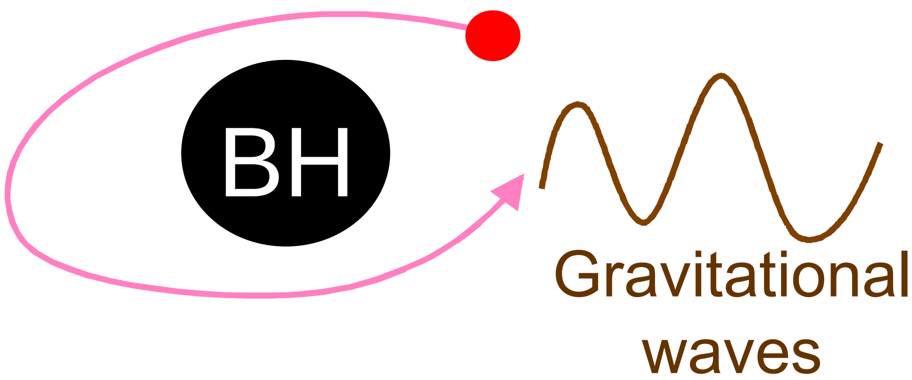
A formulation for the long-term evolution of EMRIs with linear order conservative self-force



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Order of μ in wave form

Energy balance argument is sufficient.

$$\boxed{\frac{dE_{GW}}{dt} = \frac{dE_{orbit}}{dt}}$$

$$\boxed{h(f) \approx A(f) e^{i\Psi(f)}}$$

Wave form $\equiv \dfrac{df}{dt}$ for quasi-circular orbits, for example.

$$\frac{df}{dt} = \frac{dE_{orbit}}{dt} \Big/ \frac{dE_{orbit}}{df}$$

linear perturbation

leading order ($O(\mu^{-1})$ phase)

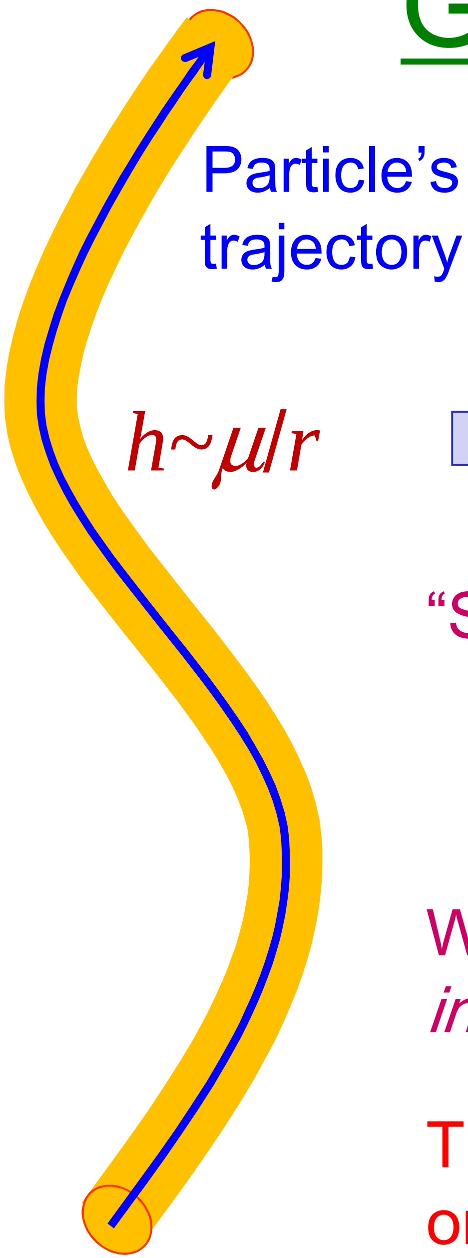
$$\frac{dE_{orbit}}{dt} = 0 + O(\mu) + O(\mu^2)$$

next leading order ($O(1)$ phase)

$$\frac{dE_{orbit}}{df} = (\text{geodesic}) + O(\mu) + O(\mu^2)$$

only up to here

Gauge invariance



Perturbation is everywhere small outside the world tube

"tube radius" >> μ

Unavoidable ambiguity in the perturbed trajectory of $O(\mu)$

"Self-force is *gauge dependent*"

$F^{\mu}_{self}(\tau, \gamma)$ has unnecessary information.

Source trajectory

While, "long term orbital evolution is *gauge invariant*"

There must be a concise description keeping only the gauge invariant information

Use of canonical transformation

$$S=\frac{1}{2}\int g^{\mu\nu}u_\mu u_\nu d\tau=\frac{1}{2}\int g^{\mu\nu}_{(0)}u_\mu u_\nu d\tau\underbrace{-\frac{1}{2}\int h^{\mu\nu}u_\mu u_\nu d\tau}$$

Interaction Hamiltonian H_{int}

It is natural to change the variables to the constants of motion in the background $P_\alpha=\left\{u^2/2,-E,L_z,Q\right\}$ and their conjugates X^α.

