A New Discontinuous Galerkin Code for Time Domain, Effective Source Self-force Calculations

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Outline

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- Self-consistent vs. geodesic evolutions.
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- Results for eccentric orbit ($e = 0.1, p = 9.9$).
- Conclusions and Outlook.

The problem.

We wish to determine the self-forced motion and field (e.g. energy and angular momentum fluxes) of a particle with scalar charge

$$\Box\psi^{\mathrm{ret}} = -4\pi q \int \delta^{(4)}(x - z(\tau))\, d\tau.$$

2 general approaches:

- Compute enough "geodesic"-based self-forces and then use this to drive the motion of the particle. (Post-processing, fast, accurate self-forces, relies on slow orbit evolution)
- Compute the "true" self-force while simultaneously driving the motion. (Slow and expensive, less accurate self-forces)

Effective source approach.

... is a general approach to self-force and self-consistent orbital evolution that doesn't use any delta functions.

Key ideas

- Compute a regular field, ψ^{R}, such that the self-force is

  $$F_\alpha = \nabla_\alpha \psi^{\mathsf{R}}|_{x=z},$$

  where $\psi^{\mathsf{R}} = \psi^{\mathsf{ret}} - \psi^{\mathsf{S}}$, and ψ^{S} can be approximated via local expansions: $\psi^{\mathsf{S}} = \tilde{\psi}^{\mathsf{S}} + O(\epsilon^n)$.

- The effective source, S, for the field equation for ψ^{R} is regular at the particle location.

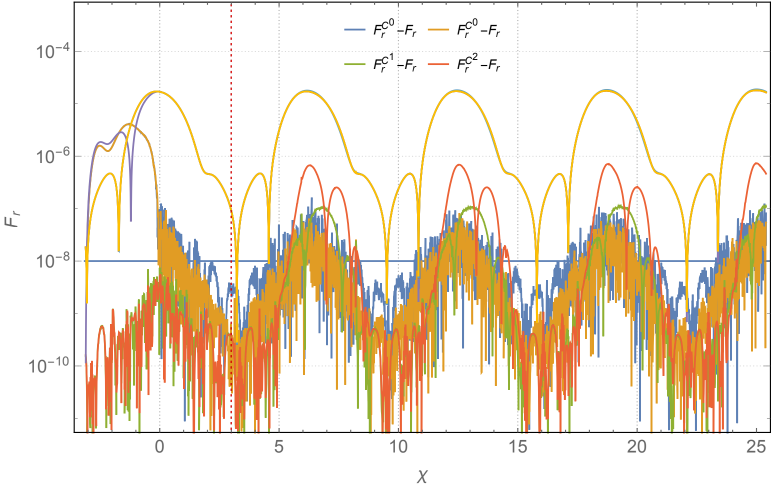
  $$\Box\psi^{\mathsf{R}} = \Box\psi^{\mathsf{ret}} - \Box\psi^{\mathsf{S}} = S(x|z,u)$$

  where $\Box\psi^{\mathsf{S}} = -4\pi q \int \delta^{(4)}(x - z(\tau))\, d\tau - S$.

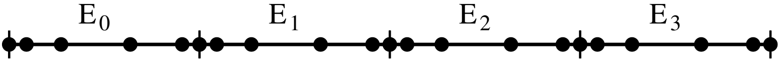
Self-consistent vs. geodesic evolutions.

- One main goal is to compare our self-consistent evolutions with Niels Warburton's geodesic evolutions.
- First attempt: 3+1 multi-patch finite difference code with a C^0 effective source.
- 3+1 accuracy limited by the non-smoothness of the source leading to high frequency noise with 2nd order convergent amplitude.
- Self-consistent evolutions agreed beautifully with geodesic evolutions within the errors (dominated by the noise).
- Next attempt: 3+1 multi-patch finite difference code with a C^2 effective source.
- Geodesic evolution agreed with the C^0 evolutions and the frequency domain result with the noise reduced by more than an order of magnitude.
- However, we found differences between C^2 and C^0 results as soon as the back-reaction was turned on.

