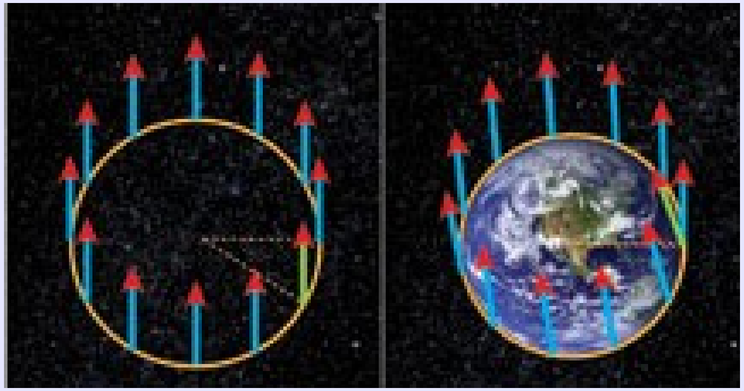
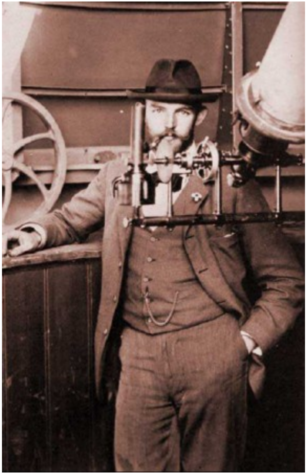
Self-force corrections to spin precession for eccentric orbits in Schwarzschild spacetime



Sarp Akcay
University College Dublin
in collaboration with
David Dempsey, Sam Dolan (U. Sheffield)

Spin precession for circular geodesics

de Sitter 1916



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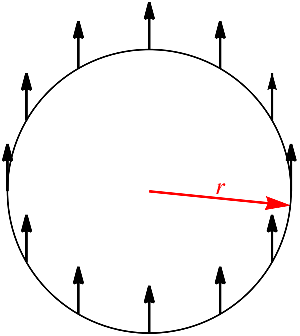
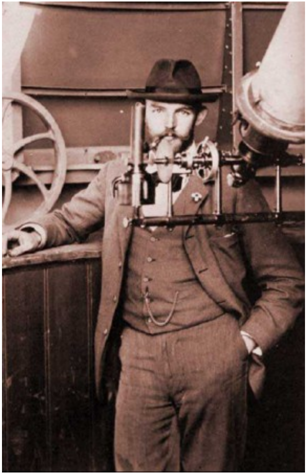
Precession due to spacetime curvature





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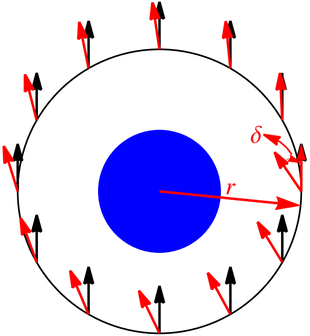

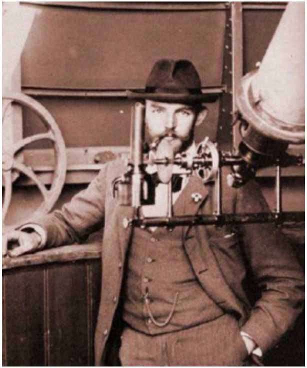

$$\frac{2\pi+\delta}{\Delta\tau}-\frac{2\pi}{\Delta\tau}=\frac{2\pi}{\Delta\tau}-\frac{2\pi}{\Delta t}$$
$$=\frac{2\pi}{\Delta\tau}\left(1-\frac{1}{u^t}\right)$$

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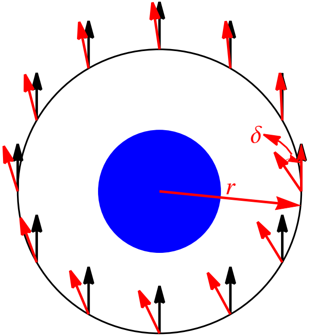

