APPLYING THE EFFECTIVE-SOURCE APPROACH TO FREQUENCY-DOMAIN SELF-FORCE CALCULATIONS

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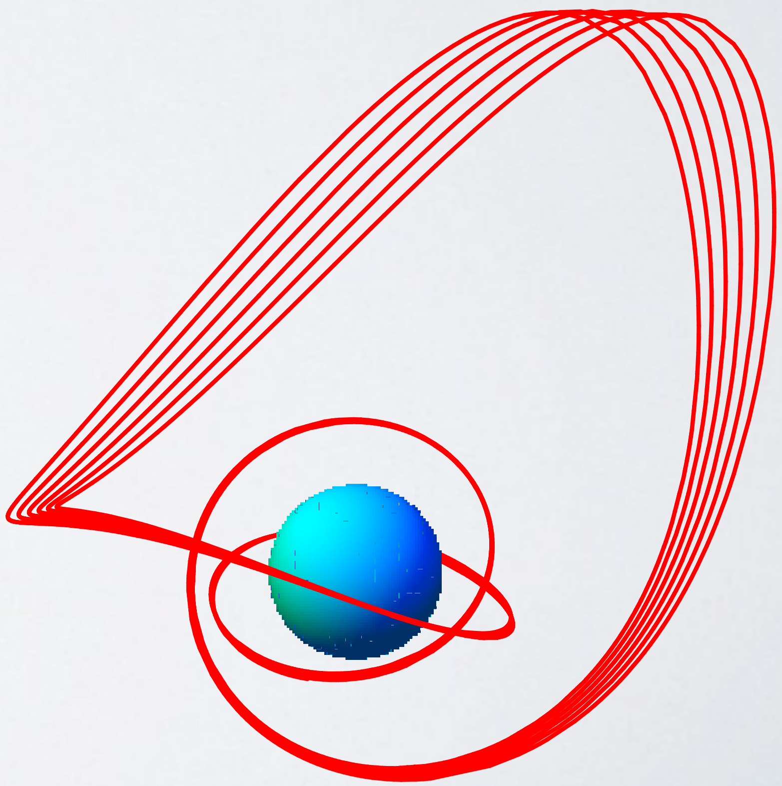
Talk outline

- Motivation
- Scalar-field example
  - Puncture
  - World-tube vs window-function
  - Relation to mode-sum
  - Results
- Lorenz-gauge gravity
  - Puncture
  - Window-function
  - Results
- Road to second-order



$$h^{(1)}_{\alpha\beta} \propto \frac{1}{r} \xrightarrow{S.H.D.} r^0$$

$$h^{(2)}_{\alpha\beta} \propto \frac{1}{r^2} \xrightarrow{S.H.D.} \log(r)$$

~~mode sum~~

effective-source

APPLYING THE EFFECTIVE-SOURCE APPROACH TO FREQUENCY-DOMAIN SELF-FORCE CALCULATIONS

- Modelling I/EMRIs
- 2nd-order self-force needed for phase evolution to O(1)
- Compute gauge invariants

- Avoid instabilities in the low multipole modes
- High accuracy: only need to solve ODEs

Effective-source: scalar-field

$$F_\alpha = \nabla_\alpha \Phi^R \qquad \Box \Phi^{\mathrm{ret}/S} = -4\pi\rho \qquad \Box\Phi^R = 0$$

Basic idea is to move the singular piece into the source

$$\Box\Phi^{\mathrm{ret}} = \Box(\Phi^R + \Phi^S) \quad \longrightarrow \quad \Box\Phi^{\mathrm{res}} = -4\pi\rho - \Box(\mathcal{W}\Phi^P)$$

Next decompose everything into spherical harmonics and frequency modes

$$\Box_{lm}\phi^{\mathrm{res}} = \kappa_{lm}\delta(r - r_0) - \Box_{lm}(\mathcal{W}\phi^P_{lm}) \equiv S^{\mathrm{eff}}_{lm}$$

Variation of parameters gives the inhomogeneous solutions

$$\phi^{\mathrm{res}}_{lm}(r) = c^{+\mathrm{res}}_{lm}(r)\tilde{\phi}^{+}_{lm}(r) + c^{-\mathrm{res}}_{lm}(r)\tilde{\phi}^{-}_{lm}(r)$$

$$c^{\pm\mathrm{res}}_{lm}(r) = \int \frac{\tilde{\phi}^{\mp}_{lm}(r')}{W(r')} S^{\mathrm{eff}}_{lm}\, dr'$$

Effective-source: scalar-field

Decompose the approximation to the Detweiler-Whiting singular field into spherical-harmonic and Fourier modes

$$\tilde{\Phi}^{\mathrm{S}} = \sum_{\ell m \omega} \Phi^{\mathrm{S}}_{\ell m \omega}(r) Y_{\ell m}(\theta, \phi) e^{-i\omega t}$$

We do this decomposition analytically with methods similar to those used in mode-sum regularisation

In a coordinate system where the world line is on the north pole

$$\Phi^{\mathrm{S}}_{l,m'=0} = -\frac{(2l+1)|\Delta r|}{2r_0(r_0-2M)}\sqrt{1-\frac{3M}{r_0}}\left[1-\frac{(r_0-M)\Delta r}{r_0(r_0-2M)}\right] + \frac{1}{\pi r_0}\sqrt{\frac{r_0-3M}{(r_0-2M)}}\left[2\mathcal{K}+\frac{(\mathcal{E}-2\mathcal{K})}{r_0}\Delta r+\frac{(2l+1)^2\mathcal{E}}{4r_0(r_0-2M)}\Delta r^2\right]$$

Spherical-harmonic modes in unrotated coordinate system (where particle is on an equatorial orbit) obtained by rotating using Wigner-D symbol

$$\Phi^{\mathrm{S}}_{lm} = \sum_{m'=-\ell}^{\ell} \Phi^{\mathrm{S}}_{lm'} D^{\ell}_{mm'}(0, \pi/2, \Omega t)$$

Effective-source: scalar-field

Standard mode-sum frequency-domain approach

$$\boxed{\left[\frac{d^2}{dr^2}+\frac{2(r-M)}{fr^2}\frac{d}{dr}+\frac{1}{f}\left(\frac{\omega^2}{f}-\frac{\ell(\ell+1)}{r^2}\right)\right]\Phi^{\rm ret}_{\ell m}=\alpha_{\ell m}\delta(r-r_0)}$$