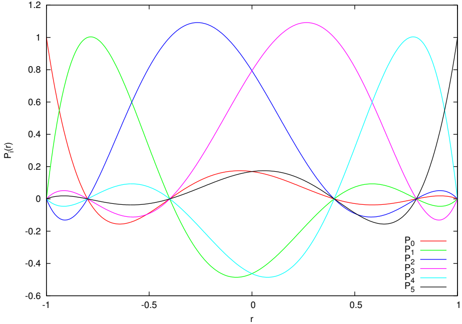
Self-consistent vs. geodesic evolutions.



Discontinuous Galerkin method.



- Split the domain into N nth order elements.
- Each element contains $n+1$ nodes.
- $u(t,x) \approx \sum_{i=0}^{n} \tilde{u}(t,x_i) P_i(x)$



- The numerical approximation is double valued at all element boundaries.
- Derivatives are approximated by multiplying the state vector in each element by a derivative matrix.
- Neighboring elements are glued together by numerical fluxes.

Discontinuous Galerkin method.

- Numerical fluxes can be constructed in many different ways in order to maintain numerical stability and to guarantee that the jumps in the solution at the element boundaries converge to zero.
- We use fluxes based on characteristic information.

The convergence properties of the DG method for smooth solutions are

- Exponential with the order n (with N kept fixed).
- polynomial with the element size $1/N$ (with n kept fixed).

As the DG scheme has discontinuities built in at the element boundaries, we retain these convergence properties even when the solution itself is non-smooth IF and only if, the non-smooth features can be placed at element boundaries.

(Hesthaven & Warburton, 2007)

Code description.

The code is a 1+1 dimensional code based on the spherical harmonic decomposition of the scalar wave equation in Schwarzschild in tortoise coordinates $r_* = r + 2M\log(r/(2M) - 1)$ with a spherically harmonic decomposed effective source.

$$-\frac{\partial^2\psi_{\ell m}}{\partial t^2} + \frac{\partial^2\psi_{\ell m}}{\partial r_*^2} - V_\ell(r)\psi_{\ell m} = S^{\rm eff}_{\ell m}.$$

As $r_* \in [-\infty, \infty]$ we split the domain into three regions. In the inner ($r_* \in [-\infty, a]$) and outer ($r_* \in [b, \infty]$) regions we introduce new coordinates (τ, ρ) by (Bernuzzi, Nagar & Zenginoğlu, 2011)

$$\begin{aligned} t &= \tau + h(\rho) \\ r_* &= \rho/\Omega(\rho) \end{aligned}$$

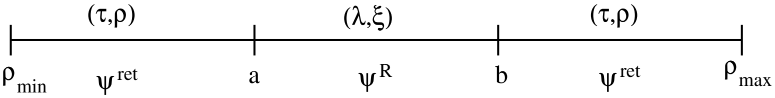
where $h(\rho)$ and $\Omega(\rho)$ are chosen suitably (hyperboloidal layers) in each region to make the inner boundary ($\rho_{\min}$) coincide with the horizon and the outer boundary ($\rho_{\max}$) coincide with $\mathscr{I}^+$.

Code description.

In the middle region ($r_* \in [a, b]$) we introduce a time dependent coordinate transformation (Field, Hesthaven & Lau, 2009)

