Conservative effects of the second-order gravitational self-force

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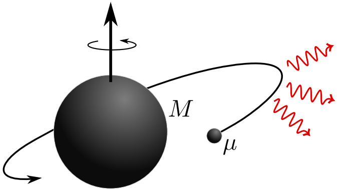
Second-order self-force

- Self-force formalism starts from an expansion in powers of μ/M
- All numerical results have been at linear order
- Second-order formalism now in place, but no numerical results

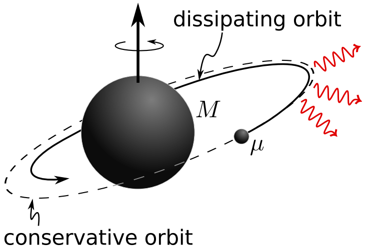


What should we calculate?

- It's well known that *dissipative* second-order effects are essential for accurate inspiral
- But local-in-time conservative effects far easier to calculate

Motivation for looking at conservative effects

- Self-force originally studied to model inspirals
- But interesting conservative effects have been emphasized in recent years
  - orbital precession
  - ISCO shift
  - Detweiler's redshift factor
  - self-tides



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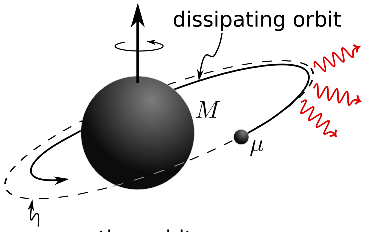
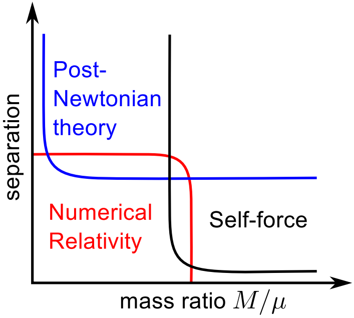


Going to second order

We can calculate order-$(\mu/M)^2$ contributions to all these effects

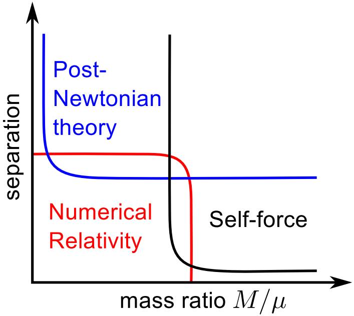
Motivation continued: interfacing between models



Conservative self-force results

- fix Effective One Body parameters
- determine high-order PN terms
- set benchmarks for NR
- show self-force has surprisingly large domain of validity [Le Tiec et al]

Motivation continued: interfacing between models



Conservative self-force results

- fix Effective One Body parameters
- determine high-order PN terms
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Going to second order

- fix $(\mu/M)^2$ terms in PN and EOB
- set stronger benchmarks for NR
- first step toward using SF to model IMRIs and comparable-mass binaries

Outline

1. Second-order formalism
2. Conservative-dissipative split
   - Options that incorporate dissipation
   - Options that neglect dissipation
3. Detweiler's redshift invariant
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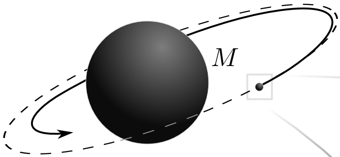
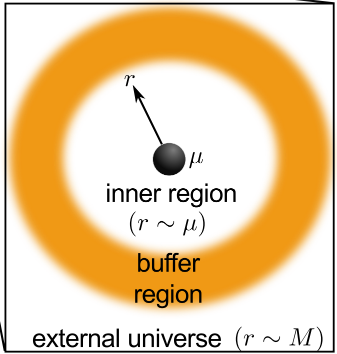
Outline

Matched asymptotic expansions

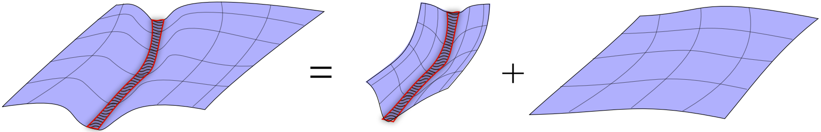




- in external universe, gravitational field of M dominates
- in inner region, gravitational field of μ dominates
- in buffer region, extract information about μ from "inner expansion", feed it into "outer expansion", define μ's worldline