Find solutions to homogeneous equation which satisfy outgoing boundary conditions on horizon and at infinity, respectively

$$\boxed{\left[\frac{d^2}{dr^2}+\frac{2(r-M)}{fr^2}\frac{d}{dr}+\frac{1}{f}\left(\frac{\omega^2}{f}-\frac{\ell(\ell+1)}{r^2}\right)\right]\tilde{\Phi}^{\rm ret\pm}_{\ell m}=0}$$

Construct inhomogeneous solutions by matching on the world line

$$\boxed{\Phi^{\rm ret\pm}_{\ell m}=c^{\pm}_{\ell m}\tilde{\Phi}^{\rm ret\pm}_{\ell m}} \qquad \boxed{c^{\pm}_{\ell m}=\alpha_{\ell m}\frac{\tilde{\Phi}^{\mp}_{\ell m}}{W}}$$

where W is the Wronskian of homogeneous solutions

Effective-source: scalar-field

Effective-source in the frequency-domain

$$\left[\frac{d^2}{dr^2} + \frac{2(r-M)}{fr^2}\frac{d}{dr} + \frac{1}{f}\left(\frac{\omega^2}{f} - \frac{\ell(\ell+1)}{r^2}\right)\right]\Phi^{\text{ret}}_{\ell m} = S^{\text{eff}}_{\ell m}$$

Find solutions to homogeneous equation which satisfy outgoing boundary conditions on horizon and at infinity, respectively

$$\left[\frac{d^2}{dr^2} + \frac{2(r-M)}{fr^2}\frac{d}{dr} + \frac{1}{f}\left(\frac{\omega^2}{f} - \frac{\ell(\ell+1)}{r^2}\right)\right]\tilde{\Phi}^{\text{ret}\pm}_{\ell m} = 0$$

Construct inhomogeneous solutions using variation of parameters

$$\Phi^{\text{ret}}_{\ell m} = c^{+}_{\ell m}(r)\tilde{\Phi}^{\text{ret}+}_{\ell m} + c^{-}_{\ell m}(r)\tilde{\Phi}^{\text{ret}-}_{\ell m}$$

$$c^{+}_{\ell m}(r) = \int_{2M}^{r} \frac{\tilde{\phi}^{-}(r')}{W(r')} S^{\text{eff}}_{\ell m}\, dr', \quad c^{-}_{\ell m}(r) = \int_{r}^{\infty} \frac{\tilde{\phi}^{+}(r')}{W(r')} S^{\text{eff}}_{\ell m}\, dr'$$

where W is the Wronskian of homogeneous solutions

Effective-source: scalar-field

Window-function and world-tube equivalence

Detweiler-Whiting singular field defined through Hadamard form Green function which is not defined globally

Need to introduce a method for restricting the singular field to a region near the particle. Two common approaches:

Window-function:

Multiply the singular field by a function which is 1 near the particle and goes to 0 far away

$$\boxed{\Box\Phi^{\mathrm{R}} = -\Box(W\Phi^{\mathrm{S}})}$$

World-tube:

World tube around the particle

Inside solve for the R-field, outside solve for the retarded field

On the world tube boundary apply the boundary condition

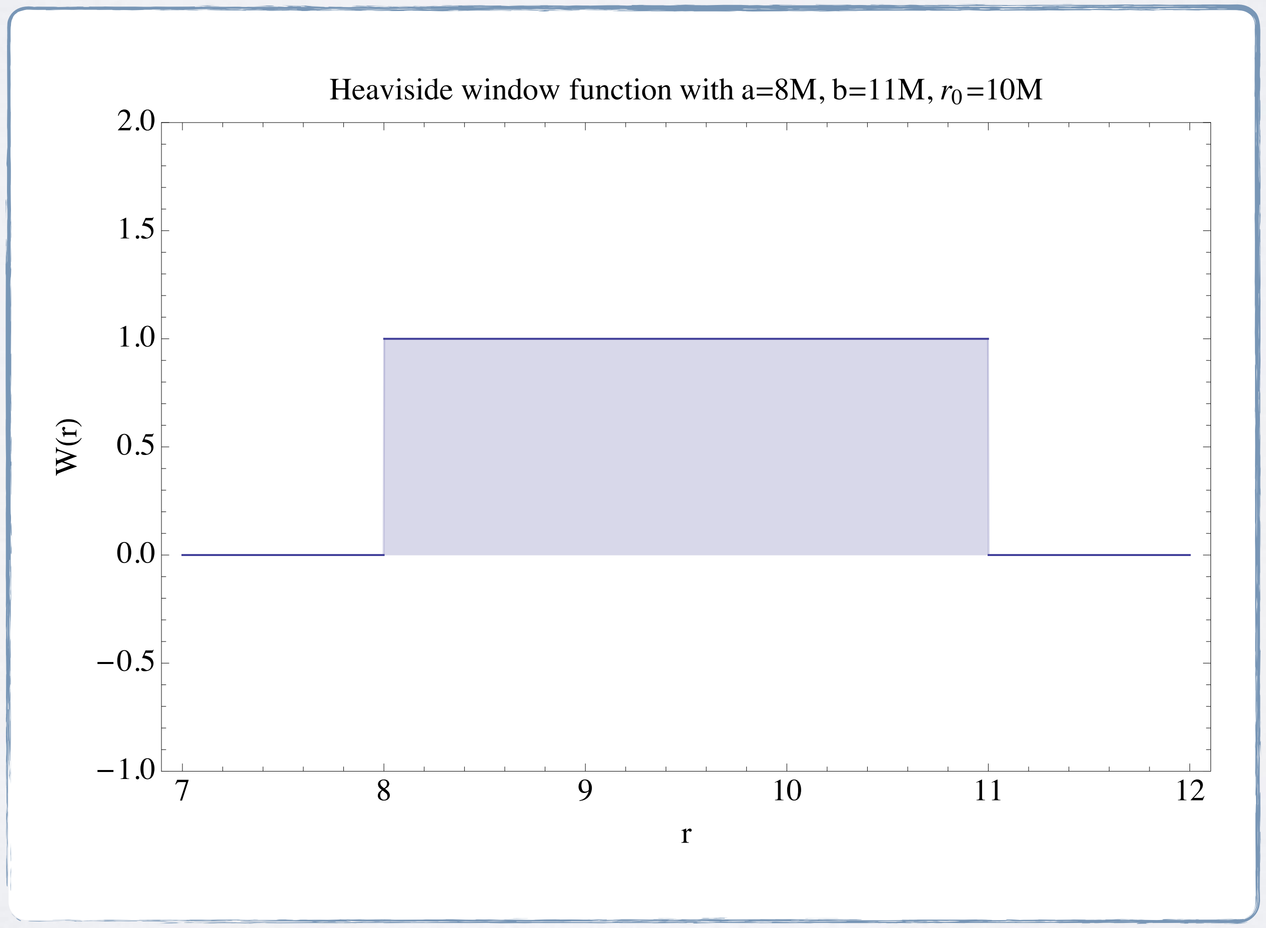
$$\boxed{\Phi^{\mathrm{ret}} = \Phi^{\mathrm{S}} + \Phi^{\mathrm{R}}}$$

