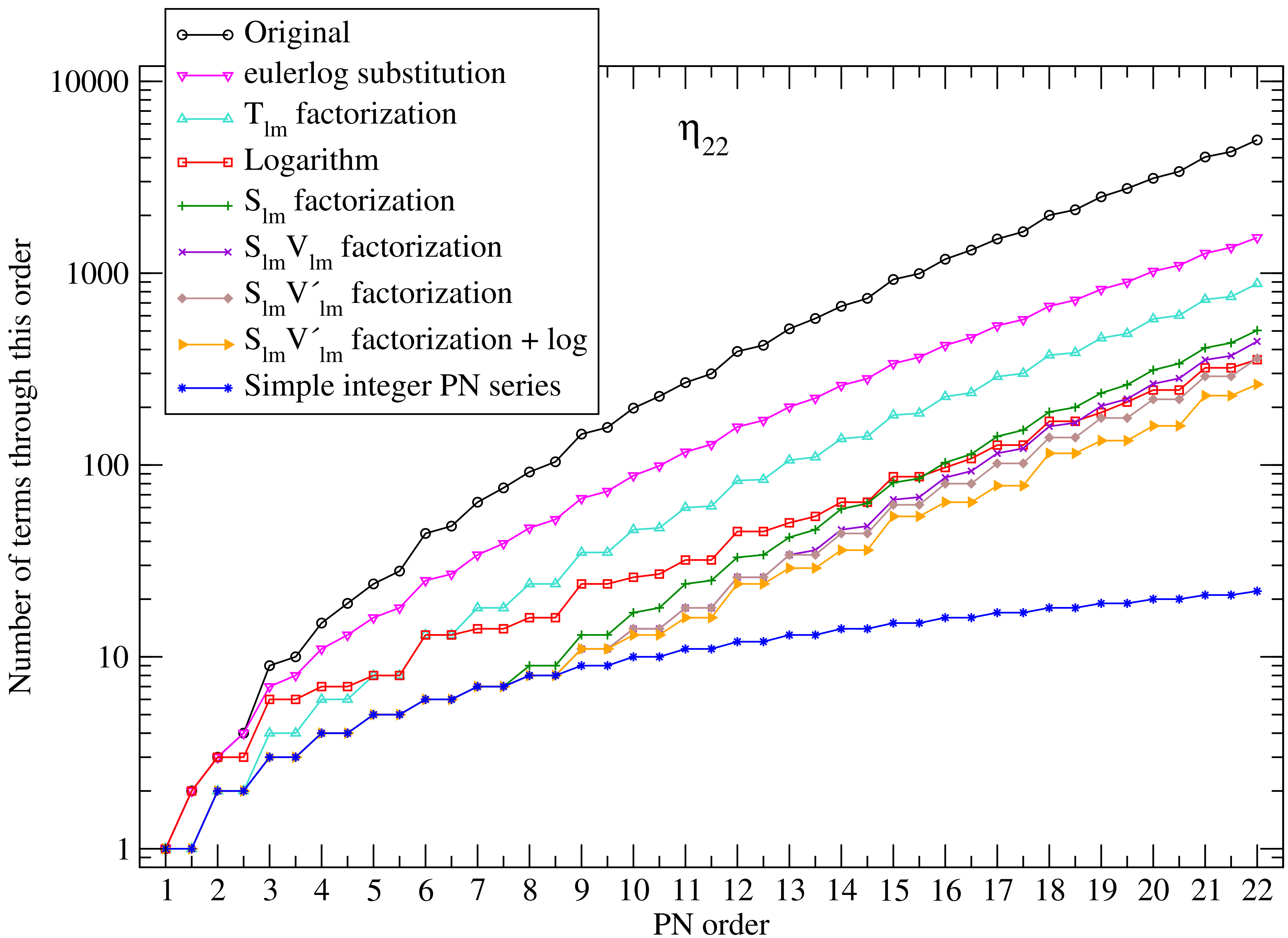
$$
\begin{aligned}
\log\eta_{22} = {} & -\frac{107}{21}v^2 + 4\pi v^3 - \frac{24779}{2646}v^4 + \left[\frac{166117624}{2546775} - \frac{8}{3}\pi^2 - \frac{1712}{105}\,\mathrm{eulerlog}_2(v)\right]v^6 + \frac{949963732033}{25029704700}v^8 + \frac{394091579209}{11945995425}v^{10} \\
& + \left[\frac{9393857895928455143 8}{116107011984088125} - \frac{366368}{11025}\pi^2 + \frac{64}{45}\pi^4 - \frac{27392}{105}\zeta(3) - \frac{108494912}{1157625}\,\mathrm{eulerlog}_2(v)\right]v^{12} \\
& + \frac{4729777868637617775596 9}{826613627084363868 75}v^{14} + \left[\frac{18189001464420142574516153}{20235501591025227507000} - \frac{512}{15}\log(2v^2)\right]v^{16} \\
& + \left[\frac{35542298255084504226966071228 48}{244082630769630017849859375} - \frac{46435822336}{121550625}\pi^2 - \frac{5861888}{165375}\pi^4 - \frac{4096}{2835}\pi^6 - \frac{111910912}{27783}\zeta(3) + \frac{438272}{105}\zeta(5)\right. \\
& \left. - \frac{193586530718464}{140390971875}\,\mathrm{eulerlog}_2(v) - \frac{8704}{63}\log(2v^2)\right]v^{18} + \left[\frac{110323183469901812677938758945966 13233}{8887717372730537736162288458437 50}\right. \\
& \left. - \frac{2723072}{3969}\log(2v^2)\right]v^{20} - \frac{16384}{75}\pi v^{21} + \left[\frac{21132071320084378552730312721148616 4929}{933210324136706462297040288135937 5} - \frac{876544}{4725}\pi^2 - \frac{16384}{15}\zeta(3)\right. \\
& \left. - \frac{401855693056}{114604875}\log(2v^2) + \frac{438272}{1575}\log^2(2v^2)\right]v^{22} + \left[\frac{173673212010237316601253379042889423718174849728 3}{549982395203443305086879617517337221367187 5}\right. \\
& - \frac{2558186604995072}{327578934375}\pi^2 - \frac{471366904832}{607753125}\pi^4 + \frac{187580416}{2083725}\pi^6 + \frac{8192}{4725}\pi^8 - \frac{9839419963916288}{140390971875}\zeta(3) \\
& - \frac{12367511552}{1157625}\zeta(5) - \frac{7012352}{105}\zeta(7) - \frac{413683856895739062732 8}{132802137890296875}\,\mathrm{eulerlog}_2(v) + \frac{131072}{225}\,\mathrm{eulerlog}_2^2(v) \\
& \left. + \frac{262144}{225}\,\mathrm{eulerlog}_2(v)\log(2v^2) - \frac{1164753374080768}{93861392625}\log(2v^2) + \frac{7450624}{6615}\log^2(2v^2)\right]v^{24} + \cdots \\
& + \Big\{\mathcal{R}_{118,109} + \cdots + \left[\mathcal{R}_{25,17} + \cdots + \mathcal{R}_{18,11}\zeta(3)\log(2v^2) + \cdots + \mathcal{R}_{20,13}\log^3(2v^2) + \cdots\right]\pi^2 + \cdots + \mathcal{R}_{13,9}\pi^8 + \cdots \\
& + \mathcal{R}_{15,6}\zeta(3)\zeta(5) + \cdots + \mathcal{R}_{12,4}\zeta(9) + \left[\mathcal{R}_{60,51} + \cdots + \mathcal{R}_{15,6}\zeta(7) + \cdots\right]\log(2v^2) + \cdots + \mathcal{R}_{19,13}\log^5(2v^2)\Big\}v^{44} + O\left(v^{45}\right)
\end{aligned}
$$