Equation of motion

- Split the field into singular and regular pieces



- Find that the equation of motion is

$$\frac{D^2 z^\mu}{d\tau^2} = -\frac{1}{2} P^{\mu\nu}(g_\nu{}^\delta - h_\nu^{R\delta})(2h^R_{\delta\beta;\gamma} - h^R_{\beta\gamma;\delta})u^\beta u^\gamma + O[(\mu/M)^3]$$

- geodesic motion in C^∞ vacuum metric $\tilde{g}_{\mu\nu} = g_{\mu\nu} + h^R_{\mu\nu}$
- $h^R_{\mu\nu} = h^{R1}_{\mu\nu} + h^{R2}_{\mu\nu}$

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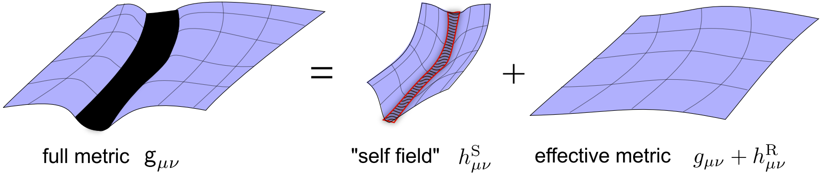

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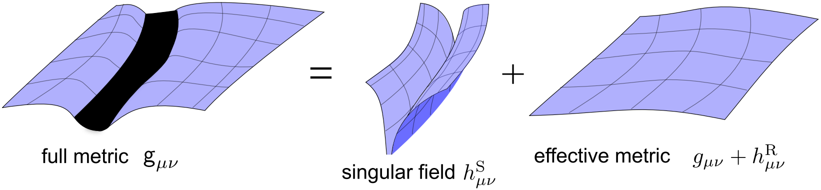

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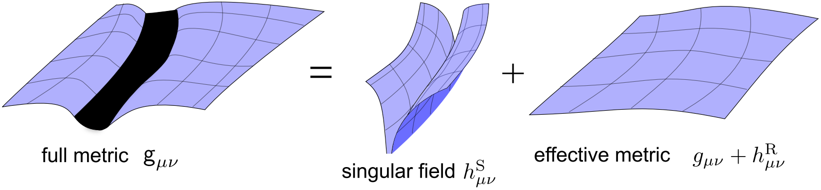

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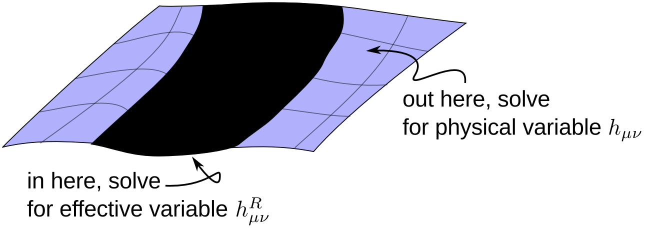
Effective-source scheme

- use a 'puncture' $h^{\mathcal{P}}_{\mu\nu} \approx h^{S}_{\mu\nu}$
- rewrite field equations for variable $h^{\mathcal{R}}_{\mu\nu} = h_{\mu\nu} - h^{\mathcal{P}}_{\mu\nu} \approx h^{R}_{\mu\nu}$
- design $h^{\mathcal{P}}_{\mu\nu}$ such that on worldline, $h^{\mathcal{R}}_{\mu\nu} = h^{R}_{\mu\nu}$ and $\partial h^{\mathcal{R}}_{\mu\nu} = \partial h^{R}_{\mu\nu}$



At second order

- for $h^{2}_{\mu\nu}$: $\delta G_{\mu\nu}[h^2] = -\delta^2 G_{\mu\nu}[h^1] \sim (\partial h^1)^2 + h^1 \partial^2 h^1$
- for $h^{2\mathcal{R}}_{\mu\nu}$: $\delta G_{\mu\nu}[h^{2\mathcal{R}}] = -\delta^2 G_{\mu\nu}[h^1] - \delta G_{\mu\nu}[h^{2\mathcal{P}}]$

Outline

1. Second-order formalism
2. Conservative-dissipative split
   - Options that incorporate dissipation
   - Options that neglect dissipation
3. Detweiler's redshift invariant
4. Asymptotically flat Lorenz gauge solutions