Effective-source: scalar-field

Window-function and world-tube equivalence

Both approaches can be shown to be equivalent in the frequency-domain by choosing a Heaviside distribution as the window function

$$W(r) = \Pi\left(\frac{\Delta r - (b+a-2r_0)/2}{b-a}\right) = \begin{cases} 1 & \left|\frac{\Delta r-(b+a-2r_0)/2}{b-a}\right| < 1/2 \\ 0 & \left|\frac{\Delta r-(b+a-2r_0)/2}{b-a}\right| > 1/2 \end{cases}$$



Effective-source: scalar-field

Window-function and world-tube equivalence

Effective-source splits into two terms, one coming from the interior of the puncture region and the other from the boundary of the puncture

$$\boxed{S^{\text{eff}}_{lm} = -\Box_{lm}(\mathcal{W}\Phi^P_{lm}) \equiv S^I_{lm}\Pi(x) + S^B_{lm}}$$

where

$$\begin{aligned}
S^I_{lm} &= -\frac{d^2\Phi^P_{lm}}{r^2} - \frac{2(r-M)}{fr^2}\frac{d\Phi^P_{lm}}{dr} + \frac{1}{f}\left(\frac{2}{f} + \frac{l(l+1)}{r^2}\right)\Phi^P_{lm} \\
S^B_{lm} &= -\left[\frac{\delta'(x_a) + \delta'(-x_b)}{(b-a)^2} + \frac{2(r-M)\left(\delta(x_a) - \delta(x_b)\right)}{fr^2(b-a)}\right]\Phi^P_{lm} - \frac{2\left(\delta(x_a) - \delta(x_b)\right)}{b-a}\frac{d\Phi^P_{lm}}{dr}
\end{aligned}$$

$$\boxed{x_a = \frac{a-r}{a-b}, \quad x_b = \frac{b-r}{a-b}}$$

Effective-source: scalar-field

Window-function and world-tube equivalence

Integrating the delta-function terms analytically, we find that the scaling coefficients are equivalent to world tube jumps

$$c^+(r) = \begin{cases} 0 & r < a \\ L_B(\phi^-/W) & a \le r < b \\ L_B(\phi^-/W) + R_B(\phi^-/W) & r \ge b \end{cases} + \Pi(x(r)) \int_a^r \frac{\tilde{\phi}^-}{W} S^I_{\text{eff}}\, dr$$

$$c^-(r) = \begin{cases} 0 & r > b \\ R_B(\phi^+/W) & b \ge r > a \\ L_B(\phi^+/W) + R_B(\phi^+/W) & r \le a \end{cases} + \Pi(x(r)) \int_r^b \frac{\tilde{\phi}^+}{W} S^I_{\text{eff}}\, dr$$

$$L_B[f(r)] = \int_{a^-}^{a^+} f(r) S^B_{\text{eff}}\, dr = \alpha(a) f(a) + \beta(a) f'(a)$$

$$R_B[f(r)] = \int_{b^-}^{b^+} f(r) S^B_{\text{eff}}\, dr = -\alpha(b) f(b) - \beta(b) f'(b)$$

$$\alpha(x) = -\frac{2(x-M)}{x(x-2M)} \Phi^P_{lm}(x) - \frac{d\Phi^P_{\ell m}}{dr}(x)$$

$$\beta(x) = \Phi^P_{lm\omega}(x)$$

Effective-source: scalar-field

Relation to mode-sum scheme

Taking the limit of the world tube width to a point, i.e., a $\rightarrow$ *r_0, b $\rightarrow r_0$ we recover the usual Barack-Ori mode-sum scheme*

Effective-source turns to jump on the world line

$$\boxed{\begin{array}{ll} c_0^{+R} \equiv L_B \left[\dfrac{\tilde{\phi}^-}{W}\right]_{a=r_0^-} , & c_0^{-R} \equiv R_B \left[\dfrac{\tilde{\phi}^+}{W}\right]_{b=r_0^+} \\ c_0^{+S} \equiv R_B \left[\dfrac{\tilde{\phi}^-}{W}\right]_{a=r_0^\pm} , & c_0^{-S} \equiv L_B \left[\dfrac{\tilde{\phi}^+}{W}\right]_{b=r_0^\pm} \end{array}}$$

Recover standard mode-sum matching condition

$$\boxed{c_0^\pm = c_0^{\pm R} + c_0^{\pm S} \alpha_{lm} \frac{\tilde{\phi}_0^\mp}{W_0}}$$

