Conservative effects of the second-order gravitational self-force

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Self-force formalism starts from an expansion in powers of $\mu/M$

All numerical results have been at linear order

Second-order formalism now in place, but no numerical results

What should we calculate?

- It’s well known that dissipative second-order effects are essential for accurate inspiral
- But local-in-time conservative effects far easier to calculate
Motivation for looking at conservative effects

- Self-force originally studied to model inspirals
- But interesting conservative effects have been emphasized in recent years
  - orbital precession
  - ISCO shift
  - Detweiler’s redshift factor
  - self-tides

Going to second order, we can calculate order- \( \left( \frac{\mu}{M} \right)^2 \) contributions to all these effects.
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Going to second order

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Motivation continued: interfacing between models

Conservative self-force results

- fix Effective One Body parameters
- determine high-order PN terms
- set benchmarks for NR
- show self-force has surprisingly large domain of validity [Le Tiec et al]
Motivation continued: interfacing between models

Conservative self-force results
- fix Effective One Body parameters
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Going to second order
- fix $(\mu/M)^2$ terms in PN and EOB
- set stronger benchmarks for NR
- first step toward using SF to model IMRIIs and comparable-mass binaries
1 Second-order formalism

2 Conservative-dissipative split
   - Options that incorporate dissipation
   - Options that neglect dissipation

3 Detweiler’s redshift invariant

4 Asymptotically flat Lorenz gauge solutions
Outline

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in external universe, gravitational field of $M$ dominates

in inner region, gravitational field of $\mu$ dominates

in buffer region, extract information about $\mu$ from "inner expansion", feed it into "outer expansion", define $\mu$’s worldline
Equation of motion

- Split the field into singular and regular pieces

\[ D^2 z^\mu \frac{d}{d\tau^2} = -\frac{1}{2} P^{\mu\nu} (g_{\nu}^{\ \delta} - h_{\nu}^{R\delta})(2h_{\delta\beta;\gamma}^{R} - h_{\beta\gamma;\delta}^{R})u^\beta u^\gamma + O[(\mu/M)^3] \]

- geodesic motion in \( C^\infty \) vacuum metric \( \tilde{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}^{R} \)
- \( h_{\mu\nu}^{R} = h_{\mu\nu}^{R1} + h_{\mu\nu}^{R2} \)
Equation of motion

- Split the field into singular and regular pieces

\[ \frac{D^2 z^\mu}{d\tau^2} = -\frac{1}{2} P^{\mu\nu} (g_{\nu}^{\delta} - h_{\nu}^{R,R}) (2 h_{\beta;\gamma}^{R} - h_{\beta;\gamma;\delta}^{R}) u^\beta u^\gamma + O[(\mu/M)^3] \]

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- Geodesic motion in $C^\infty$ vacuum metric $\tilde{g}_{\mu \nu} = g_{\mu \nu} + h^R_{\mu \nu}$
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Effective-source scheme

- use a ‘puncture’ $h^P_{\mu\nu} \approx h^S_{\mu\nu}$
- rewrite field equations for variable $h^R_{\mu\nu} = h_{\mu\nu} - h^P_{\mu\nu} \approx h^R_{\mu\nu}$
- design $h^P_{\mu\nu}$ such that on worldline, $h^R_{\mu\nu} = h^R_{\mu\nu}$ and $\partial h^R_{\mu\nu} = \partial h^R_{\mu\nu}$

At second order

- for $h^2_{\mu\nu}$: $\delta G_{\mu\nu}[h^2] = -\delta^2 G_{\mu\nu}[h^1] \sim (\partial h^1)^2 + h^1 \partial^2 h^1$
- for $h^2^R_{\mu\nu}$: $\delta G_{\mu\nu}[h^2^R] = -\delta^2 G_{\mu\nu}[h^1] - \delta G_{\mu\nu}[h^2^P]$
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Why the split is clear at first order

- can approximate the source orbit as a background geodesic —the deviation is $\delta z^\mu \sim \mu \Rightarrow$ it influences metric at second order
- can naturally define the conservative force as the piece of the force that’s time-reversal invariant along the geodesic
- can naturally define the time-symmetric piece of the linearized retarded field as the ‘half-ret. + half-adv.’ perturbation sourced by the geodesic
- And the naturally defined conservative force is identical to the force constructed from the (regular piece of the) naturally defined time-symmetric field:

$$F^\mu_{1\text{cons}} = \frac{1}{2}(F^\mu_{1\text{ret}} + F^\mu_{1\text{adv}})$$

$$F^\mu_{1\text{diss}} = \frac{1}{2}(F^\mu_{1\text{ret}} - F^\mu_{1\text{adv}})$$
Why it’s *unclear* at second order

- must account for the deviation $\delta z^\mu$ from background geodesic — which worldline do we refer to when deciding the time symmetry of the force?

- Products of form $\sim (\delta z^\mu_{\text{diss}})^2$ yield time-symmetric terms in force — should they be included in the conservative dynamics?