Generating fn: $W(x,P)=-Et+L_z\phi+\int^r\frac{\sqrt{R(r',P)}}{\Delta(r')}dr'+\int^\theta\sqrt{\Theta(\theta',P)}d\theta'$

$$u_\mu=\frac{\partial W(x,P)}{\partial x^\mu}\qquad X^\alpha=\frac{\partial W(x,P)}{\partial P_\alpha}$$

$$X^0=\pm\int^r\frac{r'^2dr'}{\sqrt{R(r',P)}}\pm\int^\theta\frac{a^2\cos^2\theta'}{\sqrt{\Theta(\theta',P)}}d\theta'$$

$$X^1=-t\pm\int^r\frac{dr'}{2\Delta\sqrt{R(r',P)}}\frac{\partial R(r',P)}{\partial E}\pm\int^\theta\frac{d\theta'}{2\sqrt{\Theta(\theta',P)}}\frac{\partial\Theta(\theta',P)}{\partial E}$$

…

$H_{(0)}=u^2/2$, and hence X^0 is τ and X^i (i =1,2,3) are all constant for background geodesics.

Radiation reaction to the constants of motion

"retarded" = "radiative" + "symmetric"

$$\frac{\text{"ret"} - \text{"adv"}}{2} \qquad \frac{\text{"ret"} + \text{"adv"}}{2}$$

no need for regularization

$$\left\langle \frac{dP_\alpha}{d\tau} \right\rangle = \left\langle \frac{\partial H_{\rm int}}{\partial X^\alpha} \right\rangle \approx \int d\tau \int d\tau' \frac{\partial}{\partial X^\alpha} G^{(ret)}(\gamma, \gamma') \Big|_{\gamma'=\gamma}$$

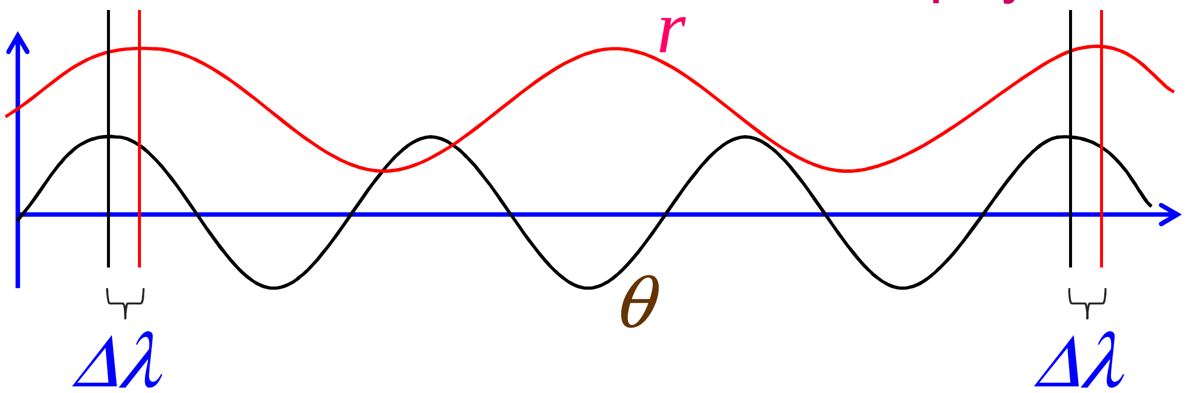
Geodesic preserving transformation (Mino transformation):

$$t \to -t,\ \phi \to -\phi,\ \tau \to -\tau \quad \Rightarrow \quad X^\mu \to -X^\mu, \quad G^{(\rm ret)} \to G^{(\rm adv)}$$

Only radiative part contributes to the change of "constants of motion" except for resonance orbits. (last Capra)

which means "Orbits with different values of X are basically equivalent."