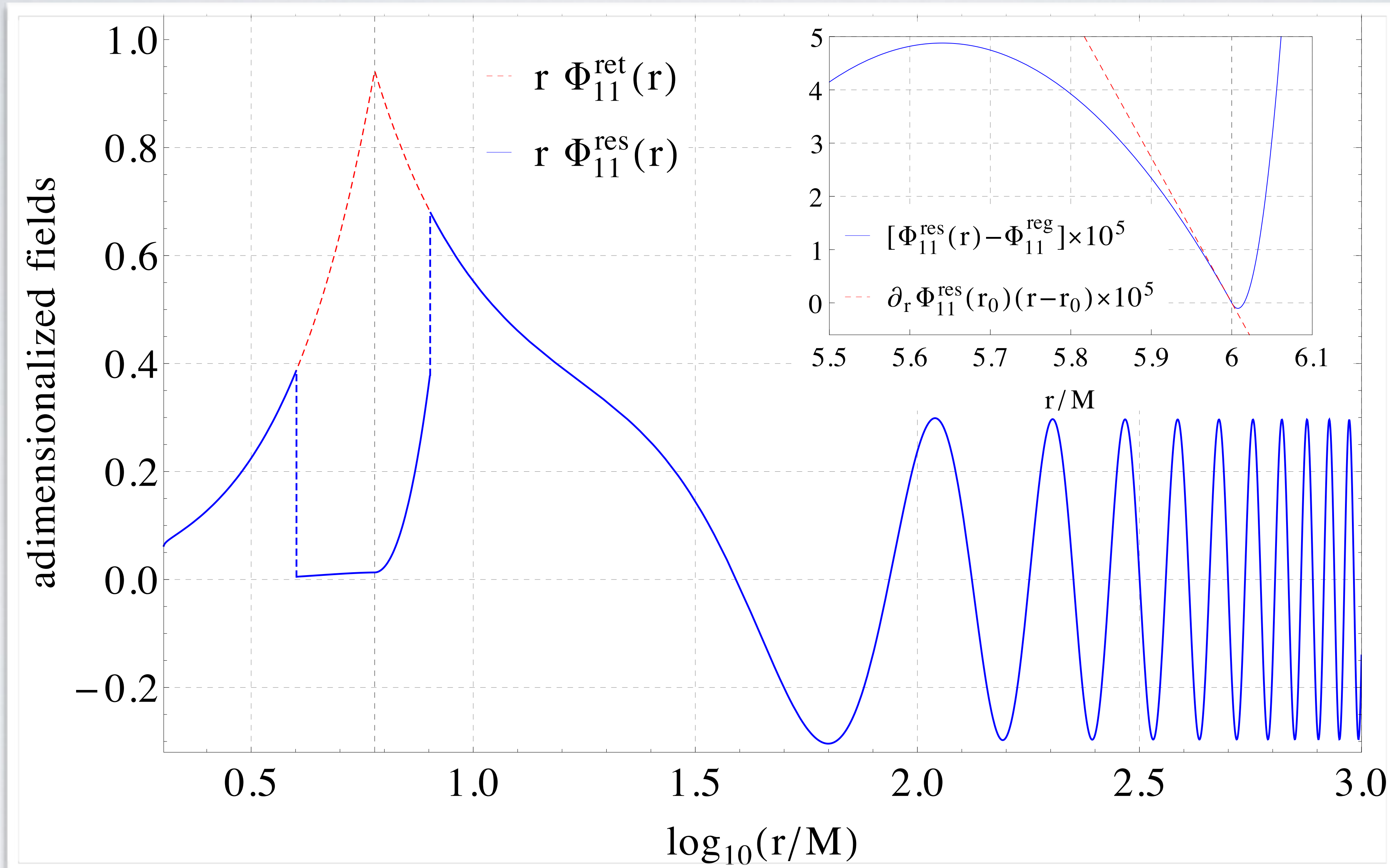
"Regularization parameters" and regularised field

$$\boxed{\phi_0^S = c_0^{+S} \tilde{\phi}_0^+ + c_0^{-S}}$$

$$\boxed{\phi_0^R = c_0^{+R} \tilde{\phi}_0^+ + c_0^{-R} \tilde{\phi}_0^-}$$

Results: scalar field

Phys. Rev. D. 89:044046
arXiv: 1311.3104



	r_0/M	eff. source$\times10^3$	mode-sum$\times10^3$	rel. diff.
Φ_0^{res}	6	5.454828078581	5.454828078597	3×10^{-12}
$\partial_r\Phi_0^{res}$	6	0.16772830795	0.16772830804	5×10^{-10}
Φ_0^{res}	10	-1.049793165979	-1.049793165983	4×10^{-12}
$\partial_r\Phi_0^{res}$	10	0.013784482250	0.013784482234	2×10^{-09}

Effective-source: Lorenz-gauge gravity

$$\Box \bar{h}_{\mu\nu} + 2R^{\alpha\ \beta}_{\ \mu\ \nu}\bar{h}_{\alpha\beta} = -16\pi T_{\mu\nu} \qquad \nabla_\mu \bar{h}^{\mu\nu} = 0$$

FD decomposition: $\bar{h}_{\mu\nu} = \dfrac{\mu}{r}\sum_{lm}\sum_{i=1}^{10} R^{(i)lm}(r) Y^{(i)lm}_{\mu\nu} e^{-im\Omega t}$

Radial field equations: $\Box_{lm} R^{(i)}_{lm} + 4\mathcal{M}^{(i)}_{\ (j)} R^{(j)}_{lm} = J^{(i)}_{lm}$

Decompose h^P into tensor harmonics and construct the effective-sources: $\bar{h}^{(i)P}_{lm} \qquad S^{(i)\mathrm{eff}}_{lm}$

Variations of parameters to find the residual field

Effective-source: Lorenz-gauge gravity

Punctures and relation to mode-sum

At first-order we can subtract the the punctures from the individual lmi-modes of the retarded field

$$\bar{h}_{\mu\nu} = \frac{\mu}{r}\sum_{lm}\sum_{i=1}^{10}(R^{(i)}_{lm} - \bar{h}^{(i)P}_{lm})Y^{(i)lm}_{\mu\nu}e^{-i\omega t}$$

Scalar:

$$\bar{h}^{(1)\mathrm{P}}_{\ell m} = 4(r_0+\Delta r)D^{\ell}_{m,0}\frac{1}{\sqrt{\pi(2\ell+1)}}\Bigg[\frac{(2\ell+1)(r_0-2M)\pi|\Delta r|}{r_0^{5/2}\sqrt{r_0-3M}} + \frac{2}{r_0^3}\sqrt{\frac{r_0-2M}{(r_0-3M)}}\Big\{2r_0(r_0-2M)\mathcal{K} + \big[(r_0-2M)\mathcal{E} - 2(r_0-4M)\mathcal{K}\big]\Delta r\Big\}\Bigg]$$

Vector:

$$\bar{h}^{(4)\mathrm{P}}_{\ell m} = 4\Big[D^{\ell}_{m,-1} - D^{\ell}_{m1}\Big]\sqrt{\frac{\ell(\ell+1)}{\pi(2\ell+1)}}\Bigg[\frac{(2\ell+1)\sqrt{M}\pi|\Delta r|}{r_0\sqrt{r_0-3M}} + 2\sqrt{\frac{r_0-2M}{Mr_0^3(r_0-3M)}}\Big\{2r_0(r_0-2M)(\mathcal{E}-\mathcal{K}) + \frac{M}{\ell(\ell+1)}\mathcal{K} + \Big[2(r_0-2M)\big((r_0-5M)\mathcal{K} - (r_0-4M)\mathcal{E}\big) + \frac{M}{\ell(\ell+1)}\big((r_0-2M)\mathcal{E} + 2M\mathcal{K}\big)\Big]\Delta r\Big\}\Bigg]$$

