Metric reconstruction via Hertz potential in the time domain

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Overview

Background and Motivation

Motivation: Long Term evolution of a particle on Kerr background

	Solve For $h^{Lorentz}$	Solve Teukolsky equation
Frequency domain	(Akcay etal, Osburn etal) • Hard to evolve orbit • Hard to get high eccentricity • Degenerate modes	(Shah etal ,van de Meent) • Hard to evolve orbit • Hard to get high eccentricity
Time domain	(Barack, Sago, Dolan) • Computationally expansive • Some unstable modes	OUR NEW METHOD: • Any orbit • No "bad modes" • Evolution

The CCK metric reconstruction procedure involves three steps:

- Solving the Teukolsky equation for ψ_s

$$T_s \psi_s = 0$$

- Finding Hertz potential that satisfies both the Teukolsky equation $T_{-s}\Psi^* = 0$ and a differential equation with ψ_s as a source

$$D_s^4 \Psi = \psi_s$$

- Operating on the Hertz potential with another differential operator to obtain metric prerturbation in a (traceless) radiation gauge

$$h_{\alpha\beta} = H_{\alpha\beta}\Psi$$

Background I : Metric reconstruction in vacuum

The Kinnersley tetrad in Boyer-Linquist coordinates is given by $e^{\alpha}_{\boldsymbol{a}} = (\ell^{\alpha}, n^{\alpha}, m^{\alpha}, \bar{m}^{\alpha})$. We can use ORG or IRG gauges defined as

$$IRG: \quad l^{\alpha} g_{\alpha\beta} = 0 \quad ORG: \quad n^{\alpha} g_{\alpha\beta} = 0$$

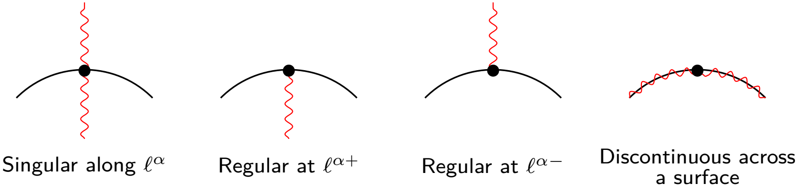
Reconstruction of the metric via Hertz potential is done via

$$H^{ORG}_{\alpha\beta} =$$
$$-\varrho^{-4}\left\{ n_{\alpha} n_{\beta} \left(\bar{\boldsymbol{\delta}} - 2\alpha\right)\left(\bar{\boldsymbol{\delta}} - 4\alpha\right) + \bar{m}_{\alpha}\bar{m}_{\beta}\left(\boldsymbol{\Delta} + 5\mu - 2\gamma\right)\left(\boldsymbol{\Delta} + \mu - 4\gamma\right)\right.$$
$$\left. - n_{(\alpha}\bar{m}_{\beta)}\left[\left(\bar{\boldsymbol{\delta}} - 2\alpha\right)\left(\boldsymbol{\Delta} + \mu - 4\gamma\right) + \left(\boldsymbol{\Delta} + 4\mu - 4\gamma\right)\left(\bar{\boldsymbol{\delta}} - 4\alpha\right)\right]\right\}$$
$$\pm \text{c.c.}$$

similar for IRG

Background II : Metric reconstruction with point particle

- Gauges: full-string, half-string, no-string[1]



- No-string reconstruction in f-domain done by Friedman, Shah etal, Merlin , van de Meent
- Here we will discuss no-string gauge in time domain

[1] Courtesy of C.Merlin

Background III : Self-force from reconstructed metric

- In either of the two half string gauges via

$$F_\alpha^{\pm} = \sum_l (F_\alpha^{l\pm} - A_\alpha^{\pm} L - B_\alpha - C_\alpha / L) + D_\alpha^{\pm}$$

  - radial limit $\pm$
  - A,B,C as in Lorenz gauge
  - $D_\alpha^{\pm}$ non zero in general (depends on u^α extension)

- In no-string gauge

$$F_\alpha = \sum_l (\bar{F}_\alpha^l - B_\alpha) \qquad \bar{F}^l = \frac{1}{2}(F^+ + F^-)$$

Our new method: Metric reconstruction in the time domain

Our Method

$\boxed{\text{Evolve Hertz Potential directly in the time domain}}$

Need:

- Evolution equation (Teukolsky equation) ✓
- Boundary conditions
- Junction condition at the particle

