Lorenz gauge GSF code: Low frequency modes and expanding e and p ranges

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University of North Carolina at Chapel Hill

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In collaboration with Erik Forseth, Charles Evans, and Seth Hopper

Method Overview

- Eccentric extreme mass ratio Schwarzschild binary
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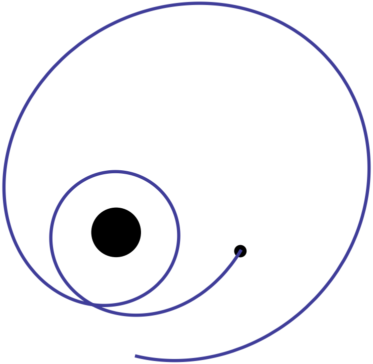

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- Lorenz gauge: $\Box \bar{p}_{\mu\nu} + 2R_{\mu\alpha\nu\beta}\bar{p}^{\alpha\beta} = -16\pi T_{\mu\nu}$
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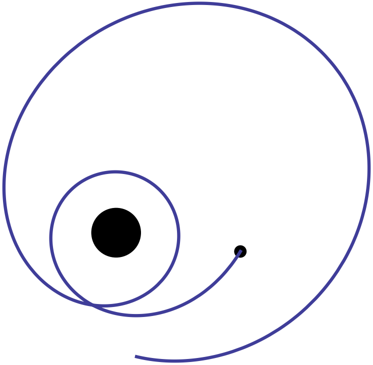

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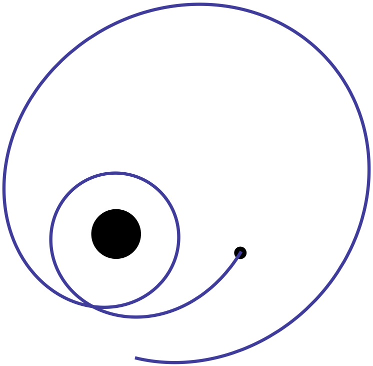

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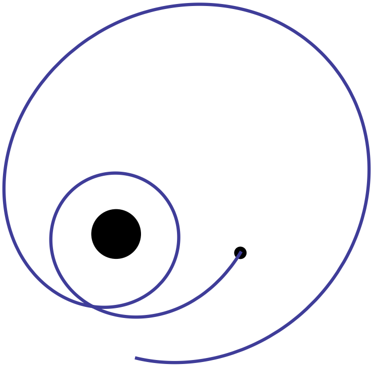

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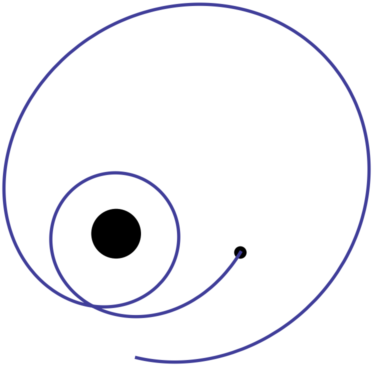

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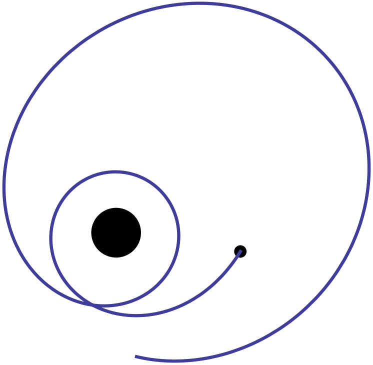

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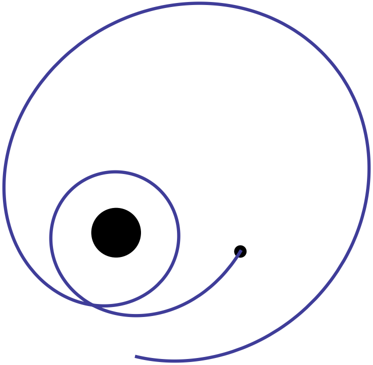

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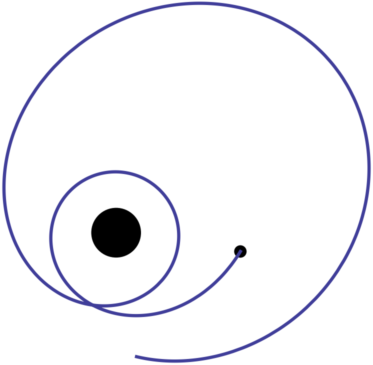

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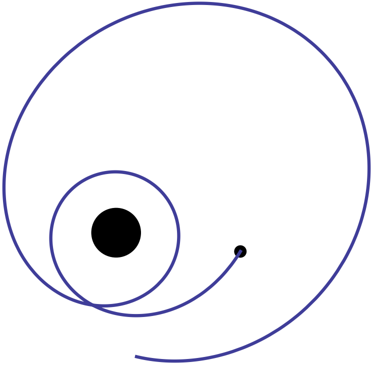
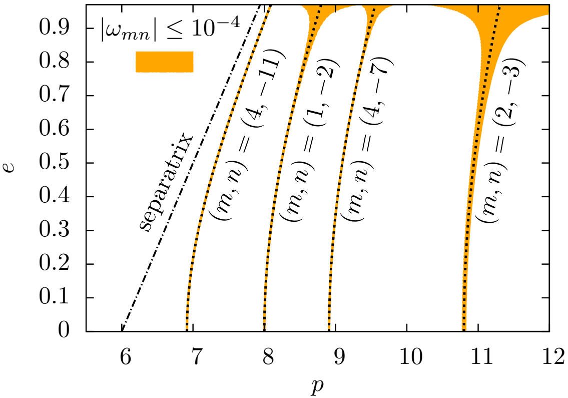
Problem: existing codes have trouble with small frequency modes. Handling these modes is the focus of this talk.