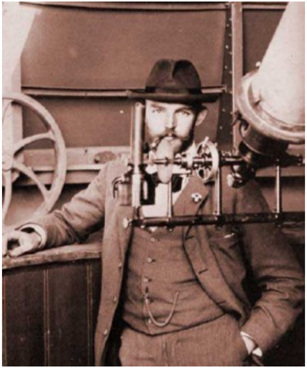

$$\frac{2\pi + \delta}{\Delta\tau} - \frac{2\pi}{\Delta\tau} = \frac{2\pi}{\Delta\tau} - \frac{2\pi}{\Delta t}$$
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$$\psi_0 \equiv \frac{\delta}{2\pi} = 1 - \frac{1}{u^t} = 1 - \sqrt{1 - \frac{3M}{r}}$$

Spin precession for circular geodesics

Mathematical picture:

Orthonormal frame: $\bar{e}^{\alpha}_{(a)} = \{u^{\alpha}, \bar{e}^{\alpha}_{1}, \bar{e}^{\alpha}_{2} \propto \hat{\theta}^{\alpha}, \bar{e}^{\alpha}_{3}\}$ on the geodesic $\bar{\gamma}$

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Consider $/\!\!/$-transport of spin $s^{\alpha} = s^i \bar{e}^{\alpha}_i$: $u^{\beta}\nabla_{\beta} s^{\alpha} = 0, \quad (s^{\alpha} u_{\alpha} = 0)$

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$$\dot{s}^i = \omega_{ij} s^j \ , \quad \omega_{ij} = (u^\gamma \nabla_\gamma \bar{e}^\alpha_i)\bar{e}_{\alpha,j} = -\omega_{ji}, \quad i,j = 1,2,3$$

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If $\bar{e}^\alpha_{(a)}$ is //-transported then $\Psi = \omega = 0 \longmapsto$ Need a second frame

Spin precession for circular geodesics

Mathematical picture:

Let $\bar{e}^{\alpha}_{(a)}$ be a Lie-transported tetrad $/\!/$ to fixed stars mod 2π

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$\lambda^{\alpha}_{(a)}$ is a rotation of $\bar{e}^{\alpha}_{(a)}$ in the $1-3$ plane by $\Psi = \omega\tau$

$$\begin{pmatrix} \lambda^{\alpha}_{1} \\ \lambda^{\alpha}_{3} \end{pmatrix} = \begin{pmatrix} \cos(\omega\tau) & -\sin(\omega\tau) \\ \sin(\omega\tau) & \cos(\omega\tau) \end{pmatrix} \begin{pmatrix} \bar{e}^{\alpha}_{1} \\ \bar{e}^{\alpha}_{3} \end{pmatrix}$$

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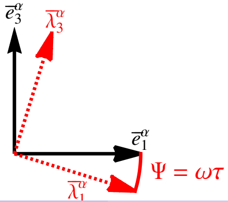

where $\Psi = \Omega_{\phi} \mathcal{T}_{\phi} = \frac{2\pi}{\bar{u}^{t}}$

Self-force correction to spin precession for circular orbits

Point particle in circular orbit: $\mu \ll M, \quad |\vec{s}| \ll \frac{G\mu^2}{c}$

Dissipation OFF

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Define

$$\delta X \equiv X(\Omega) - \bar{X}(\bar{\Omega})$$

$$\Delta Y \equiv Y(\Omega) - \bar{Y}(\Omega) = \delta Y - \frac{d\bar{Y}}{d\Omega}\delta\Omega$$

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Note: $\delta\Omega \neq 0$ but $\Delta\Omega = 0$

Self-force correction to spin precession for circular orbits

Using $\psi = 1 - \frac{\omega}{u^\phi} \Longrightarrow \delta\psi = -\sqrt{1 - \frac{3M}{r_0}}\left(\frac{\delta\omega}{\bar{\omega}} - \frac{\delta u^\phi}{\bar{u}^\phi}\right)$