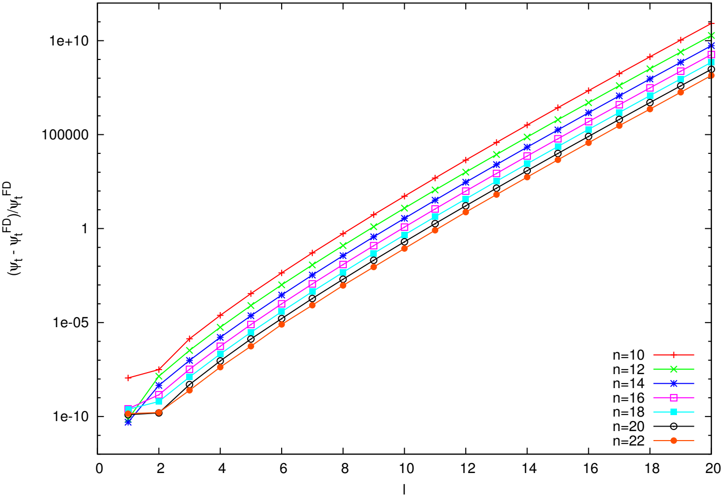
$$
\begin{aligned}
t &= \lambda \\
r_* &= a + \frac{r_*^p - a}{\xi^p - a}(\xi - a) + \\
&\quad \frac{(b - r_*^p)(\xi^p - a) - (r_*^p - a)(b - \xi^p)}{(\xi^p - a)(b - \xi^p)(b - a)}(\xi - a)(\xi - \xi^p)
\end{aligned}
$$

where r_*^p is the time-dependent particle location. This satisfies $r_*(\lambda, a) = a$, $r_*(\lambda, \xi^p) = r_*^p$, $r_*(\lambda, b) = b$.
In addition we use the world tube approach so that we in the middle region evolve $\psi_{\ell m}^{\mathrm{R}} = \psi_{\ell m}^{\mathrm{ret}} - \psi_{\ell m}^{\mathrm{S}}$, while we in the inner and outer region evolve $\psi_{\ell m}^{\mathrm{ret}}$.
The values of a and b is of course chosen to coincide with element boundaries.



Results for circular orbit ($r = 10$).

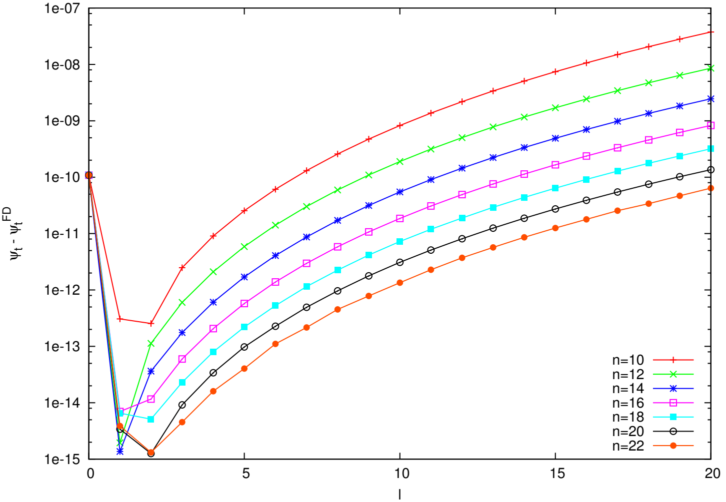
Relative error for ψ_t (data extracted at $t = 4000M$).



Note: mode amplitudes goes to zero exponentially with ℓ.

Results for circular orbit ($r = 10$).

Absolute error for ψ_t (data extracted at $t = 4000M$).