Small frequencies

- When Ω_r/Ω_ϕ is rational the orbit closes
- When the orbit nearly closes there exists an m and n such that $m\Omega_\phi + n\Omega_r \simeq 0$



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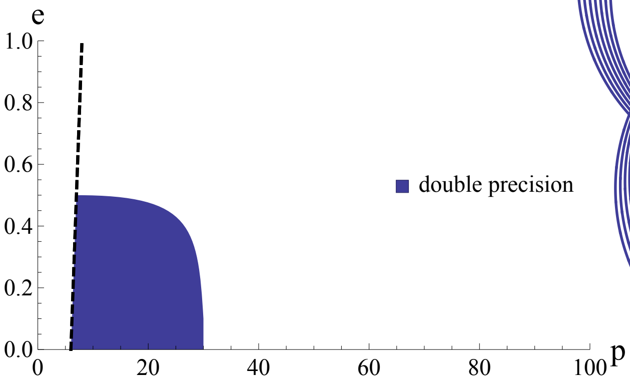

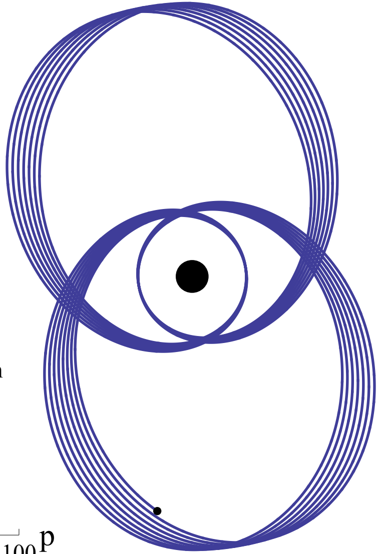

Variation of parameters and ill-conditioning

$$\left(\mathcal{D}_0^\ell + \omega^2\right) h_t^{\ell m\omega} + \omega \mathcal{L}_0^\ell h_r^{\ell m\omega} = Z_0^{\ell m\omega}$$
$$\left(\mathcal{D}_1^\ell + \omega^2\right) h_r^{\ell m\omega} + \omega \mathcal{L}_1^\ell h_t^{\ell m\omega} = Z_1^{\ell m\omega}$$

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$$\boldsymbol{\mathcal{J}} \equiv \left[\begin{array}{c} h_t^{\ell m\omega} \\ h_r^{\ell m\omega} \end{array}\right] = \boldsymbol{\mathcal{J}}_0^+ c_0^+(r) + \boldsymbol{\mathcal{J}}_1^+ c_1^+(r) + \boldsymbol{\mathcal{J}}_0^- c_0^-(r) + \boldsymbol{\mathcal{J}}_1^- c_1^-(r)$$

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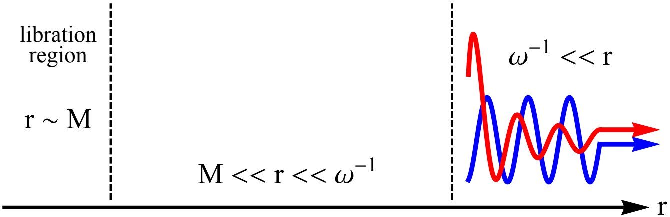

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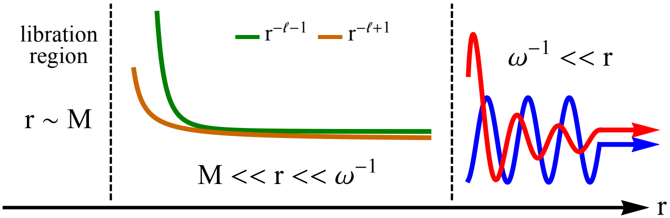

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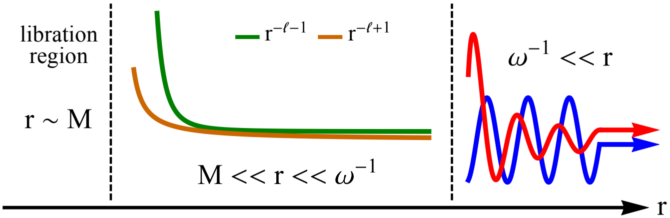
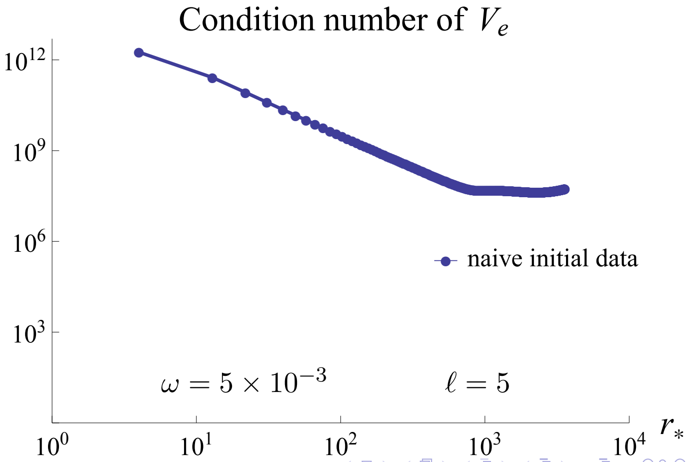