For resonance orbits, $X^3 = \Delta\lambda$ has physical meaning.



$$\lambda = \int \frac{d\tau}{r^2 + a^2 \cos^2\theta}$$

Second canonical transformation

$\{X\}$ is not a good set of variables to see the orbital phase evolution.

$$X^1 = -t \pm \int^r \frac{dr'}{2\Delta\sqrt{R(r',P)}} \frac{\partial R(r',P)}{\partial E} \pm \int^\theta \frac{d\theta'}{2\sqrt{\Theta(\theta',P)}} \frac{\partial \Theta(\theta',P)}{\partial E}$$

Small change of P, with fixed X, at a late time

$\Longrightarrow$ large variation of x

Further canonical transformation:

$(X,P) \to (q,J)$ action-angle variables

$$\tilde{W}(q,P) = q^t E - q^\phi L_z - \frac{q^r}{2\pi} \oint \frac{\sqrt{R(r',P)}}{\Delta(r')} dr' - \frac{q^\theta}{2\pi} \oint \sqrt{\Theta(\theta',P)} d\theta'$$

$$J_\mu = -\frac{\partial \tilde{W}}{\partial q^\mu} \qquad X^\alpha = -\frac{\partial \tilde{W}}{\partial P_\alpha}$$

$$J_0 = -E,\ J_\phi = L_z,\ J_r = \frac{1}{2\pi} \oint \frac{\sqrt{R(r',P)}}{\Delta(r')} dr',\ J_\theta = \frac{1}{2\pi} \oint \sqrt{\Theta(\theta',P)} d\theta'$$

Physical meaning of the angle variables

$$\pm\int^r \frac{r'^2 dr'}{\sqrt{R(r',P)}} \pm \int^\theta \frac{a^2\cos^2\theta'}{\sqrt{\Theta(\theta',P)}} d\theta' = X^0 = \frac{q^r}{2\pi}\oint \frac{r'^2 dr'}{\sqrt{R(r',P)}} + \frac{q^\theta}{2\pi}\oint \frac{a^2\cos^2\theta'}{\sqrt{\Theta(\theta',P)}} d\theta'$$

$$\mp\int^r \frac{dr'}{2\sqrt{R(r',P)}} \pm \int^\theta \frac{d\theta'}{\sqrt{\Theta(\theta',P)}} = X^3 = -\frac{q^r}{2\pi}\oint \frac{dr'}{2\sqrt{R(r',P)}} + \frac{q^\theta}{2\pi}\oint \frac{d\theta'}{2\sqrt{\Theta(\theta',P)}}$$

After n^r and n^θ cycles, $q^r = 2\pi n^r$, $q^\theta = 2\pi n^\theta$,
irrespective of $\{-E, L_z, Q\}$.

Small change in J (or in P) with fixed X^i ⟹ small variation of x

$$t \mp \int^r \frac{dr'}{2\Delta\sqrt{R(r',P)}} \mp \int^\theta \frac{d\theta'}{2\sqrt{\Theta(\theta',P)}} = q^t + \frac{q^r}{2\pi}\oint \frac{dr'}{2\Delta\sqrt{R(r',P)}} + \frac{q^\theta}{2\pi}\oint \frac{d\theta'}{2\sqrt{\Theta(\theta',P)}}$$

$t - q^t$ is a periodic function w.r.t. q^r, q^θ.

q^μ is gauge invariant in the context of long term evolution.
which allows an $O(\mu)$ error at each time,
but the error should not accumulate.

Phase velocity

"averaged change rate of q^μ" = "phase velocity"

almost directly related to the phase of observed GWs

$$\Omega^\mu = \left\langle \frac{dq^\mu}{d\tau} \right\rangle = -\frac{\partial H_{(0)}}{\partial J_\mu} - \left\langle \frac{\partial H_{\rm int}(\gamma,\gamma')}{\partial J_\mu} \right\rangle_{\gamma'=\gamma}$$

$$\underbrace{\left\langle \frac{\partial H_{\rm int}(\gamma,\gamma')}{\partial J_\mu} \right\rangle_{\gamma'=\gamma}}_{\text{averaged after force calculation}} = \left\langle \frac{\partial H^{(sym-S)}_{\rm int}(\gamma,\gamma')}{\partial J_\mu} \right\rangle_{\gamma'=\gamma} \overset{?}{=} \underbrace{\frac{1}{2}\frac{\partial}{\partial J_\mu}\left\langle H^{(sym-S)}_{\rm int}(\gamma,\gamma) \right\rangle}_{\text{differentiation of a fn of } J_\mu}$$

Mino transformation

$J_\mu \rightarrow J_\mu$

$$r.h.s. \approx \frac{\partial}{\partial J_\mu}\int d\tau \int d\tau' G(\gamma_J(\tau), \gamma_J(\tau'))$$

Here, not $\left(\frac{\partial}{\partial J_\mu}\right)_q$ but $\left(\frac{\partial}{\partial J_\mu}\right)_X$.

X^i (i =1,2,3) are all constant for the background geodesic.

$$\left(\frac{\partial}{\partial J_\mu}\right)_X = \left(\frac{\partial}{\partial J_\mu}\right)_q + \underbrace{\left(\frac{\partial q^\alpha}{\partial J_\mu}\right)_X \left(\frac{\partial}{\partial q^\alpha}\right)_J}_{\text{This term should be removed}}$$

Can we insist $\frac{dJ_\alpha}{d\tau} = \left(\frac{\partial H_{\rm int}}{\partial q^\alpha}\right)_J = 0$?