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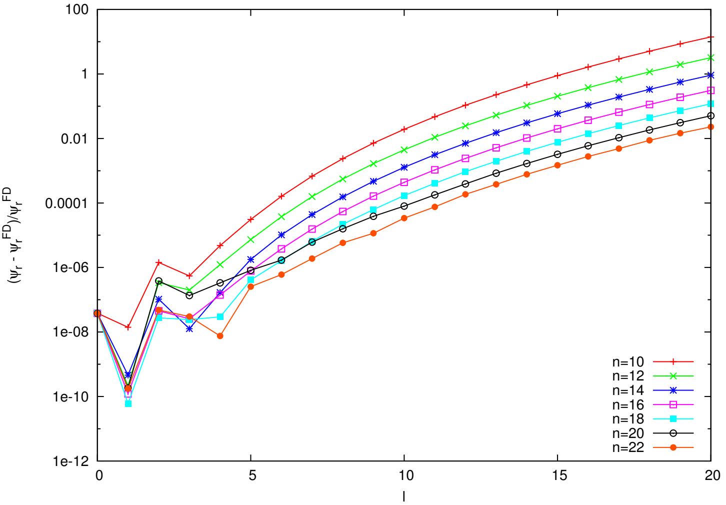
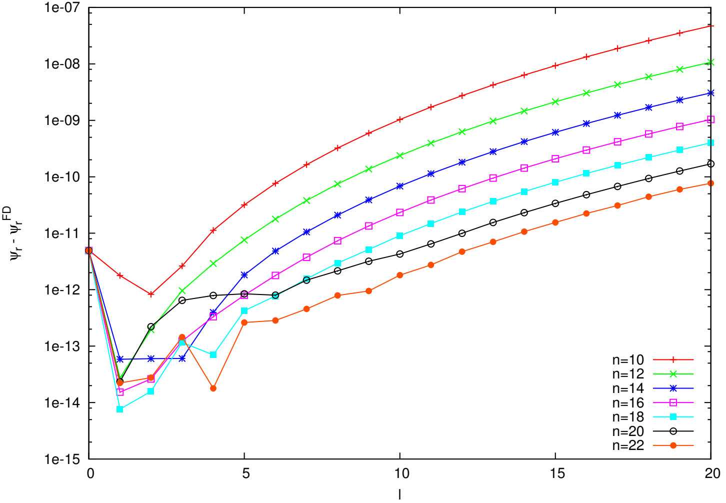
Results for circular orbit ($r = 10$).

Relative error for ψ_r (data extracted at $t = 4000M$).



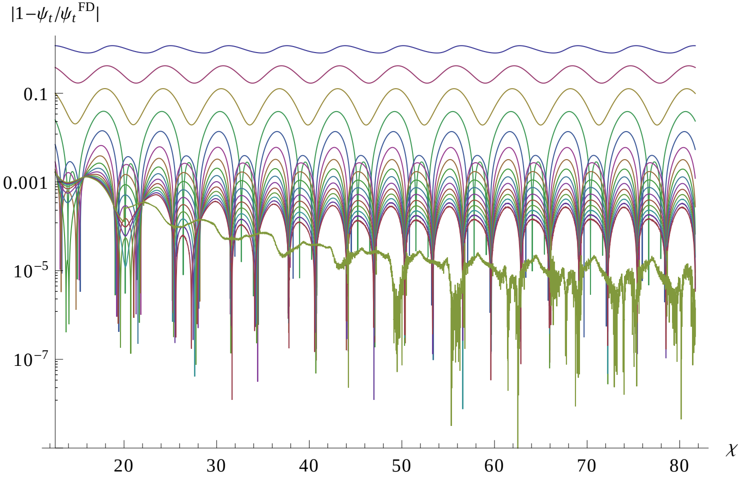
Results for circular orbit ($r = 10$).

Absolute error for ψ_r (data extracted at $t = 4000M$).



Results for eccentric orbit ($e = 0.1, p = 9.9$).

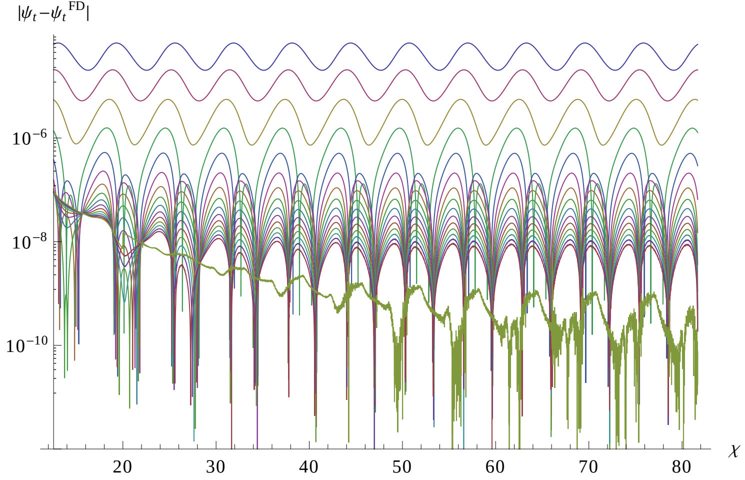
Relative error for ψ_t (data extracted at $t = 4000M$).