- likewise for products of form $\sim \delta z^\mu_{\text{diss}} (F^\mu_{1\text{ret}} - F^\mu_{1\text{adv}})$

- $h_{\mu\nu}^2$ sourced by $\sim (\partial h^1_{\mu\nu})^2 + h^1_{\mu\nu} \partial^2 h^1_{\mu\nu}$

  $\Rightarrow$ physical, retarded $h_{\mu\nu}^2$ sourced by $(\partial h^1_{\mu\nu})^2 \neq (\frac{1}{2} \partial [h^1_{\mu\nu}^{\text{ret}} + h^1_{\mu\nu}^{\text{adv}}])^2$

  $\Rightarrow$ (time-symmetric piece of $h^\text{ret}_{\mu\nu}$) $\neq$ (half-ret-plus-half-adv. $h_{\mu\nu}$)

— the conservative force taken from the retarded field will not be equal to the force generated by the half-ret-plus-half-adv. field
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Definition 1: slowly varying orbital frequency

- Restrict to quasicircular orbits in Schwarzschild
- solve *relaxed* EFE
- What do I mean by ‘incorporating dissipation’ in this case?
  —incorporate effect of $\dot{\Omega}$ on the perturbation

*First obvious definition to consider*

- slow evolution: $\dot{\Omega} \sim \mu/M$
- define conservative dynamics by freezing $\Omega$
  —but still account for perturbation sourced by $\dot{\Omega}$

*Difficulty*

- gives rise to term like $\sim \dot{\Omega} h^{1}_{\mu\nu}$ in source for $h^{2}_{\mu\nu}$
  $\Rightarrow$ source behaves as $\sim 1/r$ at large $r$
  $\Rightarrow$ infrared divergence in $h^{2}_{\mu\nu}$
Definition 2: Gralla-Wald treatment of dissipation

- Expand around a zeroth-order orbit:

\[
z^\mu \left( t, \frac{\mu}{M} \right) = z_0^\mu(t) + \frac{\mu}{M} z_1^\mu(t) + \frac{\mu^2}{M^2} z_2^\mu(t) + O \left( \frac{\mu^3}{M^3} \right)
\]

- \( z_0^\mu \) is a circular geodesic of background
- split \( z_1^\mu, z_2^\mu \) into deviations driven by conservative and dissipative forces

**Difficulty**

- get terms growing as \( \sim t^2 \) in \( h_{\mu \nu}^2 \)
- get growing but time-symmetric terms in quantities on \( z_0^\mu \) —how to classify these?
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Definition 3: balance of in-out radiation

- by ‘neglecting dissipation’ I mean setting $\dot{\Omega} = 0$
- way to get this consistently: use half-ret. + half-adv. solution at both first and second order:
  1. find $h_{\mu\nu}^1 = \frac{1}{2} (h_{\mu\nu}^{1\text{ret}} + h_{\mu\nu}^{1\text{adv}})$
  2. use this $h_{\mu\nu}^1$ in source $\sim (\partial h_{\mu\nu}^1)^2 + h_{\mu\nu}^1 \partial^2 h_{\mu\nu}^1$ for $h_{\mu\nu}^2$
  3. find $h_{\mu\nu}^2 = \frac{1}{2} (h_{\mu\nu}^{2\text{ret}} + h_{\mu\nu}^{2\text{adv}})$
- field satisfies EFE, motion is geodesic in its regular part

Problems
- infrared divergence
- not what’s done in PN, where only retarded Green’s functions are used
Definition 3: balance of in-out radiation

- by ‘neglecting dissipation’ I mean setting $\dot{\Omega} = 0$
- way to get this consistently: use half-ret. + half-adv. solution at both first and second order:
  1. find $h^{1\mu\nu}_{\mu\nu} = \frac{1}{2}(h^{1\text{ret}}_{\mu\nu} + h^{1\text{adv}}_{\mu\nu})$
  2. use this $h^{1\mu\nu}_{\mu\nu}$ in source $\sim (\partial h^{1\mu\nu}_{\mu\nu})^2 + h^{1\mu\nu}_{\mu\nu}\partial^2 h^{1\mu\nu}_{\mu\nu}$ for $h^{2\mu\nu}_{\mu\nu}$
  3. find $h^{2\mu\nu}_{\mu\nu} = \frac{1}{2}(h^{2\text{ret}}_{\mu\nu} + h^{2\text{adv}}_{\mu\nu})$
- field satisfies EFE, motion is geodesic in its regular part

Problems

- infrared divergence
- not what’s done in PN, where only retarded Green’s functions are used
Definition 4: “turning off” dissipative terms in the force

- set source orbit $\mathring{z}^\mu$ to be circular—i.e., $\mathring{\Omega} = 0$
- find retarded solution at both first and second order
- find the piece of the retarded force consistent with $\mathring{z}^\mu$