- Th ex
- Th /. [Phys Rev [and flux
- Th d co fa oes no
- Of d fa (Th y 263 terms for the 2,2 mode.)

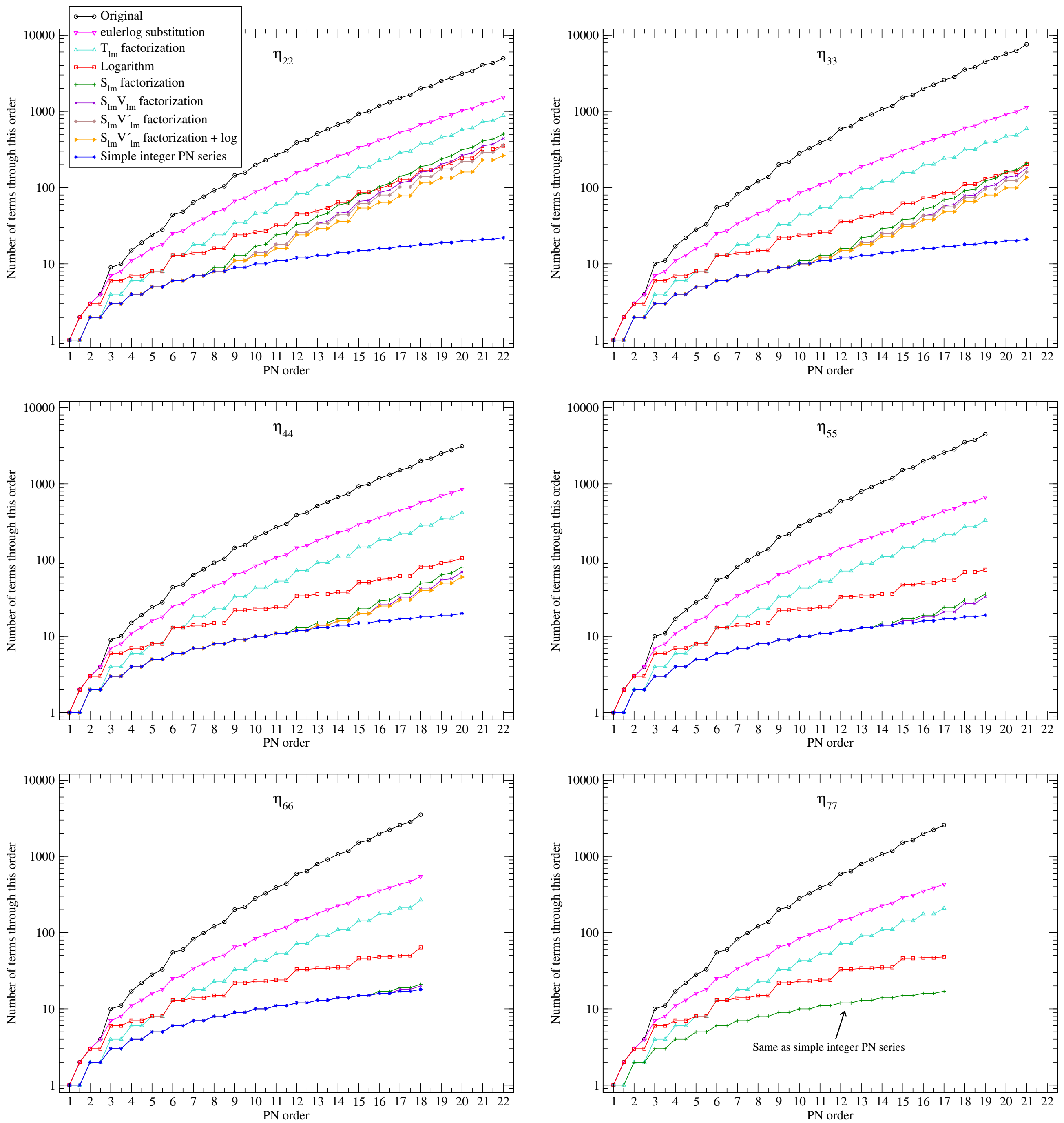
only 33 terms at $O(v^{44})$, and 354 terms total

All significant complexity [and all appearances of $\mathrm{eulerlog}_m(v)$] confined to powers of v divisible by 6