Condition Number

$$\boldsymbol{\mathcal{J}} \equiv (h_t, h_r)^\top \qquad \boldsymbol{\mathcal{H}} \equiv (h_{tt}, h_{tr}, h_{rr}, K)^\top$$

$$V_o \equiv \begin{bmatrix} \boldsymbol{\mathcal{J}}_0^+ & \boldsymbol{\mathcal{J}}_1^+ \\ \partial_r \boldsymbol{\mathcal{J}}_0^+ & \partial_r \boldsymbol{\mathcal{J}}_1^+ \end{bmatrix} \qquad V_e \equiv \begin{bmatrix} \boldsymbol{\mathcal{H}}_0^+ & \boldsymbol{\mathcal{H}}_1^+ & \boldsymbol{\mathcal{H}}_2^+ & \boldsymbol{\mathcal{H}}_3^+ \\ \partial_r \boldsymbol{\mathcal{H}}_0^+ & \partial_r \boldsymbol{\mathcal{H}}_1^+ & \partial_r \boldsymbol{\mathcal{H}}_2^+ & \partial_r \boldsymbol{\mathcal{H}}_3^+ \end{bmatrix}$$



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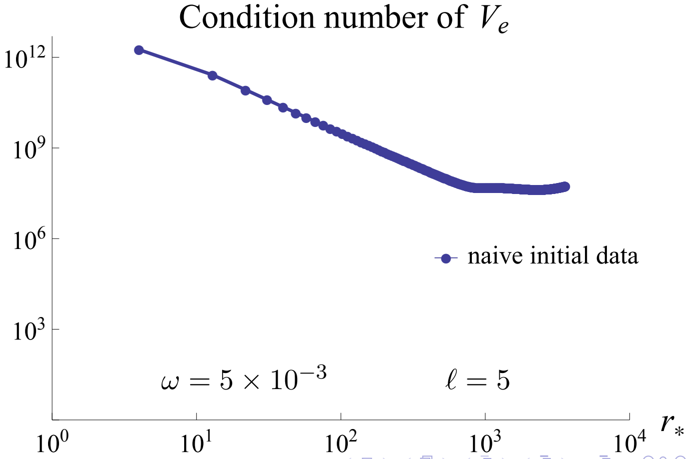
$$\boldsymbol{\mathcal{J}}_i^+ = \sum_k \begin{bmatrix} a_k^{(0)} \\ a_k^{(1)} \end{bmatrix}_i \frac{e^{i\omega r}}{r^k}$$

$$\boldsymbol{\mathcal{H}}_i^+ = \sum_k \begin{bmatrix} b_k^{(0)} \\ b_k^{(1)} \\ b_k^{(2)} \\ b_k^{(3)} \end{bmatrix}_i \frac{e^{i\omega r}}{r^k}$$



Initial conditions

- Choose initial coefficients in asymptotic expansion

$$\begin{bmatrix} a_0^{(0)} \\ a_1^{(1)} \end{bmatrix}_i = \delta_i^j \to \boldsymbol{\mathcal{J}}_i^+ \qquad \begin{bmatrix} b_0^{(0)} \\ b_1^{(0)} \\ b_2^{(0)} \\ b_0^{(3)} \end{bmatrix}_i = \delta_i^j \to \boldsymbol{\mathcal{H}}_i^+$$

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$$\begin{bmatrix} a_0^{(0)} \\ a_1^{(1)} \end{bmatrix}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \to \boldsymbol{\mathcal{J}}_0^+, \qquad \begin{bmatrix} a_0^{(0)} \\ a_1^{(1)} \end{bmatrix}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \to \boldsymbol{\mathcal{J}}_1^+$$

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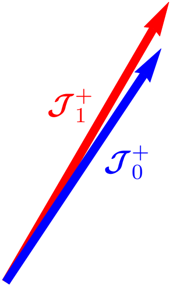
- Orthogonal initial coefficients $\neq$ orthogonal data

$$< \boldsymbol{\mathcal{J}}_0^+ | \boldsymbol{\mathcal{J}}_1^+ > \; \sim \; |\boldsymbol{\mathcal{J}}_0^+||\boldsymbol{\mathcal{J}}_1^+|$$



Orthogonalization: thin QR decomposition

$$V_o \equiv \begin{bmatrix} \boldsymbol{\mathcal{J}}_0^+ & \boldsymbol{\mathcal{J}}_1^+ \\ \partial_r \boldsymbol{\mathcal{J}}_0^+ & \partial_r \boldsymbol{\mathcal{J}}_1^+ \end{bmatrix} \qquad V_e \equiv \begin{bmatrix} \boldsymbol{\mathcal{H}}_0^+ & \boldsymbol{\mathcal{H}}_1^+ & \boldsymbol{\mathcal{H}}_2^+ & \boldsymbol{\mathcal{H}}_3^+ \\ \partial_r \boldsymbol{\mathcal{H}}_0^+ & \partial_r \boldsymbol{\mathcal{H}}_1^+ & \partial_r \boldsymbol{\mathcal{H}}_2^+ & \partial_r \boldsymbol{\mathcal{H}}_3^+ \end{bmatrix}$$