Effective-source: Lorenz-gauge gravity

Punctures and relation to mode-sum

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$$\bar{h}_{\mu\nu} = \frac{\mu}{r}\sum_{lm}\sum_{i=1}^{10}(R^{(i)}_{lm} - \bar{h}^{(i)P}_{lm})Y^{(i)lm}_{\mu\nu}e^{-i\omega t}$$

Scalar:

$$\bar{h}^{(1)\mathrm{P}}_{\ell m} = 4(r_0+\Delta r)D^{\ell}_{m,0}\frac{1}{\sqrt{\pi(2\ell+1)}}\left[\frac{(2\ell+1)(r_0-2M)\pi|\Delta r|}{r_0^{5/2}\sqrt{r_0-3M}}\right.$$
$$\left.+\frac{2}{r_0^3}\sqrt{\frac{r_0-2M}{(r_0-3M)}}\left\{2r_0(r_0-2M)\mathcal{K}+\left[(r_0-2M)\mathcal{E}-2(r_0-4M)\mathcal{K}\right]\Delta r\right\}\right]$$

Vector:

$$\bar{h}^{(4)\mathrm{P}}_{\ell m} = 4\left[D^{\ell}_{m,-1}-D^{\ell}_{m1}\right]\sqrt{\frac{\ell(\ell+1)}{\pi(2\ell+1)}}\left[\frac{(2\ell+1)\sqrt{M}\pi|\Delta r|}{r_0\sqrt{r_0-3M}}\right.$$
$$+2\sqrt{\frac{r_0-2M}{Mr_0^3(r_0-3M)}}\left\{2r_0(r_0-2M)(\mathcal{E}-\mathcal{K})+\frac{3M}{(2\ell-1)(2\ell+3)}\mathcal{K}\right.$$
$$\left.\left.+\left[2(r_0-2M)\big((r_0-5M)\mathcal{K}-(r_0-4M)\mathcal{E}\big)+\frac{3M}{(2\ell-1)(2\ell+3)}\big((r_0-2M)\mathcal{E}+2M\mathcal{K}\big)\right]\Delta r\right\}\right]$$

Effective-source: Lorenz-gauge gravity

At first-order we can re-write this as a mode-sum formula

$$\bar{h}^R_{\mu\nu} = \left[\sum_{l=0}^{\infty}\left(\sum_{mi}\frac{\mu}{r}R^{(i)}_{lm}Y^{(i)lm}_{\mu\nu}e^{-i\omega t}\right) - B\right] - D$$

Can regularize the tensor-harmonic modes directly!

Compare with scalar-harmonic regularisation formula:

$$h^{lm}_{\alpha\beta}u^\alpha u^\beta = \left\{\mathcal{G}^{l+2,m}_{(+2)} + \mathcal{G}^{l+1,m}_{(+1)} + \mathcal{G}^{lm}_{(0)} + \mathcal{G}^{l-1,m}_{(-1)} + \mathcal{G}^{l-2,m}_{(-2)}\right\}Y^{lm}$$

$$\begin{aligned}
\mathcal{G}^{lm}_{(+2)} &= r^2(u^\varphi)^2\left[\alpha^{lm}_{(-2)}\bar{h}^{(3)} - \frac{(l-2)!}{(l+2)!}\left(\gamma^{lm}_{(-2)} - \beta^{lm}_{(-2)}\right)\bar{h}^{(7)}\right], \\
\mathcal{G}^{lm}_{(+1)} &= 2imr^2(u^\varphi)^2\frac{(l-2)!}{(l+2)!}\epsilon^{lm}_{(-1)}\bar{h}^{(10)} - \frac{2ru^t u^\varphi}{l(l+1)}\delta^{lm}_{(-1)}\bar{h}^{(8)} - \frac{2ru^r u^\varphi}{fl(l+1)}\delta^{lm}_{(-1)}\bar{h}^{(9)}, \\
\mathcal{G}^{lm}_{(0)} &= \left(\bar{h}^{(1)} + f\bar{h}^{(6)}\right)(u^t)^2 + 2f^{-1}\bar{h}^{(2)}u^t u^r + f^{-2}\left(\bar{h}^{(1)} - f\bar{h}^{(6)}\right)(u^r)^2 \\
&\quad + \frac{2imr\bar{h}^{(4)}}{l(l+1)}u^t u^\varphi + \frac{2imr\bar{h}^{(5)}}{fl(l+1)}u^r u^\varphi \\
&\quad + r^2(u^\varphi)^2\left[\alpha^{lm}_{(0)}\bar{h}^{(3)} - \frac{(l-2)!}{(l+2)!}\left(\gamma^{lm}_{(0)} - \beta^{lm}_{(0)} + m^2\right)\bar{h}^{(7)}\right], \\
\mathcal{G}^{lm}_{(-1)} &= 2imr^2\frac{(l-2)!}{(l+2)!}\epsilon^{lm}_{(+1)}\bar{h}^{(10)}(u^\varphi)^2 - \frac{2r\bar{h}^{(8)}}{l(l+1)}\delta^{lm}_{(+1)}u^t u^\varphi - \frac{2r\bar{h}^{(9)}}{fl(l+1)}\delta^{lm}_{(+1)}u^r u^\varphi, \\
\mathcal{G}^{lm}_{(-2)} &= r^2(u^\varphi)^2\left[\alpha^{lm}_{(+2)}\bar{h}^{(3)} - \frac{(l-2)!}{(l+2)!}\left(\gamma^{lm}_{(+2)} - \beta^{lm}_{(+2)}\right)\bar{h}^{(7)}\right].
\end{aligned}$$