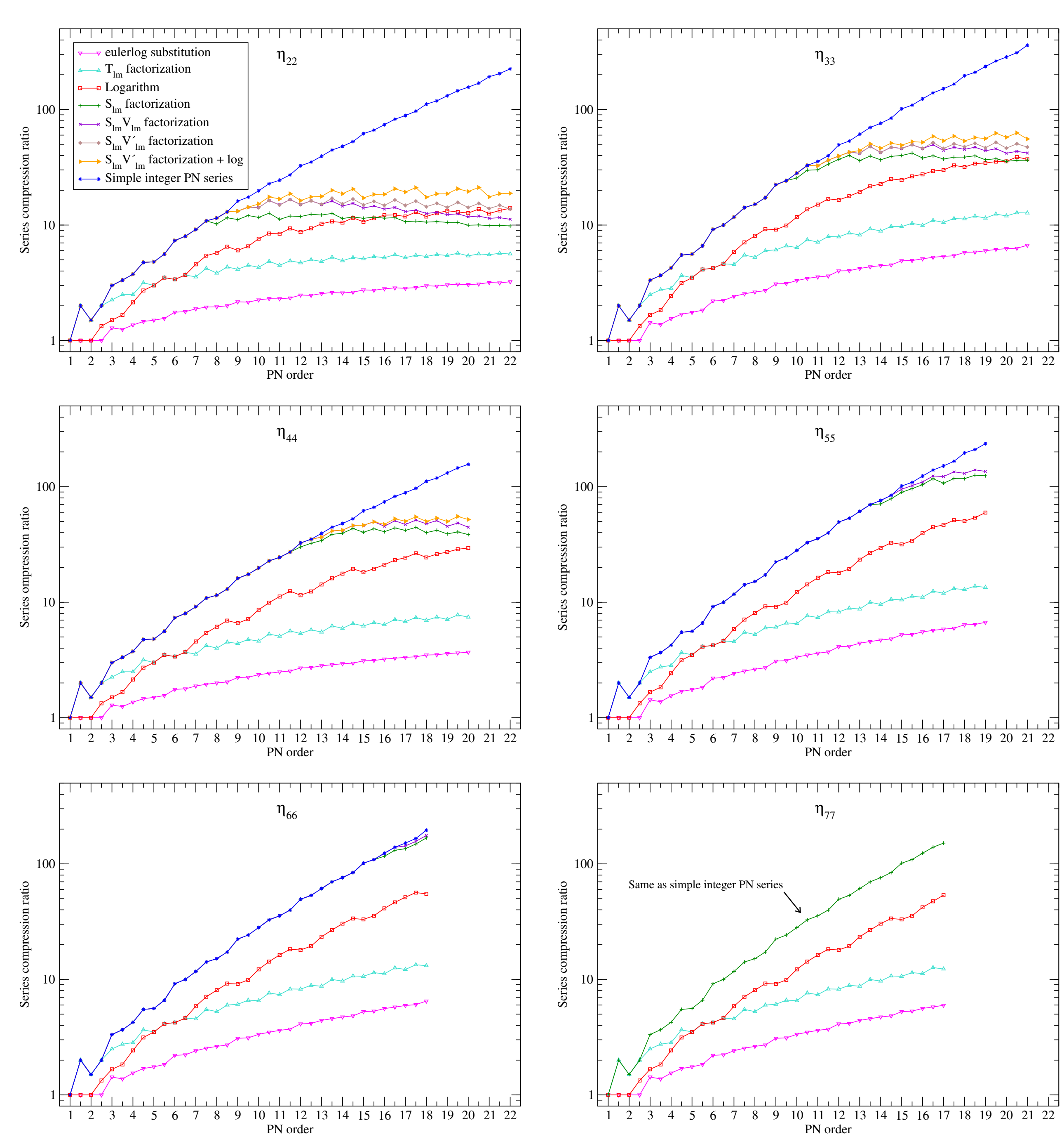
Comparing the simplifications of the (2,2) mode



Comparing the simplifications of the modes with $\ell \leq 7$



Number of terms through a given PN order



Ratio of number of terms through a given PN order in a simplification to the number of terms in the original

Why do these simplifications simplify?

- One can understand where these simplifications come from [and why they do not simplify completely] via a study of the structure of the MST formalism.
  [Indeed, the V_{lm} factorization was obtained in this manner, by analogy with the S_{lm} factorization, which was itself obtained by a study of the prime factorization of the coefficients of the PN expansion of the modes and analogy with the Damour-Nagar T_{lm} factorization.]

- All we will note here is that the MST formalism gives the flux as a product of pieces that are themselves sums of terms depending on ν and $-\nu - 1$, where the $-\nu - 1$ pieces only enter at higher orders, and generate the complexity that is not removed by the S_{lm} factorization (and is only incompletely removed by the S_{lm} V'_{lm} factorization).

- Also, the transcendentality structure of S_{lm} can be understood by noting that

$$\Gamma(1+z) = \exp\left[-\gamma z + \sum_{n=2}^{\infty} \frac{\zeta(n)}{n}(-z)^n\right]$$

so

$$S_{\ell m} = \exp\left[\bar{\nu}_{\ell m}(v)\,\mathrm{eulerlog}_m(v) + \pi m v^3 + \sigma_{S_{\ell m}}(v)\right]$$

$$\sigma_{S_{\ell m}}(v) := \sum_{n=2}^{\infty} \frac{\zeta(n)}{n}\left\{\left[-\bar{\nu}_{\ell m}(v) + 2\mathrm{i} m v^3\right]^n - \left[-2\bar{\nu}_{\ell m}(v)\right]^n\right\}$$

There is a similar expression for V_{lm}, involving $\log(2v^2)$ in addition to $\mathrm{eulerlog}_m(v)$

Why do these simplifications simplify?

- One can ... they do not simplify comp...
  [Indeed, the V_{lm} f... d by a study of the prime factori... torization.]

- All we wil... roduct of pieces ... v - 1, where the ... he complexi... incomple...

This expression also helps explain why expanding the logarithm produces such a significant simplification.