$$V_{e/o} = Q_{e/o} R_{e/o} \qquad Q^\dagger Q = I$$

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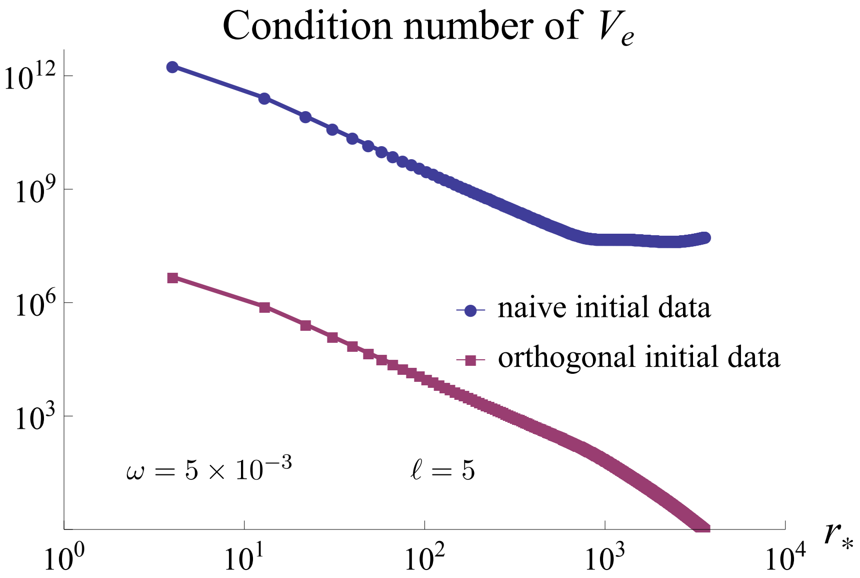
$$V_{e/o} = Q_{e/o} R_{e/o} \qquad Q^\dagger Q = I$$

- The columns of $Q_{e/o}$ are an orthogonal basis
- The ill-conditioning of $V_{e/o}$ causes this basis to have low accuracy
- Instead use $R_{e/o}^{-1}$ to choose new starting coefficients

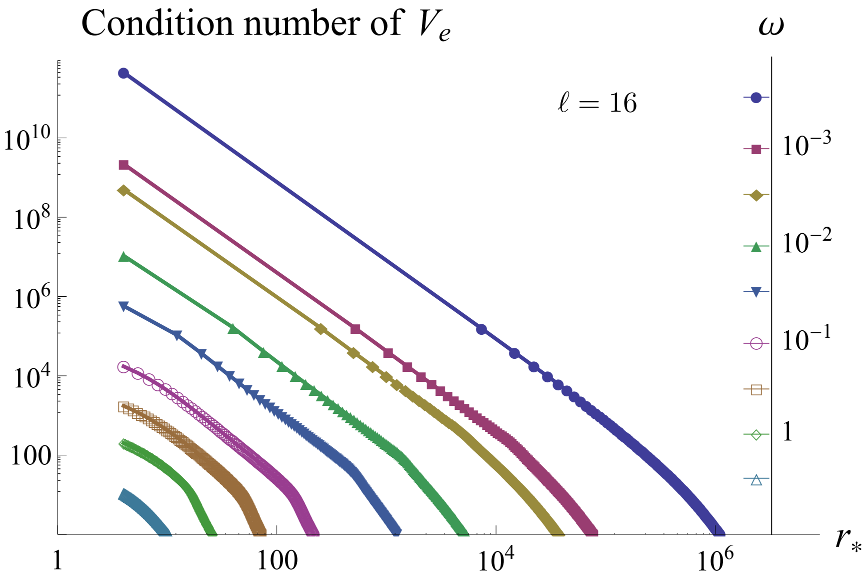
$$V_{e/o} R_{e/o}^{-1} = Q_{e/o}$$

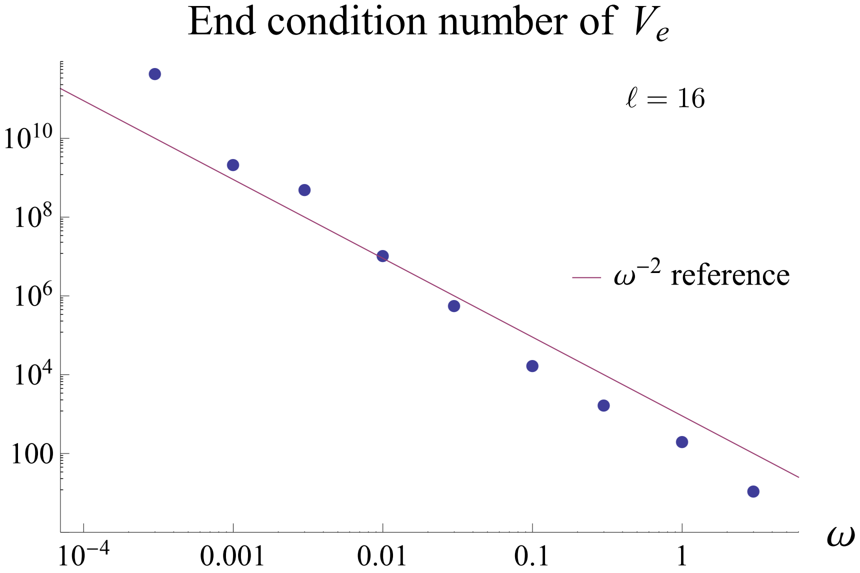
$$\begin{bmatrix} a_0^{(0)} \\ a_1^{(1)} \end{bmatrix}_i^{\text{new}} = R_o^{-1} \qquad \begin{bmatrix} b_0^{(0)} \\ b_1^{(0)} \\ b_2^{(0)} \\ b_0^{(3)} \end{bmatrix}_i^{\text{new}} = R_e^{-1}$$