Effective-source: Lorenz-gauge gravity

Effective-source and window-function

Use a Gaussian window-function: it's effectively compact for our purposes

$$\mathcal{W} = e^{-8\Delta r^2}$$

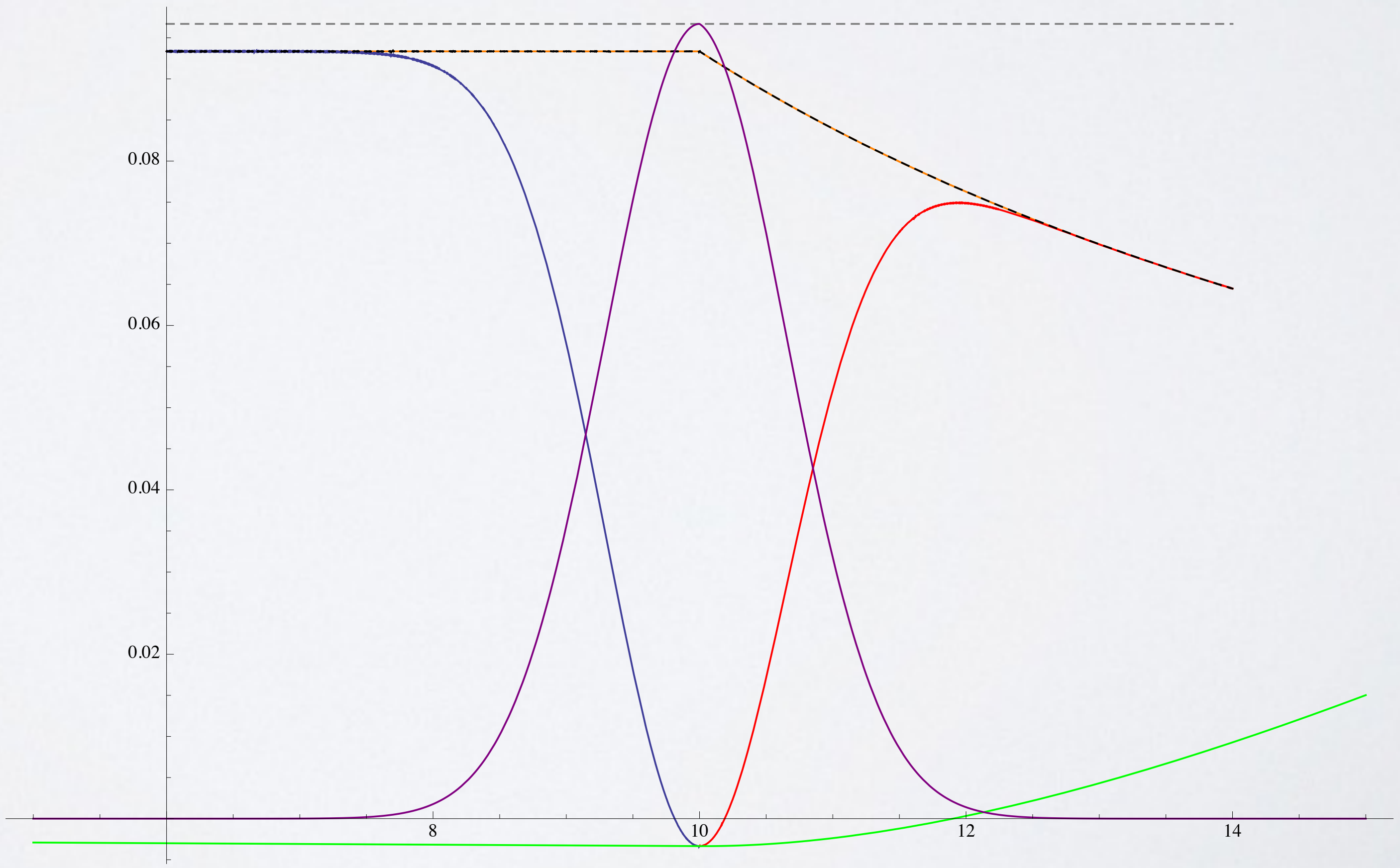
Radial field equations:

$$\Box_{lm} R^{(i)}_{lm} + 4\mathcal{M}^{(i)}_{\ (j)} R^{(j)}_{lm} = J^{(i)}_{lm}$$

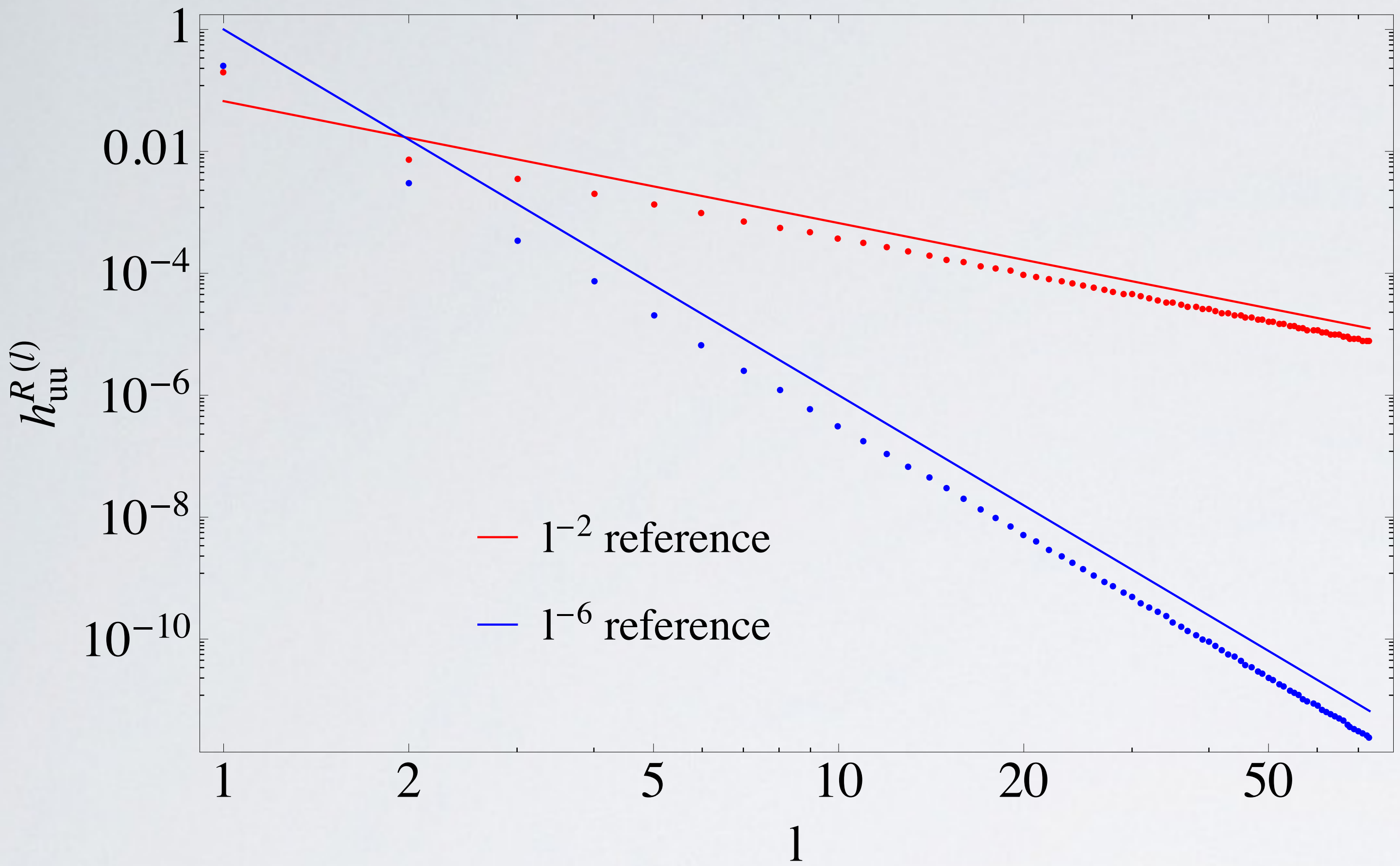
Construct effective-source:

$$S^{(i)\mathrm{eff}}_{lm} = \Box_{lm} \bar{h}^{(i)P}_{lm} + 4\mathcal{M}^{(i)}_{\ (j)} \bar{h}^{(i)P}_{lm}$$

Variation of parameters with multiple fields:



Effective-source: Lorenz-gauge gravity



With zero-width world-tube

Regular: -0.48925800245472

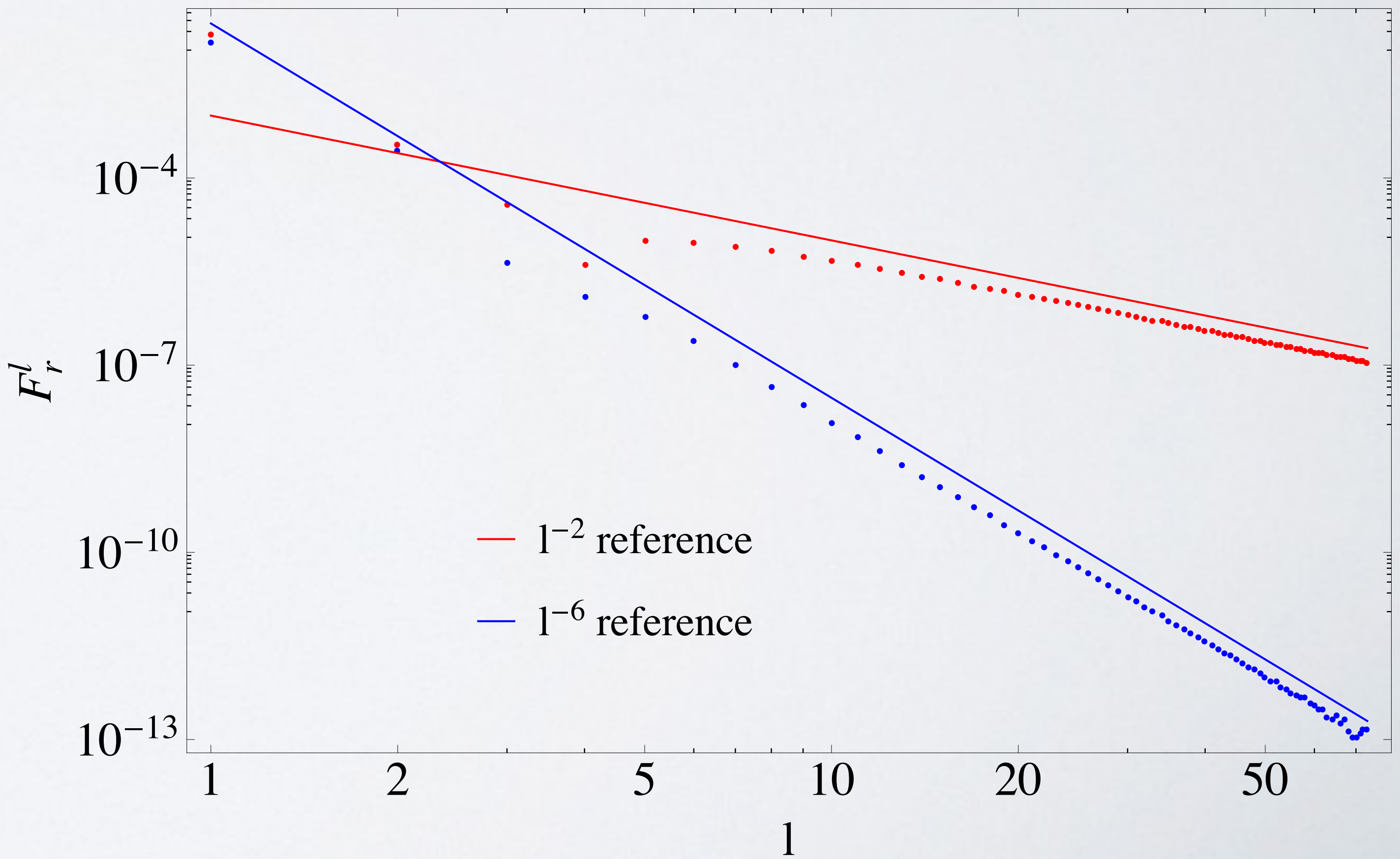