Additionally, we can compare the expressions for T_{lm} and S_{lm} to see why the latter removes so many more transcendentals

$$|T_{\ell m}| = e^{\pi m v^3} \frac{|\Gamma(1+\ell-2imv^3)|}{\Gamma(1+\ell)} \qquad S_{\ell m} := (2mv)^{\bar{\nu}_{\ell m}(v)} e^{\pi m v^3} \frac{\Gamma[1+\bar{\nu}_{\ell m}(v)-2imv^3]}{\Gamma[1+2\bar{\nu}_{\ell m}(v)]}$$

(not an integer)

$$|T_{\ell m}|^2 = \frac{4\pi m v^3}{1-e^{-4\pi m v^3}} \prod_{k=1}^{\ell} \left[1 + \left(\frac{2mv^3}{k}\right)^2\right]$$

- Also, the transcendentality structure of S_{lm} can be understood by noting that

$$\Gamma(1+z) = \exp\left[-\gamma z + \sum_{n=2}^{\infty} \frac{\zeta(n)}{n}(-z)^n\right]$$

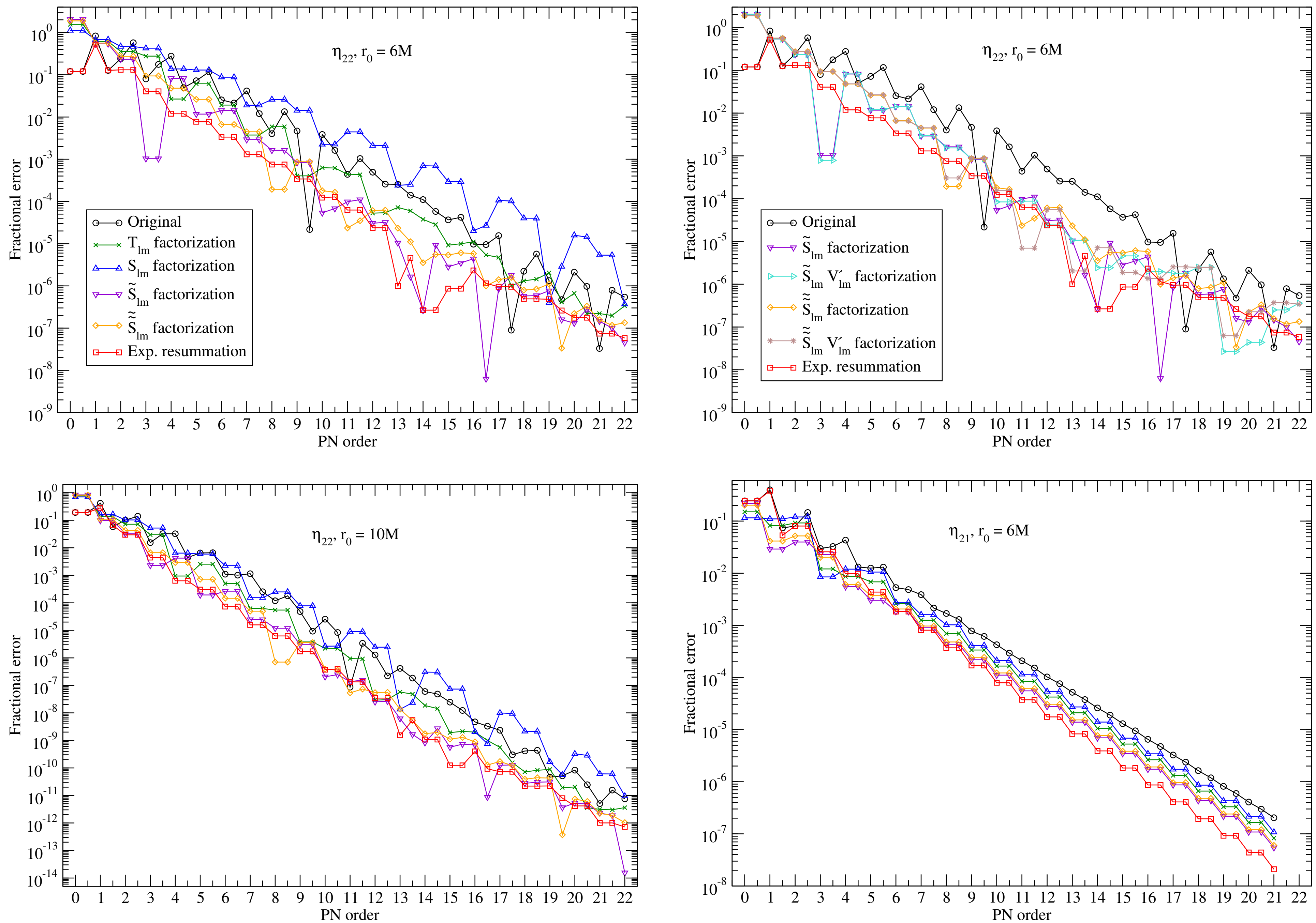
so

$$S_{\ell m} = \exp\left[\bar{\nu}_{\ell m}(v)\,\mathrm{eulerlog}_m(v) + \pi m v^3 + \sigma_{S_{\ell m}}(v)\right]$$

$$\sigma_{S_{\ell m}}(v) := \sum_{n=2}^{\infty} \frac{\zeta(n)}{n}\left\{\left[-\bar{\nu}_{\ell m}(v) + 2imv^3\right]^n - \left[-2\bar{\nu}_{\ell m}(v)\right]^n\right\}$$