Orthogonal initial conditions



Condition number




End condition number of V_e
$\ell = 16$
ω^{-2} reference
10^{10}
10^{8}
10^{6}
10^{4}
100
10^{-4}
0.001
0.01
0.1
1
ω

Adaptive stepsize accuracy tolerance

Integrate Wronskian and normalization functions in shared ODE system

$$\vec{y} = \begin{bmatrix} \boldsymbol{\mathcal{J}}_0^+ \\ \partial_r \boldsymbol{\mathcal{J}}_0^+ \\ \boldsymbol{\mathcal{J}}_1^+ \\ \partial_r \boldsymbol{\mathcal{J}}_1^+ \\ \boldsymbol{\mathcal{J}}_0^- \\ \partial_r \boldsymbol{\mathcal{J}}_0^- \\ \boldsymbol{\mathcal{J}}_1^- \\ \partial_r \boldsymbol{\mathcal{J}}_1^- \\ - - - \\ c_0^+ \\ c_1^+ \\ c_0^- \\ c_1^- \end{bmatrix}$$

Adaptive stepsize accuracy tolerance

Integrate Wronskian and normalization functions in shared ODE system

$$\vec{y} = \begin{bmatrix} \boldsymbol{\mathcal{J}}_0^+ \\ \partial_r \boldsymbol{\mathcal{J}}_0^+ \\ \boldsymbol{\mathcal{J}}_1^+ \\ \partial_r \boldsymbol{\mathcal{J}}_1^+ \\ \boldsymbol{\mathcal{J}}_0^- \\ \partial_r \boldsymbol{\mathcal{J}}_0^- \\ \boldsymbol{\mathcal{J}}_1^- \\ \partial_r \boldsymbol{\mathcal{J}}_1^- \\ - - - \\ c_0^+ \\ c_1^+ \\ c_0^- \\ c_1^- \end{bmatrix}$$

$\Big\}$ Demand high accuracy Wronskian components $\sim 10^{-15}$

Adaptive stepsize accuracy tolerance

Integrate Wronskian and normalization functions in shared ODE system

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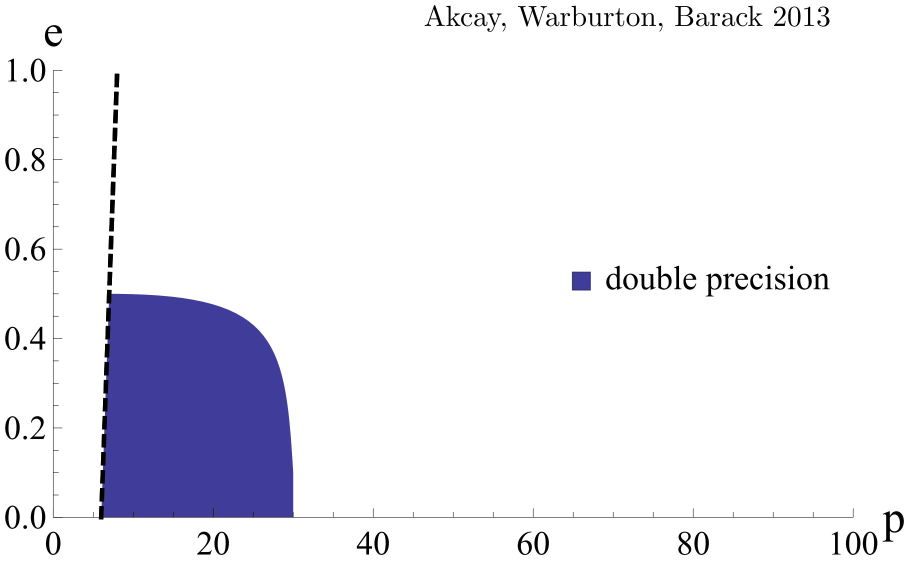
} Demand high accuracy Wronskian components $\sim 10^{-15}$

} Ignore errors unless accuracy falls below $\sim 10^{-10}$

Parameter space open to double precision



Brute force solution: 128-bit floating point numbers

- Compute smaller frequency modes by increasing digits of accuracy such that

$$\textit{condition number} \times \%\, \textit{accuracy} \ll 1$$

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Brute force solution: 128-bit floating point numbers

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- Quad precision homgeneous solutions with accuracy tolerance $\sim 10^{-12}\,(M\omega)^2$

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Brute force solution: 128-bit floating point numbers