Fitted $\ell \in [10, 15]$ to $1/((2l-3)(2l-1)(2l+3)(2l+5))$

Results for eccentric orbit ($e = 0.1, p = 9.9$).

Absolute error for ψ_t (data extracted at $t = 4000M$).



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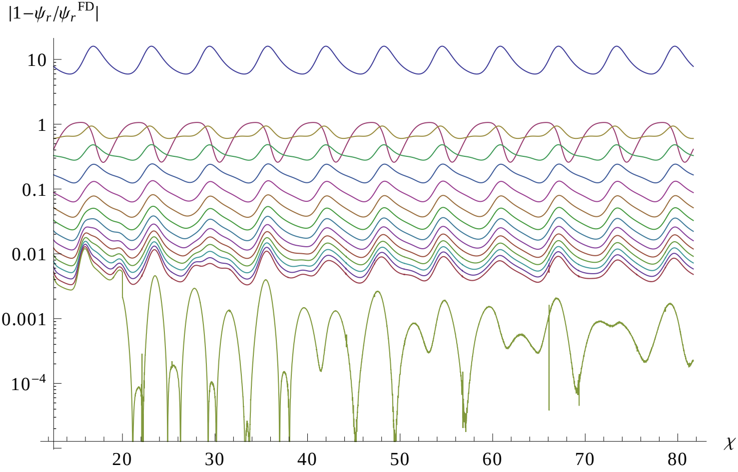

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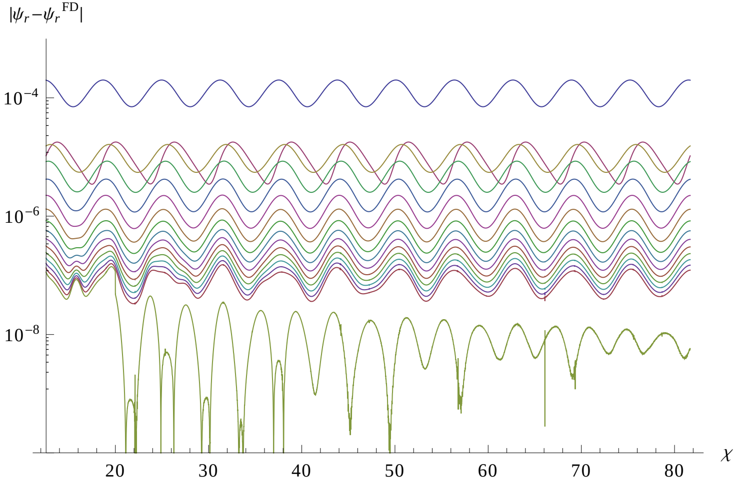

Conclusions and Outlook.

Discontinuous Galerkin is a powerful numerical method that allows us to overcome the non-smoothness of the effective source.

The accuracy has been improved and computational cost reduced by at least 2 orders of magnitude.

What is needed before self-consistent evolutions can be done?

- Make the world-tube smaller.
- Implement high l-mode fitting.
- Add self-force terms to osculating orbit equations.

What else is in the pipeline.

- Implement flux calculation at horizon and $\mathscr{I}^+$.
- Coupled modes for Kerr.
- Gravitational perturbations in Lorenz-gauge.

We hope that we now have enough accuracy to perform the comparison between the self-consistent and geodesic evolutions.