Why the split is clear at first order

- can approximate the source orbit as a background geodesic —the deviation is $\delta z^\mu \sim \mu \Rightarrow$ it influences metric at second order
- can naturally define the conservative force as the piece of the force that's time-reversal invariant along the geodesic
- can naturally define the time-symmetric piece of the linearized retarded field as the 'half-ret. + half-adv.' perturbation sourced by the geodesic
- And the naturally defined conservative force is identical to the force constructed from the (regular piece of the) naturally defined time-symmetric field:

$$F^\mu_{1\text{cons}} = \frac{1}{2}(F^\mu_{1\text{ret}} + F^\mu_{1\text{adv}})$$
$$F^\mu_{1\text{diss}} = \frac{1}{2}(F^\mu_{1\text{ret}} - F^\mu_{1\text{adv}})$$

Why it's *unclear* at second order

- must account for the deviation δz^μ from background geodesic —which worldline do we refer to when deciding the time symmetry of the force?
- Products of form $\sim (\delta z^\mu_{\rm diss})^2$ yield time-symmetric terms in force —should they be included in the conservative dynamics?
- likewise for products of form $\sim \delta z^\mu_{\rm diss}(F^\mu_{1\rm ret} - F^\mu_{1\rm adv})$
- $h^2_{\mu\nu}$ sourced by $\sim (\partial h^1_{\mu\nu})^2 + h^1_{\mu\nu}\partial^2 h^1_{\mu\nu}$

  $\Rightarrow$ physical, retarded $h^2_{\mu\nu}$ sourced by $(\partial h^{1\rm ret}_{\mu\nu})^2 \neq (\frac{1}{2}\partial[h^{1\rm ret}_{\mu\nu} + h^{1\rm adv}_{\mu\nu}])^2$

  $\Rightarrow$ (time-symmetric piece of $h^{\rm ret}_{\mu\nu}$) $\neq$ (half-ret-plus-half-adv. $h_{\mu\nu}$)

  —the conservative force taken from the retarded field will not be equal to the force generated by the half-ret-plus-half-adv. field

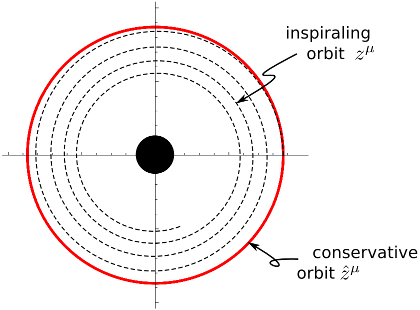
Outline

Definition 1: slowly varying orbital frequency

- Restrict to quasicircular orbits in Schwarzschild
- solve *relaxed* EFE
- What do I mean by 'incorporating dissipation' in this case?
  —incorporate effect of $\dot{\Omega}$ on the perturbation

First obvious definition to consider

- slow evolution: $\dot{\Omega} \sim \mu/M$
- define conservative dynamics by freezing Ω
  —but still account for perturbation sourced by $\dot{\Omega}$



Difficulty

- gives rise to term like $\sim \dot{\Omega}\, h^1_{\mu\nu}$ in source for $h^2_{\mu\nu}$
  - $\Rightarrow$ source behaves as $\sim 1/r$ at large r
  - $\Rightarrow$ infrared divergence in $h^2_{\mu\nu}$

Definition 2: Gralla-Wald treatment of dissipation

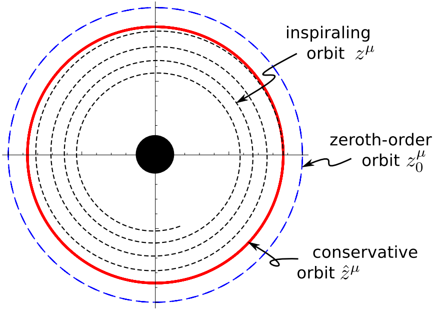
- Expand around a zeroth-order orbit:

$$z^\mu\left(t, \frac{\mu}{M}\right) = z_0^\mu(t) + \frac{\mu}{M} z_1^\mu(t) + \frac{\mu^2}{M^2} z_2^\mu(t) + O\left(\frac{\mu^3}{M^3}\right)$$

- z_0^μ is a circular geodesic of background
- split z_1^μ, z_2^μ into deviations driven by conservative and dissipative forces

Difficulty

- get terms growing as $\sim t^2$ in $h^2_{\mu\nu}$
- get growing but time-symmetric terms in quantities on z_0^μ
—how to classify these?