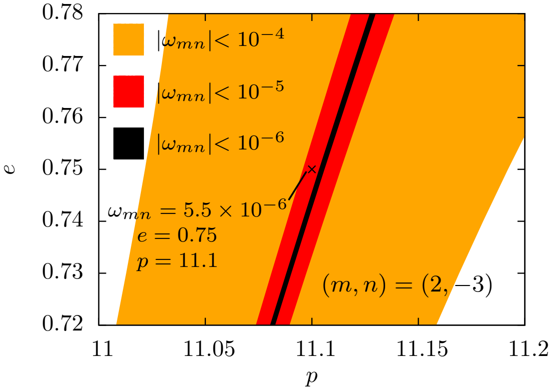
- Compute smaller frequency modes by increasing digits of accuracy such that

$$\textit{condition number} \times \%\,\textit{accuracy} \ll 1$$

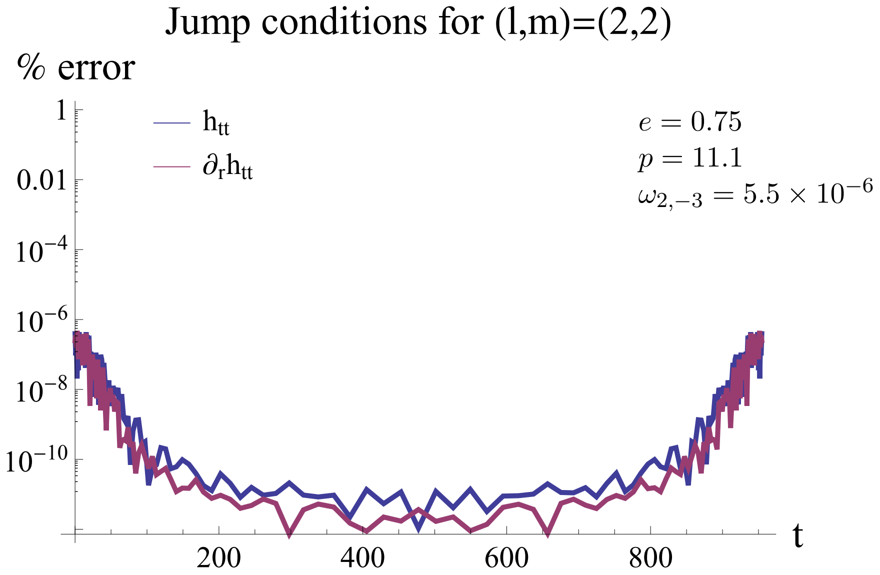
- Quad precision homgeneous solutions with accuracy tolerance $\sim 10^{-12}\,(M\omega)^2$

- Test orbit:

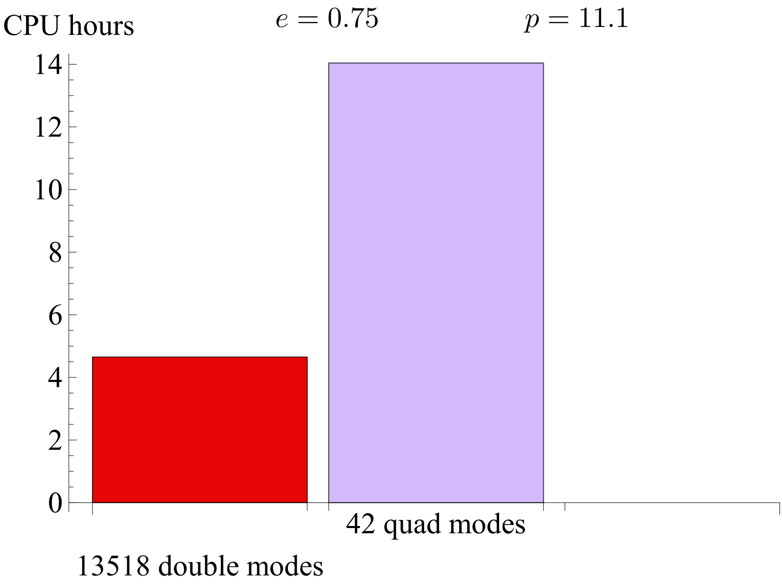
$e = 0.75$
$p = 11.1$



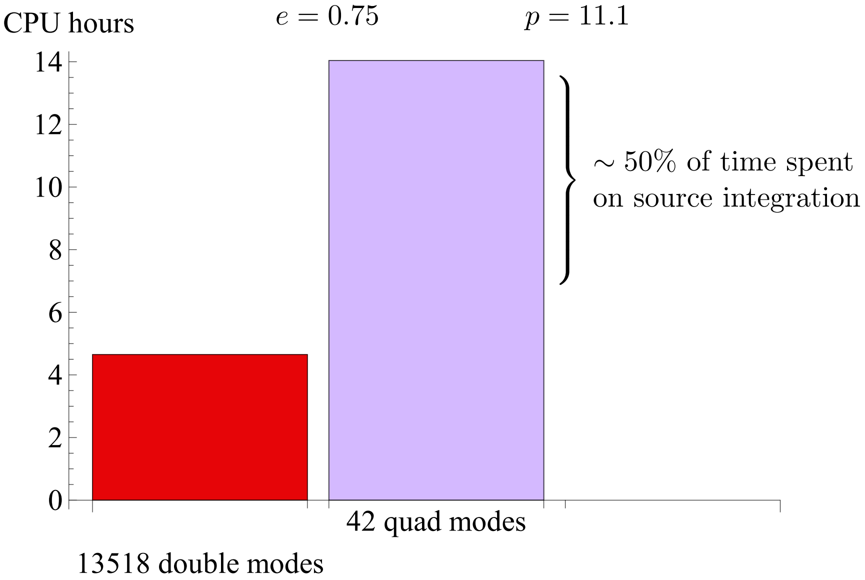
Small frequency mode: test jump conditions