Perturbation of generating function

Yes! "J_μ=constant" can be realized by an appropriate gauge transformation, $x^\mu \rightarrow x^\mu - \xi^\mu$, without secular growth of ξ^μ.

$$\delta_\xi J_\alpha = -\left(\frac{\partial J_\alpha}{\partial u_\nu}\right)_x \left(\frac{\partial\left(\xi^\rho u_\rho\right)}{\partial x^\nu}\right)_J$$

The same transformation can be achieved by $\delta\tilde{W}\left(x(q,P), J(P)\right) = \xi^\mu u_\mu$

$$-\left(\frac{\partial\left(\delta\tilde{W}\right)}{\partial q^\alpha}\right)_J = -\left(\frac{\partial\left(\delta\tilde{W}\right)}{\partial x^\nu}\right)_J \left(\frac{\partial x^\nu}{\partial q^\alpha}\right)_J \qquad \because \left(\frac{\partial q^\alpha}{\partial x^\nu}\right)_J = \frac{\partial^2 \Phi(x,J)}{\partial J_\alpha \partial x^\nu} = \left(\frac{\partial u_\nu}{\partial J_\alpha}\right)_x$$

Additional $\delta\tilde{W}$ makes J_μ constant. $\Rightarrow$ $\left(\frac{\partial H_{\mathrm{int}}}{\partial q^\alpha}\right)_J = 0$

$$\Rightarrow \left\langle \frac{\partial H_{\mathrm{int}}^{(sym-S)}(\gamma,\gamma')}{\partial J_\mu} \right\rangle_{\gamma'=\gamma} = \frac{1}{2}\frac{\partial}{\partial J_\mu}\left\langle H_{\mathrm{int}}^{(sym-S)}(\gamma,\gamma)\right\rangle$$

$$\Rightarrow \Omega^\mu = \Omega^\mu_{(0)}\left(J + \delta_\xi J\right) - \frac{1}{2}\frac{\partial}{\partial J_\mu}\left\langle H_{\mathrm{int}}^{(sym-S)}(\gamma,\gamma)\right\rangle$$

L.h.s. is gauge inv. (δq caused by $\delta\tilde{W}$ is purely oscillatory)

while r.h.s. may look a gauge-independent fn. of J.

J was originally gauge dependent but $J + \delta_\xi J$ must be gauge invariant.

Why does $\dot{J}=0$ fix the gauge completely?

$$\delta_\xi J_\alpha = -\left(\frac{\partial J_\alpha}{\partial u_\nu}\right)_x \left(\frac{\partial\left(\xi^\rho u_\rho\right)}{\partial x^\nu}\right)_J$$

Let's consider the above gauge transformation at the reflection point of Mino transformation,

where $u_r = u_\theta = 0$.

From the symmetry, $\left(\xi^\rho u_\rho\right) = 0$ at the reflection point.

$$\Rightarrow \left(\frac{\partial\left(\xi^\rho u_\rho\right)}{\partial r}\right)_J = \left(\frac{\partial\left(\xi^\rho u_\rho\right)}{\partial \theta}\right)_J = 0$$

Normalization condition $\Rightarrow$ $u^\nu \left(\frac{\partial\left(\xi^\rho u_\rho\right)}{\partial x^\nu}\right)_J = 0$ $\Rightarrow$ $u^t \delta_\xi E = u^\phi \delta_\xi L$

By considering two approximate reflection points corresponding to r_{min} and r_{max}, we can conclude that constant shift of J_α is not allowed.

Conclusion

We discussed the effect of long-term evolution due to first order self-force.

Radiative part requires no regularization.

The contribution of symmetric part is concisely encoded in gauge-invariant interaction Hamiltonian:

$$\left\langle H_{\rm int}^{(sym-S)} \right\rangle (J) \approx \int d\tau \int d\tau' u^\mu(\tau) u^\nu(\tau) G_{\mu\nu\rho\sigma}^{(sym-S)}(x(\tau), x(\tau')) u^\rho(\tau') u^\sigma(\tau')$$

Evolution of Q in the resonance case can be also described by a similar quantity:

$$\left\langle H_{\rm int}^{(sym-S)} \right\rangle (J, \Delta\lambda)$$

Instead of the direct computation of the self-force, alternative simple regularization based on $H_{\rm int}$ might be handy.

$\left\{ \begin{array}{l} \text{Scalar quantity.} \\ \text{Lower order differentiation.} \\ \text{Time integral can be performed first.} \end{array} \right.$