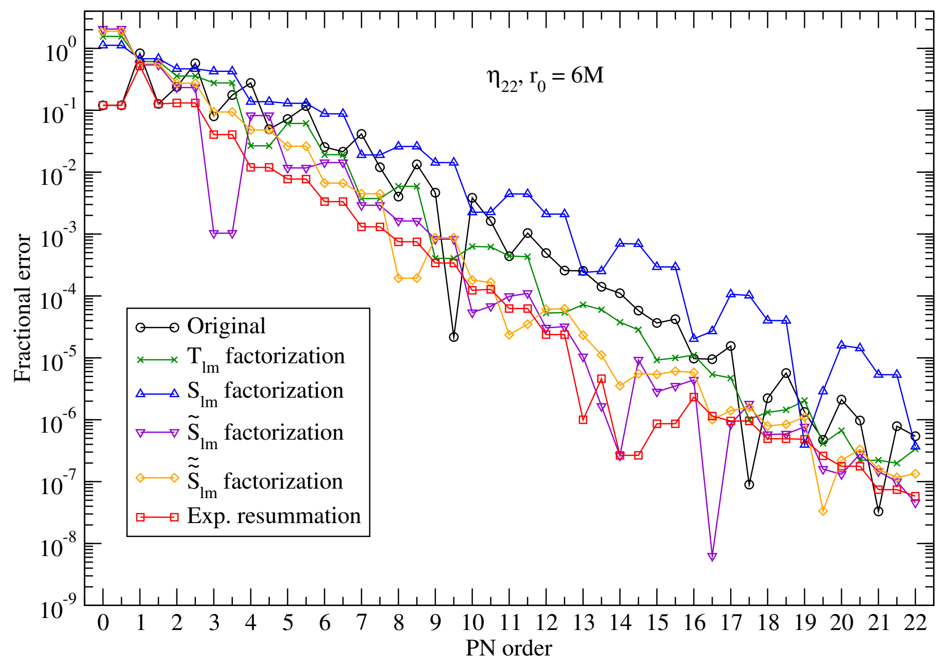
There is a similar expression for V_{lm}, involving $\log(2v^2)$ in addition to $\mathrm{eulerlog}_m(v)$

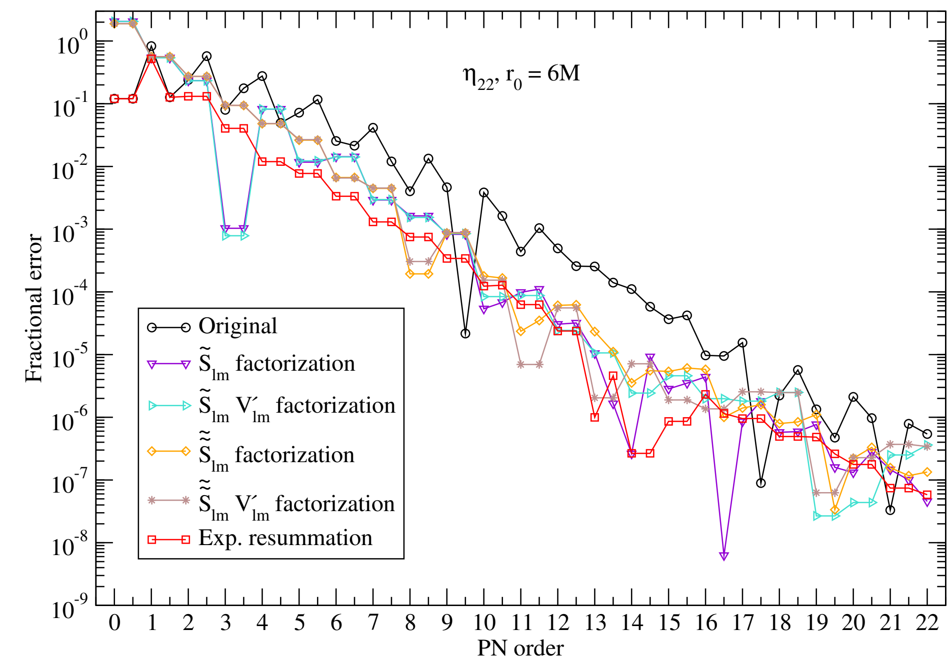
How do these factorizations and resummations improve convergence?

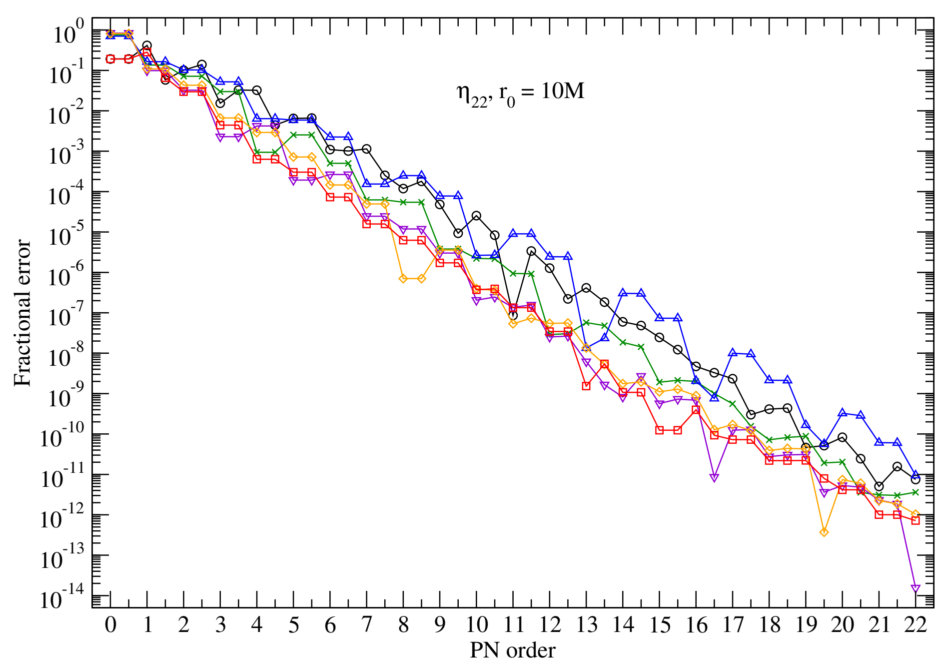


Comparison with numerical results from Fujita and Tagoshi [Prog. Theor. Phys. **112**, 415 (2004)].

How do these factorizations and resummations improve convergence?