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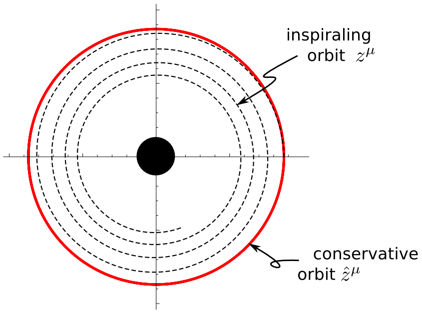
Definition 3: balance of in-out radiation

- by 'neglecting dissipation' I mean setting $\dot{\Omega} = 0$
- way to get this consistently: use half-ret.+half-adv. solution at both first and second order:
  1. find $h^1_{\mu\nu} = \frac{1}{2}(h^{1\text{ret}}_{\mu\nu} + h^{1\text{adv}}_{\mu\nu})$
  2. use this $h^1_{\mu\nu}$ in source $\sim (\partial h^1_{\mu\nu})^2 + h^1_{\mu\nu}\partial^2 h^1_{\mu\nu}$ for $h^2_{\mu\nu}$
  3. find $h^2_{\mu\nu} = \frac{1}{2}(h^{2\text{ret}}_{\mu\nu} + h^{2\text{adv}}_{\mu\nu})$
- field satisfies EFE, motion is geodesic in its regular part

Problems

- infrared divergence
- not what's done in PN, where only retarded Green's functions are used



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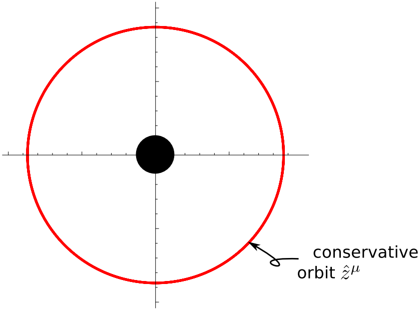


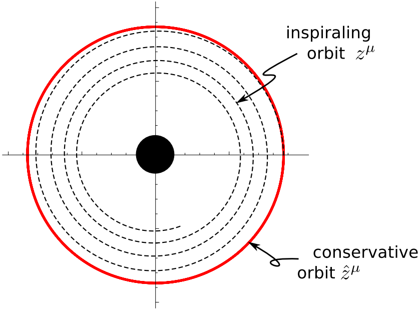
Definition 4: “turning off” dissipative terms in the force

- set source orbit $\hat{z}^\mu$ to be circular—i.e., $\dot{\Omega} = 0$
- find retarded solution at both first and second order
- find the piece of the retarded force consistent with $\hat{z}^\mu$

—i.e., satisfy $\dfrac{D^2\hat{z}^\mu}{d\tau^2} = F^\mu_{\rm cons}$

- simply set

$$F^\mu_{\rm cons} = \delta^\mu_r F^r$$
$$F^\mu_{\rm diss} = \delta^\mu_t F^t + \delta^\mu_\phi F^\phi$$



Note

- because not all of F^μ is included, $h^2_{\mu\nu}$ here is not a solution to the EFE
- $\hat{z}^\mu$ is not a geodesic of the effective metric

Definition 5: geodesic in time-symmetrized effective metric

- set source orbit $\hat{z}^\mu$ to be circular (i.e., $\dot{\Omega} = 0$)
- find retarded solution at both first and second order
- rather than simply setting $F^t = 0 = F^\phi$, find time-symmetrized effective metric $\hat{\tilde{g}}_{\mu\nu} = g_{\mu\nu} + \hat{h}^R_{\mu\nu}$ in which $\hat{z}^\mu$ is a geodesic
- Time-symmetrization: take regular piece of retarded field,

$$h^{Rn}_{\mu\nu} = \sum_{i\ell m} h^{Rn}_{i\ell m} e^{-im\Omega t} Y^{i\ell m}_{\mu\nu}$$

  and let $h^{Rn}_{i\ell m} \to \frac{1}{2}(h^{Rn}_{i\ell m} + h^{Rn*}_{i\ell m})$
- Why? Outgoing $\leftrightarrow$ ingoing waves: $h^n_{ilm} \leftrightarrow h^{n*}_{ilm}$