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and $\omega = \Gamma_{\alpha\beta\gamma} e_3^\alpha u^\beta e_1^\gamma, \quad \Gamma = \Gamma[g], \quad g = \bar{g} + h^R$

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Obtain

$$\delta\omega = \frac{1}{2}\bar{\omega}\left(h_{uu} + h_{33} - h_{11}\right) + \beta_{03}\bar{\Gamma}_{313} + \delta\Gamma_{310}$$

$$\delta\psi = \frac{1}{\bar{u}^t}\left(\frac{1}{2}\left(h_{11} - h_{33}\right) - r_0^{3/2}\delta\Gamma_{310} + \frac{1}{2} r_0 F_r\right)$$

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Finally

$$\boxed{\Delta\psi = \delta\psi + \underbrace{\frac{r_0}{2\bar{u}^t} F_r + \delta_\alpha\psi}_{-\frac{d\bar{\psi}}{d\Omega}\delta\hat{\Omega}}}$$

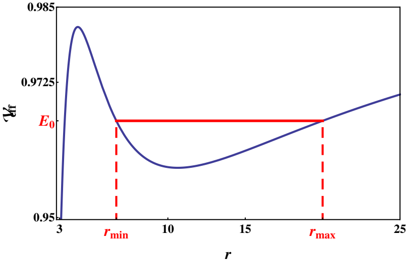
with $\bar{u}^t = \left(1 - \frac{3M}{r_0}\right)^{-1/2}$

See Dolan et al. 2014 for details

Eccentric geodesics in Schwarzschild spacetime

2-D parameter space: $(E_0, L_0) \leftrightarrow (r_{\mathrm{min}}, r_{\mathrm{max}}) \leftrightarrow (p, e)$

$$p = 10, e = 0.5$$



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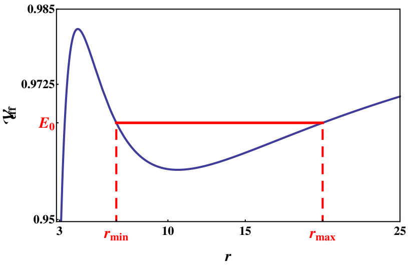

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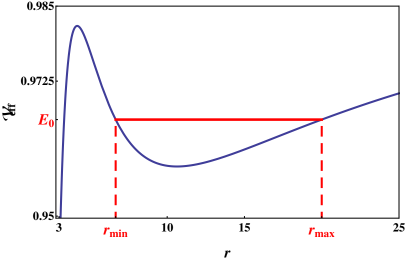

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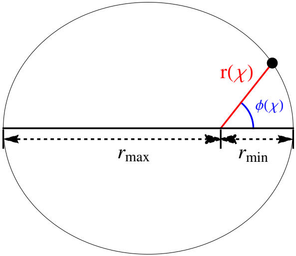
Radially periodic: $t \in [0, T_r]$

Parametrize using $\chi \in [0, 2\pi]$



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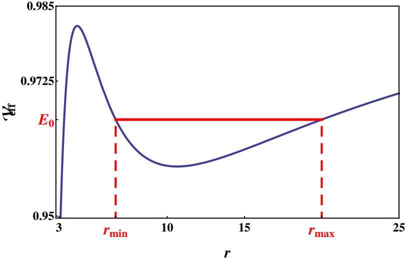

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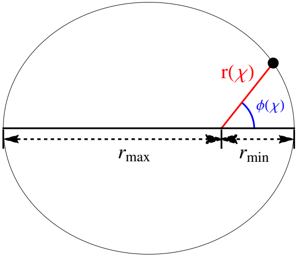

$$r(\chi) = \frac{p}{1 + e\cos\chi}$$

$$\Phi > 2\pi$$

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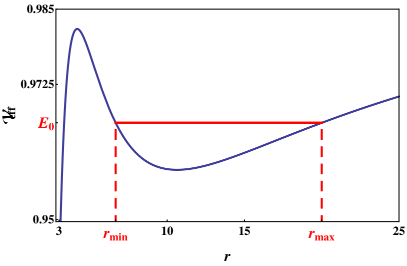