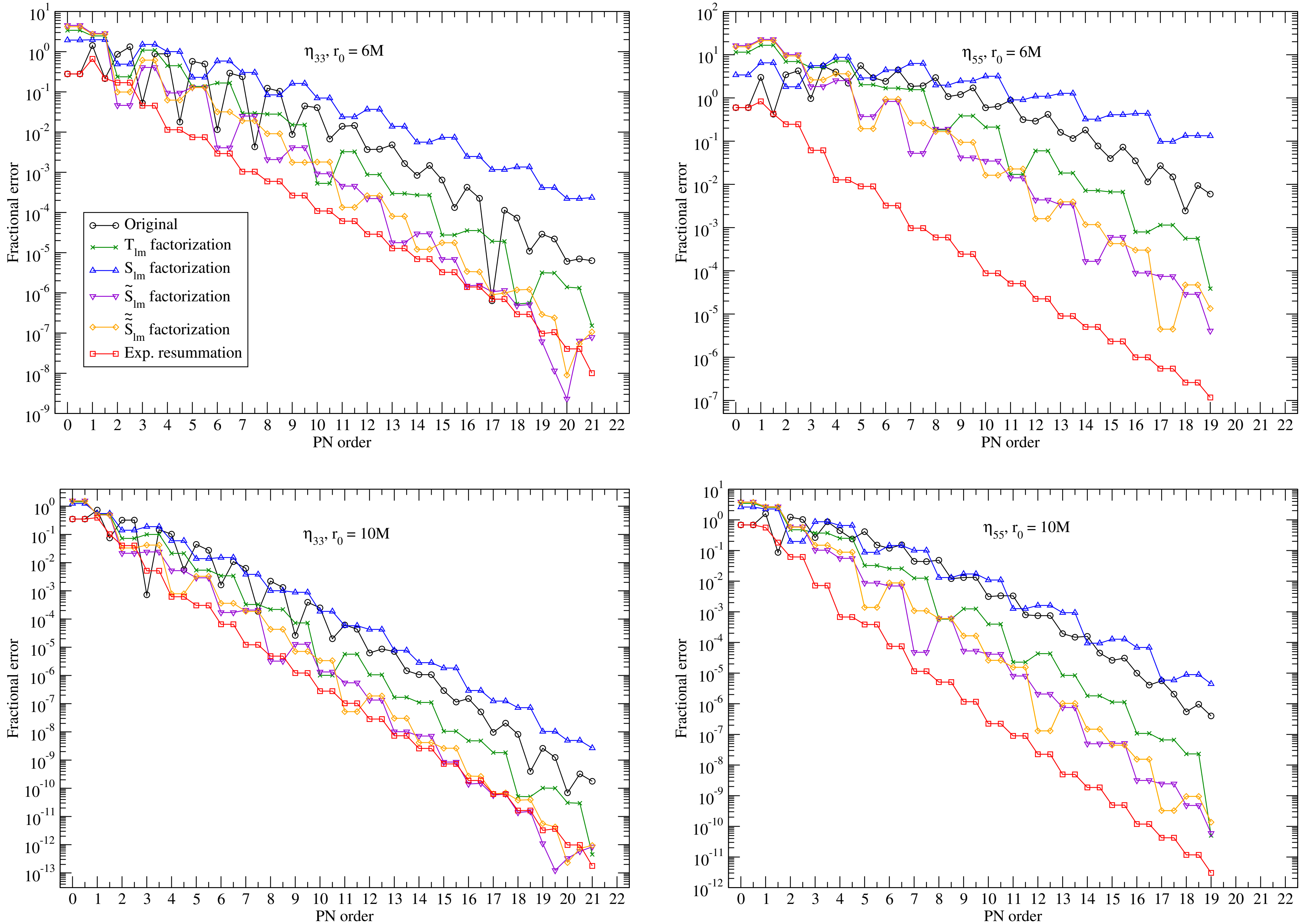
Because the original S_{lm} factorization actually *worsens* the rate of convergence in most cases, we introduced the following additional versions, with ℓ contributions one could reasonably expect from the MST formalism

$$\tilde{S}_{\ell m} := (2mv)^{\bar{\nu}_{\ell m}(v)} e^{\pi m v^3} \frac{(2\ell+1)!}{(\ell-2)!} \frac{\Gamma[-1+\ell+\bar{\nu}_{\ell m}(v) - 2imv^3]}{\Gamma[2+2\ell+2\bar{\nu}_{\ell m}(v)]}$$

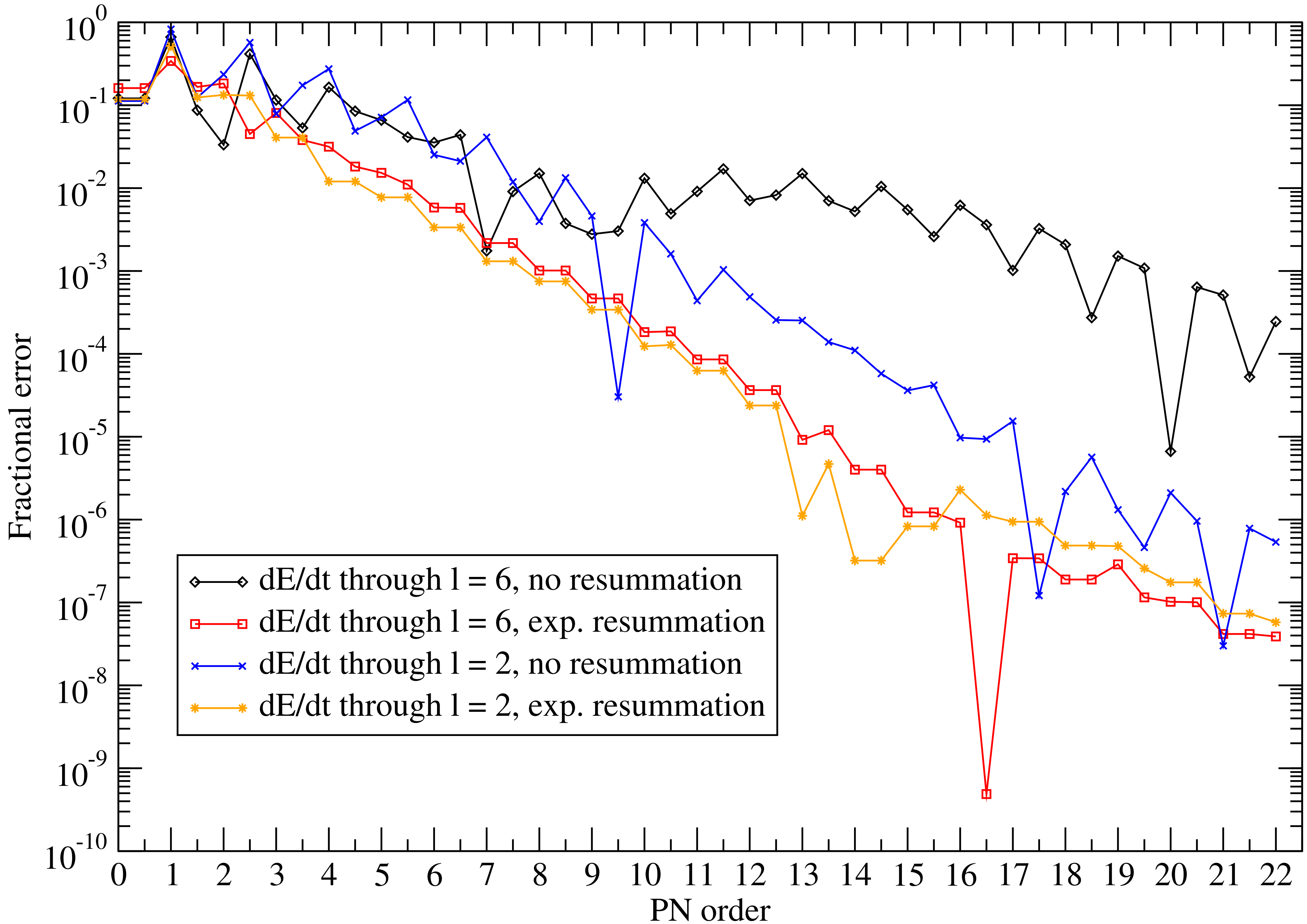
$$\tilde{\tilde{S}}_{\ell m} := (2mv)^{\bar{\nu}_{\ell m}(v)} e^{\pi m v^3} \frac{(2\ell)!}{\ell!} \frac{\Gamma[1+\ell+\bar{\nu}_{\ell m}(v) - 2imv^3]}{\Gamma[1+2\ell+2\bar{\nu}_{\ell m}(v)]}$$