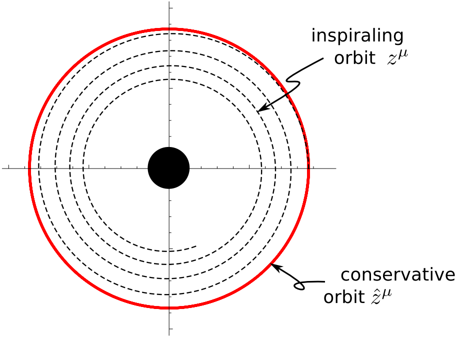
Note

- $\hat{\tilde{g}}_{\mu\nu}$ is *not* a solution to the vacuum EFE
- $\hat{z}^\mu$ is *not* same as in Definition 4
  —but metrics differ only by amount $\sim \mu^3/M^3$

Outline

An invariant quantity: Detweiler's redshift variable

- use any of the definitions that set $\dot{\Omega} = 0$
- 'redshift factor': ratio of times $\tilde{u}^t = \frac{dt}{d\tilde{\tau}}$ in effective metric



In terms of quantities on the conservative orbit

$$\tilde{u}^t = \frac{1}{\sqrt{1-\frac{3M}{r}}}\left\{1 + \frac{1}{2}(h^R_{uu} - F_r r) + \frac{1}{8}\left[3(h^R_{uu})^2 - 2rF_r h^R_{uu} - r^2(F_r)^2\right]\right\}$$

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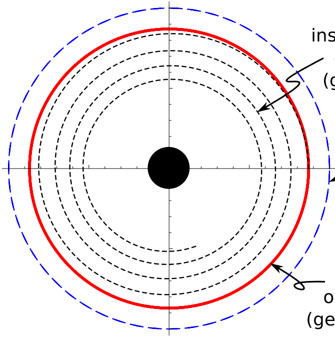

In terms of quantities on the zeroth-order orbit

$$\tilde{u}^t = \frac{1}{\sqrt{1-\frac{3M}{r_0}}}\left\{1+\frac{1}{2}\frac{\mu}{M}h^{R1}_{u_0u_0}+\left(\frac{\mu}{M}\right)^2\left[\frac{1}{2}h^{R2}_{u_0u_0}+\frac{3}{8}\left(h^{R1}_{u_0u_0}\right)^2-\frac{r_0^2(r_0-3M)}{6M}(F_{1r})^2\right]\right\}$$

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The historical problem

There are two well-known monopole solutions in the Lorenz gauge:

Solution 1: 'correct' mass

- 'physically correct' total mass-energy $M + \mu E$
- but $\lim_{r\to\infty} h_{\mu\nu} =$ const. $\neq 0$

Solution 2: asymptotically flat

- $\lim_{r\to\infty} h_{\mu\nu} = 0$
- but the total mass-energy is $M + \mu E - \delta M \neq M + \mu E$

- Historically, Solution 1 has been used as the 'physically correct' solution
- Can't use it here: It would produce a source $\sim h^1_{\mu\nu}\partial^2 h^1_{\mu\nu} \sim 1/r$ $\Rightarrow$ badly divergent $h^2_{\mu\nu}$
- Transforming it to an asymptotically flat gauge also leads to catastrophe

Avoiding the historical problem

The two solutions differ only by an overall shift in the mass

$\Rightarrow$ The shift can be absorbed into the definition of M

$\Rightarrow$ *The solutions are physically indistinguishable*

Consider: a binary containing a black hole of mass m_1 and a small object of mass m_2

Solution 1: 'correct' mass

corresponds to expansion around background of mass

$$M = m_1$$

Solution 2: asymptotically flat

corresponds to expansion around background of mass

$$M = m_1 + \delta M$$

- Both solutions describe the same binary. So we can use Solution 2
- When comparing numerical results to other calculations, simply note that $M \neq m_1$

Summary and conclusions

Short-term objectives and current status

- The first things to calculate at second order are invariant conservative quantities
  $\Rightarrow$ compare to PN, fix PN and EOB parameters
- Numerical implementation now in progress to calculate $\tilde{u}^t$ on a circular orbit

But what should we mean by 'the conservative dynamics' at second order?

- e.g., for a (quasi)circular orbit, the cleanest definitions involve setting $\dot{\Omega} = 0$ everywhere in the calculation

  —Would a different definition be more useful as input for PN/EOB?