Ψ Boundary conditions

ψ_4 and ψ_0 satisfy the vacuum Teukolsky equation for spin-weights ∓ 2 when Ψ^{ORG} or Ψ^{IRG} satisfy the spin-weight ± 2 Teukolsky equations.
The asymptotic behaviour will be
(Ingoing)

$$\Psi^{IRG} \propto e^{-i\omega v}/r^5 \quad r \to \infty \quad , \boxed{\Psi^{IRG} \propto e^{-i\omega v} \Delta^{-s} \quad r \to 2M}$$

(Outgoing)

$$\boxed{\Psi^{IRG} \propto e^{-i\omega u}/r \quad r \to \infty} \quad , \Psi^{IRG} \propto e^{-i\omega u} \quad r \to 2M$$

Similar for Ψ^{ORG}.

Regularity condition at the Horizon

$\Delta^s \Psi$ is smooth with $\Delta = (r - r_-)(r - r_+)$

Junction condition for Ψ

Obtained from inversion relation and field equation in vacuum

$$D^4\Psi = \psi_0, \qquad T\Psi = 0$$

given the junction conditions for ψ_0

1. On either sides of the world line

$$(\ell^\mu \partial_\mu)^4 \Psi^\pm = \psi_0^\pm$$

2. Take Right-Left difference (denoted by $[\dots]$)
3. use $T\Psi^\pm = 0$ to write LHS in terms of $[\Psi]$ and $[\Psi_v]$
4. Solve for $[\Psi]$ and $[\Psi_v]$ in terms of $[\psi_0]$ and $[\psi_{0,v}]$ (known)

Cases

Case1: Static particle in flat space

- The inversion relation reduces to

$$\frac{d^4}{dr^4}\Psi^* = \psi_0$$

- By taking derivatives of $T\Psi^* = 0$ repeatedly express $\frac{d^4}{dr^4}\Psi^*$ and $\frac{d^5}{dr^5}\Psi^*$ in terms of Ψ^* and Ψ^*_r.
- Take 'Right-Left" difference to obtain

$$-\frac{16(l^2+l+15)}{r^3}[\partial_r\Psi^*] + \frac{(l^2+l+60)(l+3)(l-2)}{r^4}[\Psi^*] = [\psi_0]$$

$$\frac{1800+(l^2+l)(l^2+l+198)}{r^4}[\partial_r\Psi^*] - \frac{20(l^2+l+24)(l^2+l-6)}{r^5}[\Psi^*] = [\partial_r\psi_0]$$

This can be inverted algebraically to derive $[\Psi^*]$ and $[\Psi^*_r]$

Case2: Schwarzschild for circular orbits

- In this case the inversion relations can be written as

$$[\Psi_{vvvv}] = [\psi_0]$$

- Mixed derivative jumps can be derived via TE in vacuum

$$[\Psi_{vu}] = -(A^u_p[\Psi_u] + A^v_p[\Psi_v] + V_p[\Psi])$$

- Taking τ derivatives we derive the set

$$[\dot{\Psi}] = \dot{u}[\Psi_u] + \dot{v}[\Psi_v] = 0$$

$$[\dot{\Psi}_v] = \dot{u}[\Psi_{vu}] + \dot{v}[\Psi_{vv}] = 0$$

$$[\ddot{\Psi}] = 2\dot{v}\dot{u}[\Psi_{vu}] + \dot{u}^2[\Psi_{uu}] + \dot{v}^2[\Psi_{vv}] = 0$$

This can be done till the 5th derivative, all the jumps of the form $[\Psi_{v...vu...u}]$ will be written in terms of $[\Psi_v], [\Psi]$ so we can solve as the previous case

Case3: Schwarzschild for generic orbits

- Same inversion relation as previous case
- Second derivatives terms can be derived by solving

$$[\dot{\Psi}] = \dot{u}[\Psi_u] + \dot{v}[\Psi_v]$$

$$[\dot{\Psi_v}] = \dot{u}[\Psi_{vu}] + \dot{v}[\Psi_{vv}]$$

$$[\ddot{\Psi}] = \ddot{v}[\Psi_v] + \ddot{u}[\Psi_u] + 2\dot{v}\dot{u}[\Psi_{vu}] + \dot{u}^2[\Psi_{uu}] + \dot{v}^2[\Psi_{vv}]$$

Same procedure to the next order looking at $\frac{d}{d\tau}[\Psi_{vv}]$ and $\frac{d^3}{d\tau^3}[\Psi]$.