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TWO frequencies

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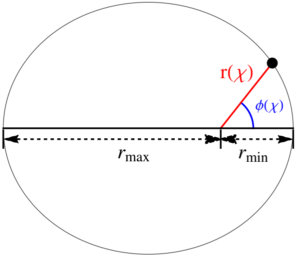

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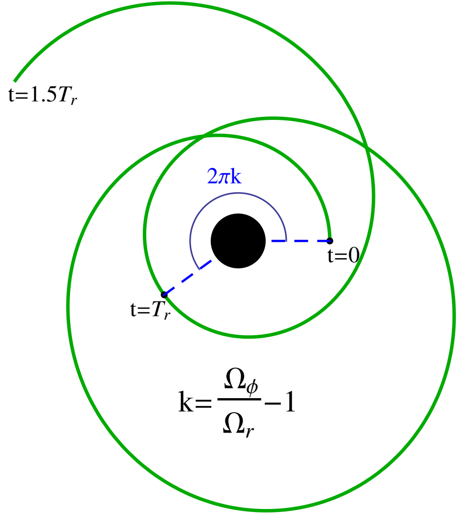

Self-force correction to spin precession for eccentric orbits

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NO conserved E or L: $E = E_0 + \delta E$, $L = L_0 + \delta L$
Shifted 4-velocity: $u^\alpha = \bar{u}^\alpha + \delta u^\alpha$

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Associated orbits: $z^\alpha(\tau) \leftrightarrow \bar{z}^\alpha(\bar{\tau})$ with $\{p, e, \chi\}$ fixed

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Gauge invariant: $\int \psi(\Omega_r, \Omega_\phi)$

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Defining $\psi \equiv 1 - \frac{\Psi}{\Phi}$ where $\Psi = \int \dot{\Psi} d\tau$

Gauge invariant correction

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$$\Delta\psi \equiv \psi(\Omega_r, \Omega_\phi, \mu) - \psi(\Omega_r, \Omega_\phi, 0)$$

$$= \boxed{\delta\psi - \frac{\partial\bar{\psi}}{\partial\Omega_r}\delta\hat{\Omega}_r - \frac{\partial\bar{\psi}}{\partial\Omega_\phi}\delta\hat{\Omega}_\phi} \quad (1)$$

where $\delta\hat{\Omega}_i = \delta\Omega - \alpha\bar{\Omega}_i$ and

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$$\boxed{\delta\psi = -\frac{\bar{\Psi}}{\bar{\Phi}}\left(\frac{\delta\Psi}{\bar{\Psi}} - \frac{\delta\Phi}{\bar{\Phi}}\right)} \qquad (2)$$

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Recall: we need TWO frames

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1. //-transported tetrad: $\dot{\lambda}_i^\alpha = 0 \quad \{u^\alpha, \lambda_1^\alpha, \lambda_2^\alpha, \lambda_3^\alpha\}$
2. Marck's tetrad (Marck 1983): $\{u^\alpha, \bar{e}_1^\alpha, \bar{e}_2^\alpha \propto \hat{\theta}, \bar{e}_3^\alpha\}$

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Rotation in the $1-3$ plane by Ψ

$$\begin{pmatrix} \lambda_1^\alpha \\ \lambda_3^\alpha \end{pmatrix} = \begin{pmatrix} \cos\Psi & -\sin\Psi \\ \sin\Psi & \cos\Psi \end{pmatrix} \begin{pmatrix} \bar{e}_1^\alpha \\ \bar{e}_3^\alpha \end{pmatrix}$$

Using $\dot{\lambda}_1 = \dot{\lambda}_3 = 0$ we get $\dot{\Psi} = \bar{e}_{3\alpha}\dot{\bar{e}}_1^\alpha = \bar{g}_{\alpha\beta}\bar{e}_3^\beta\left(\bar{u}^\gamma\bar{\nabla}_\gamma\bar{e}_1^\alpha\right)$