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Note
- because not all of $F^\mu$ is included, $h^2_{\mu\nu}$ here is not a solution to the EFE
- $\mathring{z}^\mu$ is not a geodesic of the effective metric
set source orbit $\hat{z}^\mu$ to be circular (i.e., $\dot{\Omega} = 0$)
find retarded solution at both first and second order
rather than simply setting $F^t = 0 = F^\phi$, find time-symmetrized effective metric $\hat{g}_{\mu\nu} = g_{\mu\nu} + \hat{h}^R_{\mu\nu}$ in which $\hat{z}^\mu$ is a geodesic
Time-symmetrization: take regular piece of retarded field,

$$h^{Rn}_{\mu\nu} = \sum_{ilm} h^{Rn}_{ilm} e^{-im\Omega t} Y^{ilm}_{\mu\nu}$$

and let $h^{Rn}_{ilm} \rightarrow \frac{1}{2} (h^{Rn}_{ilm} + h^{Rn*}_{ilm})$
Why? Outgoing $\leftrightarrow$ ingoing waves: $h^n_{ilm} \leftrightarrow h^{n*}_{ilm}$

Note
$\hat{g}_{\mu\nu}$ is not a solution to the vacuum EFE
$\hat{z}^\mu$ is not same as in Definition 4
— but metrics differ only by amount $\sim \mu^3 / M^3$
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An invariant quantity: Detweiler’s redshift variable

- use any of the definitions that set \( \dot{\Omega} = 0 \)
- ‘redshift factor’: ratio of times \( \tilde{u}^t = \frac{dt}{d\tilde{\tau}} \) in effective metric

In terms of quantities on the conservative orbit

\[
\tilde{u}^t = \frac{1}{\sqrt{1 - \frac{3M}{r}}} \left\{ 1 + \frac{1}{2} (h^R_{uu} - F_r r) + \frac{1}{8} \left[ 3(h^R_{uu})^2 - 2rF_r h^R_{uu} - r^2 (F_r)^2 \right] \right\}
\]
An invariant quantity: Detweiler’s redshift variable

- use any of the definitions that set $\dot{\Omega} = 0$
- ‘redshift factor’: ratio of times $\tilde{u}^t = \frac{dt}{d\tilde{\tau}}$ in effective metric

In terms of quantities on the zeroth-order orbit

$$\tilde{u}^t = \frac{1}{\sqrt{1 - \frac{3M}{r_0}}} \left\{ 1 + \frac{1}{2} \frac{\mu}{M} h^{R1}_{u0u0} + \left( \frac{\mu}{M} \right)^2 \left[ \frac{1}{2} h^{R2}_{u0u0} + \frac{3}{8} (h^{R1}_{u0u0})^2 - \frac{r_0^2 (r_0 - 3M)}{6M} (F_1r)^2 \right] \right\}$$
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There are two well-known monopole solutions in the Lorenz gauge:

<table>
<thead>
<tr>
<th>Solution 1: ‘correct’ mass</th>
<th>Solution 2: asymptotically flat</th>
</tr>
</thead>
<tbody>
<tr>
<td>‘physically correct’ total mass-energy ( M + \mu E )</td>
<td>( \lim_{r \to \infty} h_{\mu\nu} = 0 )</td>
</tr>
<tr>
<td>but ( \lim_{r \to \infty} h_{\mu\nu} = \text{const.} \neq 0 )</td>
<td>but the total mass-energy is ( M + \mu E - \delta M \neq M + \mu E )</td>
</tr>
</tbody>
</table>

- Historically, Solution 1 has been used as the ‘physically correct’ solution
- Can’t use it here: It would produce a source \( \sim h_{\mu\nu}^1 \partial^2 h_{\mu\nu}^1 \sim 1/r \)
  \( \Rightarrow \) badly divergent \( h_{\mu\nu}^2 \)
- Transforming it to an asymptotically flat gauge also leads to catastrophe
Avoiding the historical problem

The two solutions differ only by an overall shift in the mass

⇒ The shift can be absorbed into the definition of $M$

⇒ The solutions are physically indistinguishable

Consider: a binary containing a black hole of mass $m_1$ and a small object of mass $m_2$

**Solution 1: ‘correct’ mass**

corresponds to expansion around background of mass

$$M = m_1$$

**Solution 2: asymptotically flat**

corresponds to expansion around background of mass

$$M = m_1 + \delta M$$

- Both solutions describe the same binary. So we can use Solution 2
- When comparing numerical results to other calculations, simply note that $M \neq m_1$
Summary and conclusions

Short-term objectives and current status

- The first things to calculate at second order are invariant conservative quantities
  ⇒ compare to PN, fix PN and EOB parameters
- Numerical implementation now in progress to calculate $\tilde{u}^t$ on a circular orbit

But what should we mean by ‘the conservative dynamics’ at second order?

- e.g., for a (quasi)circular orbit, the cleanest definitions involve setting $\dot{\Omega} = 0$ everywhere in the calculation

—Would a different definition be more useful as input for PN/EOB?