Residual: -0.48925800172

Difference: -1.50172*10^-9

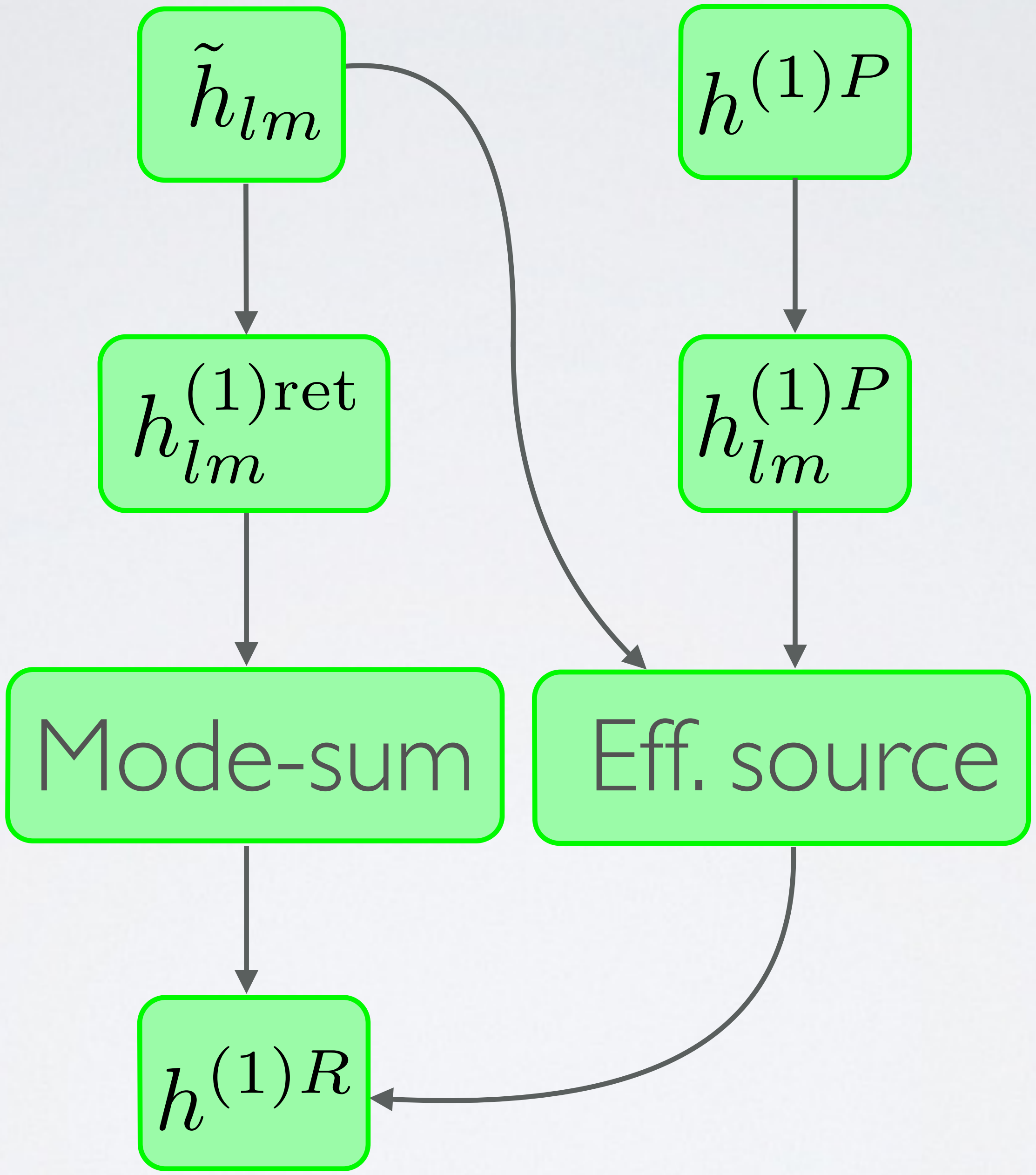
Regular: 0.016736836920

Residual: 0.016736836875

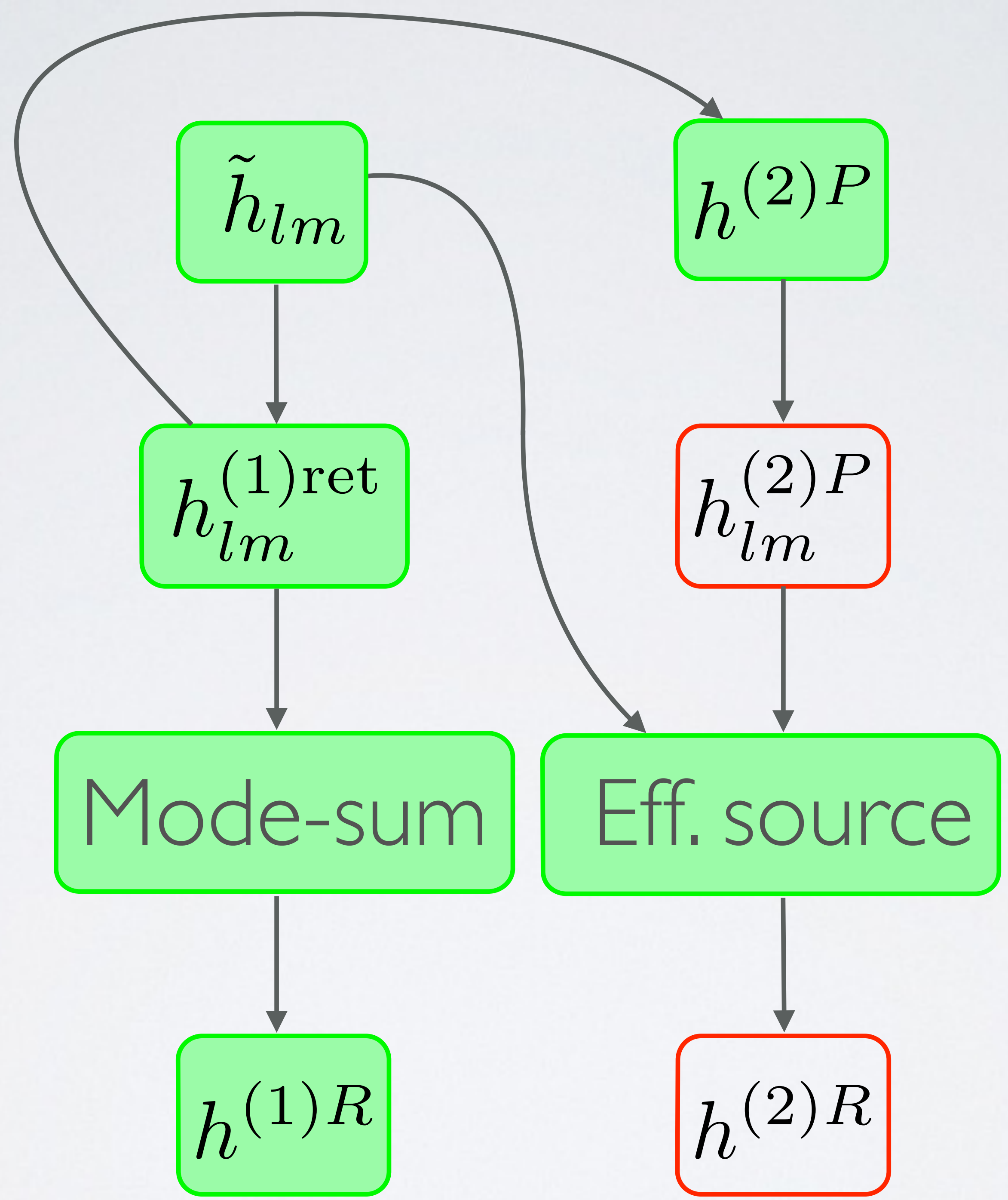
Difference: -2.72627*10^-9



Road to 2nd-order



Road to 2nd-order



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