Comparison with numerical results from Fujita and Tagoshi [Prog. Theor. Phys. **112**, 415 (2004)].

Mode convergence (cont.)



Improvement of the convergence of the full energy flux (summed through $\ell = 6$) at the ISCO from exponential resummation of the individual modes



Conclusions

- While the post-Newtonian expansion exhibits combinatorial complexity at higher orders for many quantities, it is possible to significantly simplify these results with an appropriate factorization, at least for the modes of the 22PN expression for the energy flux at infinity for a point particle in a circular orbit around a Schwarzschild black hole, the highest-order compact binary PN result known.

- In the best case (for the modes with $\ell \geq 7$), this factorization produces the maximum simplification one can hope for, reducing the complete 22PN results to a simple integer PN series with rational coefficients, reducing the size of the expressions by a factor of up to ~150.

- Even for the modes with smaller ℓ, this factorization still reduces the series to an integer PN series with rational coefficients for lower orders (to 8PN for the dominant 2,2 mode), and substantially reduces the complexity of the higher orders (by a total factor of ~10 for the 2,2 mode, ~20 if one uses the factorization combined with the logarithm).

Conclusions (cont.)

- The exponential resummation also simplifies the modes, for small ℓ almost as much, or even slightly more than, the factorization. It also improves the convergence of the series the most (both in terms of speed and monotonicity), giving ~4 orders of magnitude improvement for the full energy flux at the ISCO, due to the improvement in the convergence of the higher modes.

- The simplified expressions for the modes of the energy flux I have calculated are freely available online.

Outlook

- In addition to seeing about further simplifying the remaining terms in the series, (by, e.g., looking at the ν and $-\nu - 1$ pieces separately) it should be possible to apply these sorts of techniques to many other quantities that have been calculated to high PN order using the MST formalism, such as the horizon flux in Schwarzschild, and both fluxes in Kerr (for circular orbits), in addition to the redshift observable and the spin precession frequency, which aren't radiative quantities, but still get complexity from tail contributions: One sees very similar sorts of structures in the expansions of these quantities.

- It is likely that these sorts of simplifications could aid in determining analytic forms for some high-order PN coefficients that have so far only been determined numerically (to extremely high accuracy), either by eliminating $\log v$ terms and transcendentals, or by indicating their expected form.

Outlook (cont.)

- This sort of analysis could also lend insight into the physical content of the MST formalism, which has remained rather opaque to date. For instance, it appears that the renormalized angular momentum ν encodes tail effects.

- Finally, these sorts of studies of the structure of high-order PN expansions might even be able to discover some of the same deep connections to other branches of mathematics that have been found in similar studies of expansions of QFT amplitudes. (Indeed, the same sorts of loop integrals can be used to describe both calculations.)

Extra Slides

Why do these simplifications simplify?
A brief look at the workings of the MST BHPT formalism

- The fundamental insight of the formalism is that it is possible to write the Teukolsky equation [or the Regge-Wheeler equation] in a form that allows it to be expressed as a series in Coulomb wave functions.

- One obtains this form by introducing a “convenient zero” involving a parameter ν, which is fixed by demanding that the series converge. [$\bar{\nu}_{\ell m}(v) := \nu - \ell$; one determines ν from the solution of a continued fraction equation, which one demands reduce to ℓ for v → 0.]

$$\nu = \ell + \sum_{k=1}^{\infty} [\nu_\ell]_k \, (2mv^3)^{2k}$$

Overview of the MST BHPT formalism (cont.)