Case3: Schwarzschild for generic orbits

We solve iteratively till the 5th order where $[\Psi_{vvvv}]$ and $[\Psi_{vvvvv}]$ will be written in terms of $([\Psi], \frac{d}{d\tau}[\Psi], \ldots, \frac{d^4}{d\tau^4}[\Psi])$ and $([\Psi_v], \ldots, \frac{d^4}{d\tau^4}[\Psi_v])$ so we obtain the system

$$\sum_{n=0}^{3} a_n \frac{d^n}{d\tau^n}[\Psi] + \sum_{n=0}^{3} b_n \frac{d^n}{d\tau^n}[\Psi_v] = [\psi_0]$$

$$\sum_{n=0}^{4} \tilde{a}_n \frac{d^n}{d\tau^n}[\Psi] + \sum_{n=0}^{4} \tilde{b}_n \frac{d^n}{d\tau^n}[\Psi_v] = [\psi_{0,v}]$$

where the RHS is known. This is a system of coupled ODEs of 3rd and 4th order to be solved numerically.

Case4: Kerr Case

- Use the decomposition in spin-weighted spherical harmonics

$$\Psi = \sum_{lm} Y_{slm} \Psi_{lm}$$

The 1+1D Teukolsky equation has mixed l-modes terms

$$[\Psi^l_{uv}] + A^u[\Psi^l_u] + A^v[\Psi^l_v] + V[\Psi^l] + \sum_{\substack{q=l-2 \\ l\neq n}}^{l+2} \mathcal{I}_{l,q}[\Psi^q] = 0$$

- Truncate the number of l modes and write the system in a band diagonal matrix form

$$[\begin{pmatrix} \tilde{T}_2 & \mathcal{I}_{2,3} & \mathcal{I}_{2,4} & 0 \\ \mathcal{I}_{3,2} & \tilde{T}_3 & \mathcal{I}_{3,4} & \cdots \\ \mathcal{I}_{4,2} & \mathcal{I}_{4,3} & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots \end{pmatrix} \begin{pmatrix} \vdots \\ \Psi^l \\ \vdots \end{pmatrix}] = 0$$

This equation will be used to reduce the order as previously and write $\{[\Psi^l], [\Psi^l_v]\}$ in terms of their τ derivatives

Case4: Kerr Case

- The inversion relation reduces to

$$[(\partial_v + \frac{ima}{\Delta})^4 \Psi^l] = \sum_{n=0}^{4} c_n [\partial_v^n \Psi^l] = [\psi_0^l]$$

- Each term can we written in terms of $\{[\Psi^l], [\Psi^l_v]\}$ and its τ derivatives. As before this are coupled ODEs

$$\sum_{n=0}^{4} A_n \frac{d^n}{d\tau^n}[\vec{\Psi}] + \sum_{n=0}^{3} B_n \frac{d^n}{d\tau^n}[\vec{\Psi}_v] = [\vec{\psi}_0]$$

$$\sum_{n=0}^{4} \tilde{A}_n \frac{d^n}{d\tau^n}[\vec{\Psi}] + \sum_{n=0}^{3} \tilde{B}_n \frac{d^n}{d\tau^n}[\vec{\Psi}_v] = [\vec{\psi}_{0,v}]$$

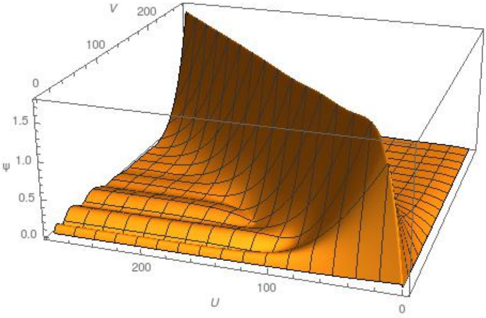
with $A, B, \tilde{A}, \tilde{B}$ band diagonal matrices. The system can be solved numerically.

Implementation

Implementation

Numerics

1. Use (u,v) coordinates grid
2. Set initial condition to zero on (u,v) axis
3. Evolve via finite difference scheme
4. Discard junk radiation and record late time behaviour



Time evolution in (u, v) for circular orbit in Schwarzschild for ψ_0

Outlook

Conclusion and Outlook

- We propose to solve Teukolsky equation for Ψ in time-domain via finite difference scheme using the derived jumps
- Numerical solutions can be used to reconstruct metric perturbation in a no-string radiation gauge and derive the self force based on Pound etal (2013)
- We are currently implementing numerical code that uses this method

ありがとうございます.

(Thank you)