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Recall: we need TWO frames

1. $/\!\!/$-transported tetrad: $\dot{\lambda}_i^\alpha = 0 \quad \{u^\alpha, \lambda_1^\alpha, \lambda_2^\alpha, \lambda_3^\alpha\}$
2. Marck's tetrad (Marck 1983): $\{u^\alpha, \bar{e}_1^\alpha, \bar{e}_2^\alpha \propto \hat{\theta}, \bar{e}_3^\alpha\}$

Rotation in the $1-3$ plane by Ψ

$$\begin{pmatrix} \lambda_1^\alpha \\ \lambda_3^\alpha \end{pmatrix} = \begin{pmatrix} \cos\Psi & -\sin\Psi \\ \sin\Psi & \cos\Psi \end{pmatrix} \begin{pmatrix} \bar{e}_1^\alpha \\ \bar{e}_3^\alpha \end{pmatrix}$$

Using $\dot{\lambda}_1 = \dot{\lambda}_3 = 0$ we get $\dot{\Psi} = \bar{e}_{3\alpha}\dot{\bar{e}}_1^\alpha = \bar{g}_{\alpha\beta}\bar{e}_3^\beta \left(\bar{u}^\gamma \bar{\nabla}_\gamma \bar{e}_1^\alpha\right)$
Thus

$$\boxed{\delta\dot{\Psi} = \delta\left(g_{\alpha\beta} e_3^\beta u^\gamma \nabla_\gamma e_1^\alpha\right)}$$

Self-force correction to spin precession for eccentric orbits

More work yields

$$\boxed{\delta\dot{\Psi} = \frac{1}{2}\dot{\bar{\Psi}}\left(h_{uu} + h_{33} - h_{11}\right) + c_{01}\bar{\Gamma}_{311} + c_{03}\bar{\Gamma}_{313} + \underbrace{\frac{dc_{13}}{d\tau}}_{\text{averages to 0}} + \delta\Gamma_{310}}$$

$$\left.\begin{aligned}\delta\Psi &= \int_0^{2\pi} \frac{d\bar{\Psi}}{d\chi}\left(\frac{\delta\dot{\Psi}}{\dot{\bar{\Psi}}} - \frac{\delta\dot{R}}{\bar{u}^r}\right) d\chi \\ \delta\Phi &= \int_0^{2\pi} \frac{d\bar{\Phi}}{d\chi}\left(\frac{\delta L}{L_0} - \frac{\delta\dot{R}}{\bar{u}^r}\right) d\chi\end{aligned}\right\}$$

$\delta\Phi$: Barack & Sago 2011

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Numerical Results

Small eccentricity for now: $e = 0.05, 0.1$
Compare with circular-orbits $\mapsto$ Extract PN expansion of $\mathcal{O}(e^2)$ term

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e.g.,
$$\begin{aligned}
\Delta\psi_{\text{circ}}(p=10) &= 0.00593855^{*} \\
\Delta\psi(p=10, e=0.1) &= -0.0503747 \\
\lim_{e\to 0} \Delta\psi(p=10, e) &= -0.0506715
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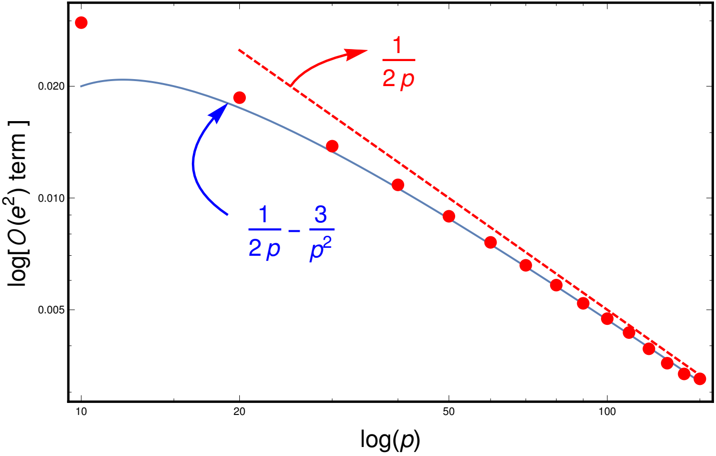
Difference is gauge invariant