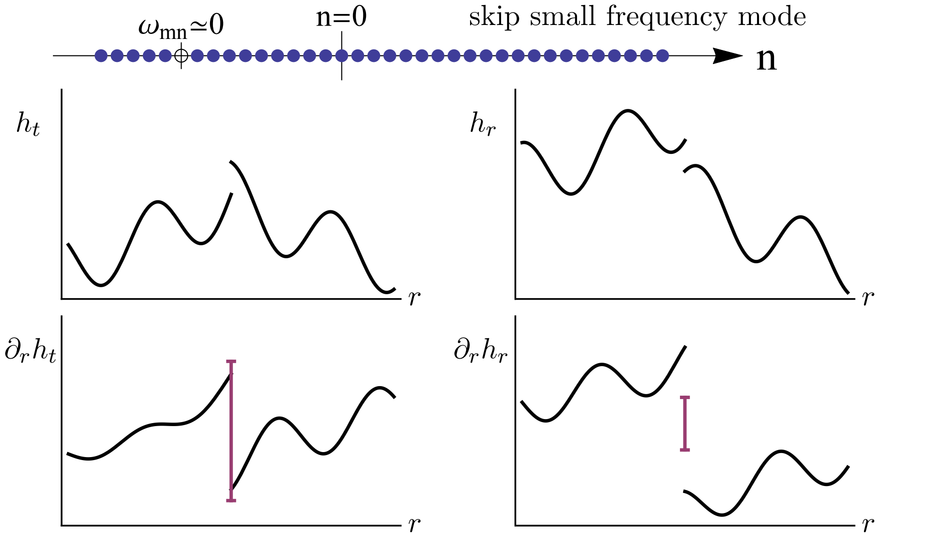
Quad precision is slow



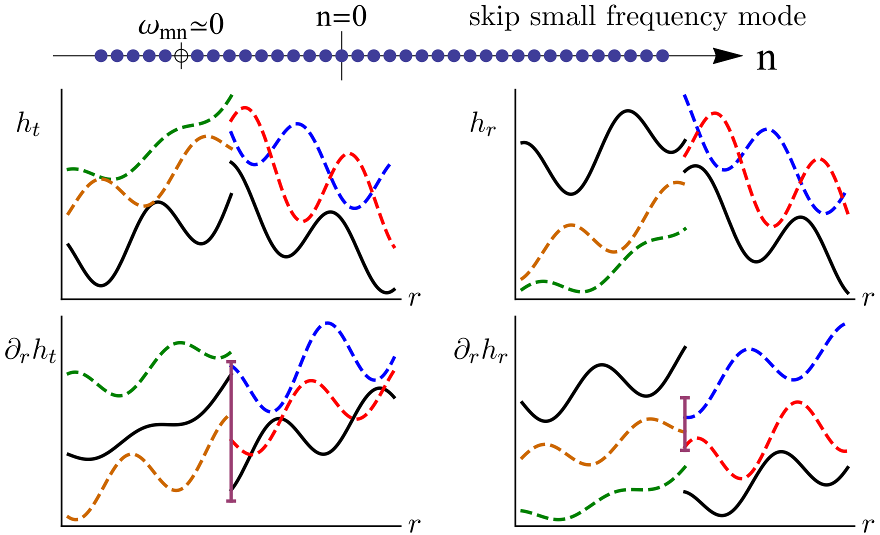
Quad precision is slow



Small frequency EHS constants: Alternative method



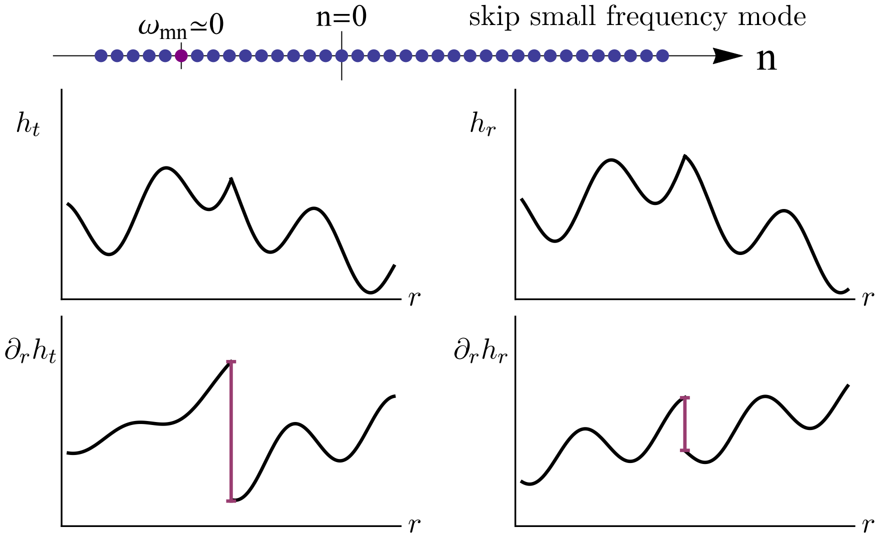
Small frequency EHS constants: Alternative method



$$\mathcal{J}^-_{\omega\simeq 0} = C^-_0\,\mathcal{J}^-_0 + C^-_1\,\mathcal{J}^-_1$$

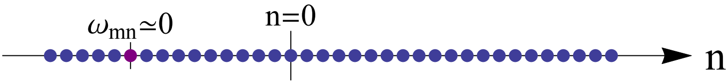
$$\mathcal{J}^+_{\omega\simeq 0} = C^+_0\,\mathcal{J}^+_0 + C^+_1\,\mathcal{J}^+_1$$