- One obtains η_{lm} as $|Z_{lm\omega}|^2$, where

$$Z_{\ell m\omega} = \frac{\mathcal{L}_{\ell m\omega} R^{\rm in}_{\ell m\omega}}{B^{\rm inc}_{\ell m\omega}}$$

Linear differential operator that differentiates w.r.t. the particle's radial coordinate

$$R^{\rm in}_{\ell m\omega} = K_\nu R^\nu_{\rm C} + K_{-\nu-1} R_{\rm C}^{-\nu-1}$$

$$B^{\rm inc}_{\ell m\omega} = \frac{1}{\omega}\left[K_\nu - {\rm i}e^{-{\rm i}\pi\nu}\frac{\sin\pi(\nu+{\rm i}\epsilon)}{\sin\pi(\nu-{\rm i}\epsilon)}K_{-\nu-1}\right]A^\nu_+ \epsilon^{-{\rm i}\epsilon}$$

$$A^\nu_+ = 2^{-3-{\rm i}\epsilon}e^{-\pi\epsilon/2}e^{{\rm i}\pi(\nu+3)/2}\frac{\Gamma(\nu+3+{\rm i}\epsilon)}{\Gamma(\nu-1-{\rm i}\epsilon)}\sum_{n=-\infty}^{\infty} a^\nu_n$$

$$K_\nu = \frac{e^{{\rm i}\epsilon}(2\epsilon)^{-2-\nu-N}\Gamma(3-2{\rm i}\epsilon)\Gamma(N+2\nu+2)}{\Gamma(N+\nu+3+{\rm i}\epsilon)\Gamma(N+\nu+1+{\rm i}\epsilon)\Gamma(N+\nu-1+{\rm i}\epsilon)}$$
$$\times\left[\sum_{n=N}^{\infty}(-1)^n\frac{\Gamma(n+N+2\nu+1)}{(n-N)!}\frac{\Gamma(n+\nu-1+{\rm i}\epsilon)\Gamma(n+\nu+1+{\rm i}\epsilon)}{\Gamma(n+\nu-3-{\rm i}\epsilon)\Gamma(n+\nu+1-{\rm i}\epsilon)}a^\nu_n\right]$$
$$\times\left[\sum_{n=-\infty}^{N}\frac{(-1)^n}{(N-n)!(N+2\nu+2)_n}\frac{(\nu-1-{\rm i}\epsilon)_n}{(\nu+3+{\rm i}\epsilon)_n}a^\nu_n\right]^{-1}$$

Confluent hypergeometric function

$$R^\nu_{\rm C} = (\omega r_0)^2\left(1-\frac{\epsilon}{\omega r_0}\right)^{2-{\rm i}\epsilon}e^{-{\rm i}\omega r_0}\sum_{n=-\infty}^{\infty}(-{\rm i})^n(2\omega r_0)^{n+\nu}\frac{(\nu-1-{\rm i}\epsilon)_n}{(\nu+3+{\rm i}\epsilon)_n}\frac{\Gamma(n+\nu+3+{\rm i}\epsilon)}{\Gamma(2n+2\nu+2)}a^\nu_n\Phi(n+\nu+3+{\rm i}\epsilon, 2n+2\nu+2; 2{\rm i}\omega r_0)$$

Overview of the MST BHPT formalism (cont.)

- One obtains η_{lm} as $|Z_{lm\omega}|^2$, where

$$Z_{\ell m\omega} = \frac{\mathcal{L}_{\ell m\omega} R^{\rm in}_{\ell m\omega}}{R^{\rm inc}\ldots} \qquad R^{\rm in}_{\ell m\omega} = K_\nu R^\nu_{\rm C} + K_{-\nu-1} R^{-\nu-1}_{\rm C}$$

Linear diff…
that diffe…
the particle'…

$$Z_{\ell m\omega} \sim \frac{R^\nu_{\rm C}}{A^\nu_+}\left[1 + \frac{K_{-\nu-1}}{K_\nu}\frac{R^{-\nu-1}_{\rm C}}{R^\nu_{\rm C}}\right]\left[1 - {\rm i}e^{-{\rm i}\pi\nu}\frac{\sin\pi(\nu+{\rm i}\epsilon)}{\sin\pi(\nu-{\rm i}\epsilon)}\frac{K_{-\nu-1}}{K_\nu}\right]^{-1}$$

S_{lm}

V_{lm}

$$|A^\nu_+| \sim e^{-\pi\epsilon/2}$$

$$\frac{K_{-\nu-1}}{K_\nu} \sim (2\epsilon)^{1+2\nu}\left[\frac{\Gamma(1-2\nu)\Gamma(1+\nu-{\rm i}\epsilon)}{\Gamma(1+2\nu)\Gamma(1-\nu-{\rm i}\epsilon)}\right]^2\frac{\Gamma(1+\nu+{\rm i}\epsilon)}{\Gamma(1-\nu+{\rm i}\epsilon)}$$

$A^\nu_+\epsilon^{-{\rm i}\epsilon}$

$\sum_{n=-\infty}^{\infty} a^\nu_n$

$$K_\nu = \frac{\ldots}{\Gamma(N+\nu+3+{\rm i}\epsilon)\Gamma(N+\nu+1+{\rm i}\epsilon)\Gamma(N+\nu-1+{\rm i}\epsilon)}$$
$$\times\left[\sum_{n=N}^{\infty}(-1)^n\frac{\Gamma(n+N+2\nu+1)}{(n-N)!}\frac{\Gamma(n+\nu-1+{\rm i}\epsilon)\Gamma(n+\nu+1+{\rm i}\epsilon)}{\Gamma(n+\nu-3-{\rm i}\epsilon)\Gamma(n+\nu+1-{\rm i}\epsilon)}a^\nu_n\right]$$
$$\times\left[\sum_{n=-\infty}^{N}\frac{(-1)^n}{(N-n)!(N+2\nu+2)_n}\frac{(\nu-1-{\rm i}\epsilon)_n}{(\nu+3+{\rm i}\epsilon)_n}a^\nu_n\right]^{-1}$$

$$R^\nu_{\rm C} = (\omega r_0)^2\left(1-\frac{\epsilon}{\omega r_0}\right)^{2-{\rm i}\epsilon}e^{-{\rm i}\omega r_0}\sum_{n=-\infty}^{\infty}(-{\rm i})^n(2\omega r_0)^{n+\nu}\frac{(\nu-1-{\rm i}\epsilon)_n}{(\nu+3+{\rm i}\epsilon)_n}\frac{\Gamma(n+\nu+3+{\rm i}\epsilon)}{\Gamma(2n+2\nu+2)}a^\nu_n\Phi(n+\nu+3+{\rm i}\epsilon, 2n+2\nu+2; 2{\rm i}\omega r_0)$$