Numerical Results $e = 0.1$

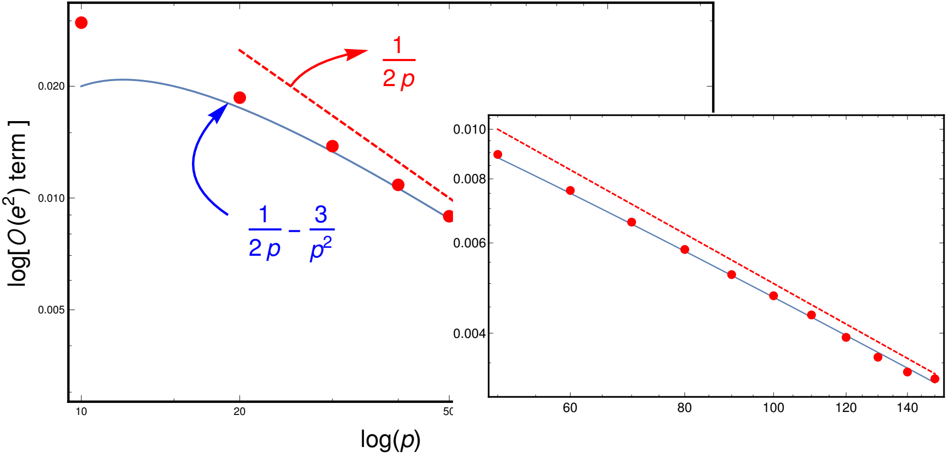
PRELIMINARY

p	$\Delta\psi$	$\lim_{e\to 0} \Delta\psi$	$(\Delta\psi - \lim_{e\to 0} \Delta\psi)/e^2$
10	$-5.0374746183 \times 10^{-2}$	-5.06715×10^{-2}	2.96711×10^{-2}
20	$-4.105502714 \times 10^{-2}$	-4.12414×10^{-2}	1.8634×10^{-2}
30	$-2.97664644 \times 10^{-2}$	-2.99044×10^{-2}	1.3792×10^{-2}
40	$-2.309158147 \times 10^{-2}$	-2.32001×10^{-2}	1.085×10^{-2}
50	$-1.880823903 \times 10^{-2}$	-1.88976×10^{-2}	8.93×10^{-3}
60	$-1.58482057 \times 10^{-2}$	-1.59242×10^{-2}	7.595×10^{-3}
70	$-1.3686455636 \times 10^{-2}$	-1.37523×10^{-2}	6.588×10^{-3}
80	$-1.204035626 \times 10^{-2}$	-1.20966×10^{-2}	5.826×10^{-3}
90	$-1.074624358 \times 10^{-2}$	-1.07983×10^{-2}	5.204×10^{-3}
100	$-9.702092699 \times 10^{-3}$	-9.74941×10^{-3}	4.732×10^{-3}
110	$-8.84229541 \times 10^{-3}$	-8.88571×10^{-3}	4.341×10^{-3}
120	$-8.122988585 \times 10^{-3}$	-8.16224×10^{-3}	3.925×10^{-3}
130	$-7.511573415 \times 10^{-3}$	-7.54748×10^{-3}	3.59×10^{-3}
140	$-6.985103217 \times 10^{-3}$	-7.01869×10^{-3}	3.359×10^{-3}
150	$-6.526467808 \times 10^{-3}$	-6.55905×10^{-3}	3.258×10^{-3}

Numerical Results $e = 0.1$



Numerical Results $e = 0.1$



What's next?

- Sort out the disagreement with 1PN

  Bug(s)? Implementation? Formulation?

- Cover Schwarzschild parameter space

$$0 < e \lesssim 0.5, \;\; 10 \lesssim p \lesssim 150$$

- Comparison with PN as was done for ΔU in gr/qc-1503.01374

- Future: Equatorial, eccentric Kerr calculation using MST, CCK reconstruction, metric completion (see van de Meent's talk)