Small frequency EHS constants: Alternative method



$$\mathcal{J}^-_{\omega\simeq 0} = C_0^- \mathcal{J}_0^- + C_1^- \mathcal{J}_1^- \qquad \mathcal{J}^+_{\omega\simeq 0} = C_0^+ \mathcal{J}_0^+ + C_1^+ \mathcal{J}_1^+$$

Small frequency EHS constants: Alternative method



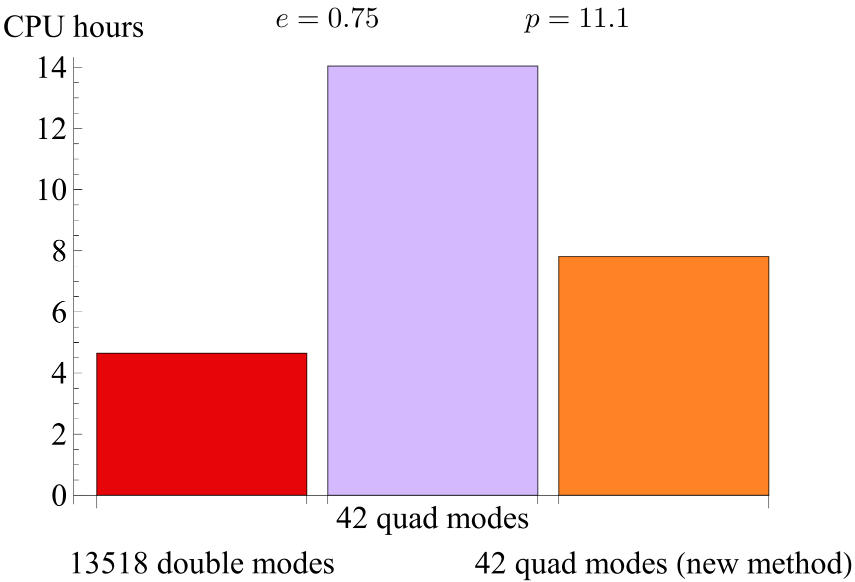
- Skip mode with very small frequency ω_0
- Solve for its normalization coefficients with jump conditions

$$e^{-i\omega_0 t}\begin{bmatrix} \mathcal{J}_0^+ & \mathcal{J}_1^+ & -\mathcal{J}_0^- & -\mathcal{J}_1^- \\ \partial_r\mathcal{J}_0^+ & \partial_r\mathcal{J}_1^+ & -\partial_r\mathcal{J}_0^- & -\partial_r\mathcal{J}_1^- \end{bmatrix}\begin{bmatrix} C_0^+ \\ C_1^+ \\ C_0^- \\ C_1^- \end{bmatrix} =$$

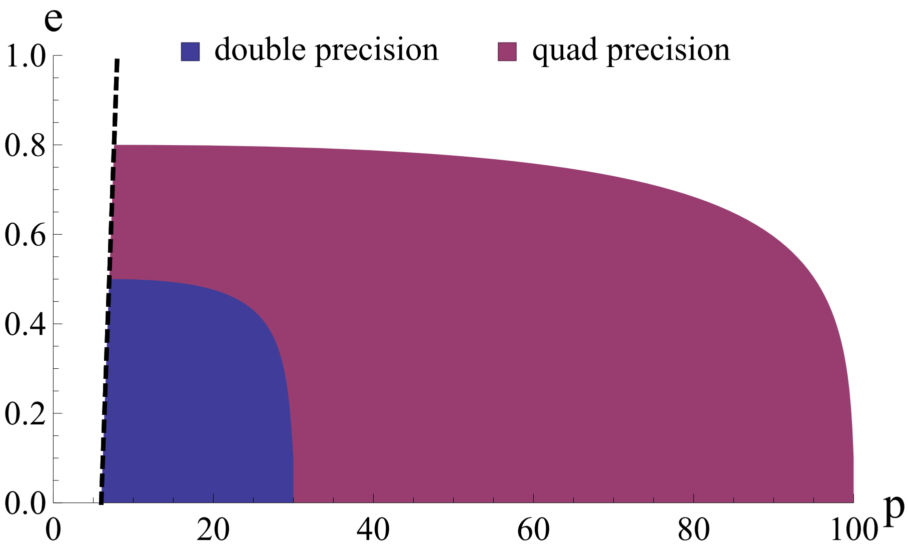
$$\begin{bmatrix} 0 \\ 0 \\ J_0(t) \\ J_1(t) \end{bmatrix} - \sum_{\omega\neq\omega_0}\left(\begin{bmatrix} \mathcal{J}_\omega^+ \\ \partial_r\mathcal{J}_\omega^+ \end{bmatrix} - \begin{bmatrix} \mathcal{J}_\omega^- \\ \partial_r\mathcal{J}_\omega^- \end{bmatrix}\right) e^{-i\omega t}$$

- Does not circumvent ill-conditioning but improves speed

Quad precision speed improvement



Parameter space open to double/quad precision



Conclusions

- Existing methods based on double precision codes cannot handle orbits with small frequency modes
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Conclusions

- Existing methods based on double precision codes cannot handle orbits with small frequency modes
- This restricts the available region of orbital parameter space
- We conquer ill-conditioned small frequency modes via brute force (quad precision)
- Much larger region of orbital parameter space available

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UNC The Graduate School

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