Finding self-force quantities in a post-Newtonian expansion

Eccentric orbits on a Schwarzschild background

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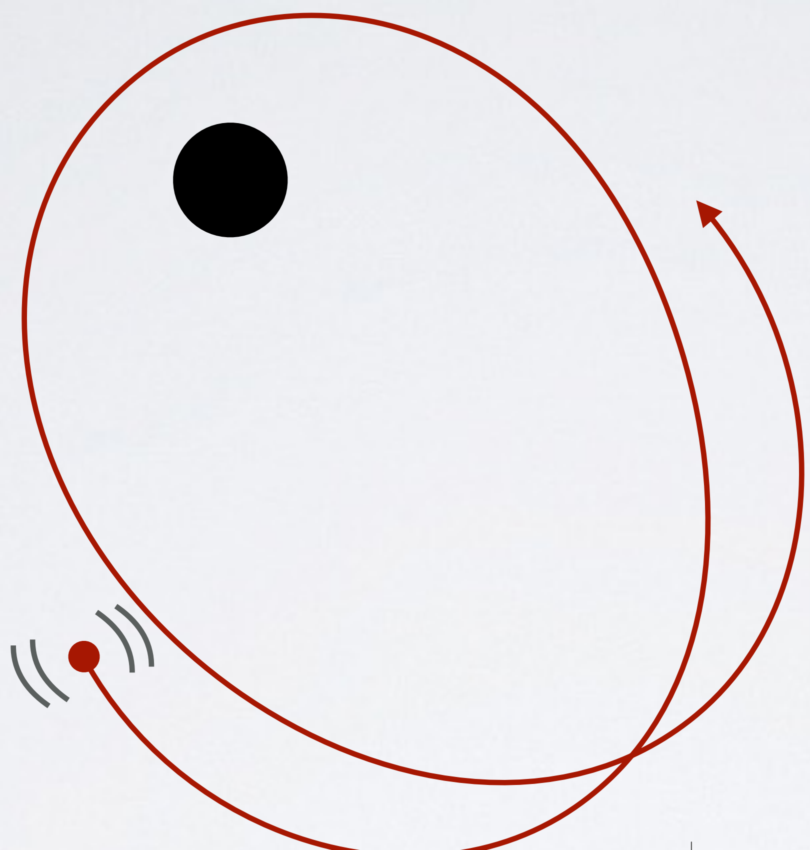
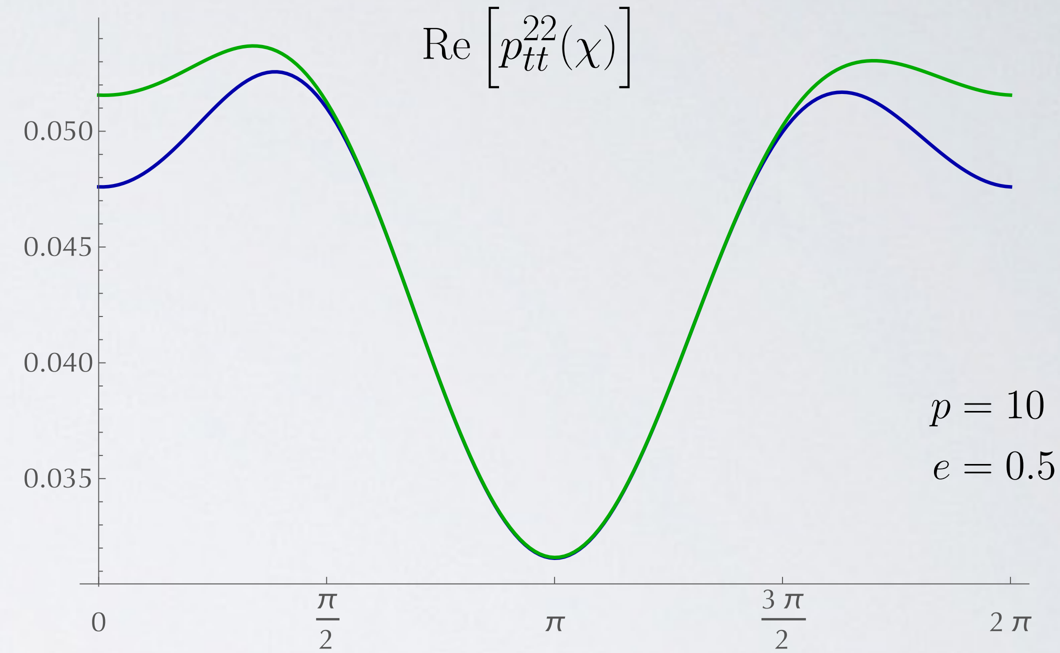
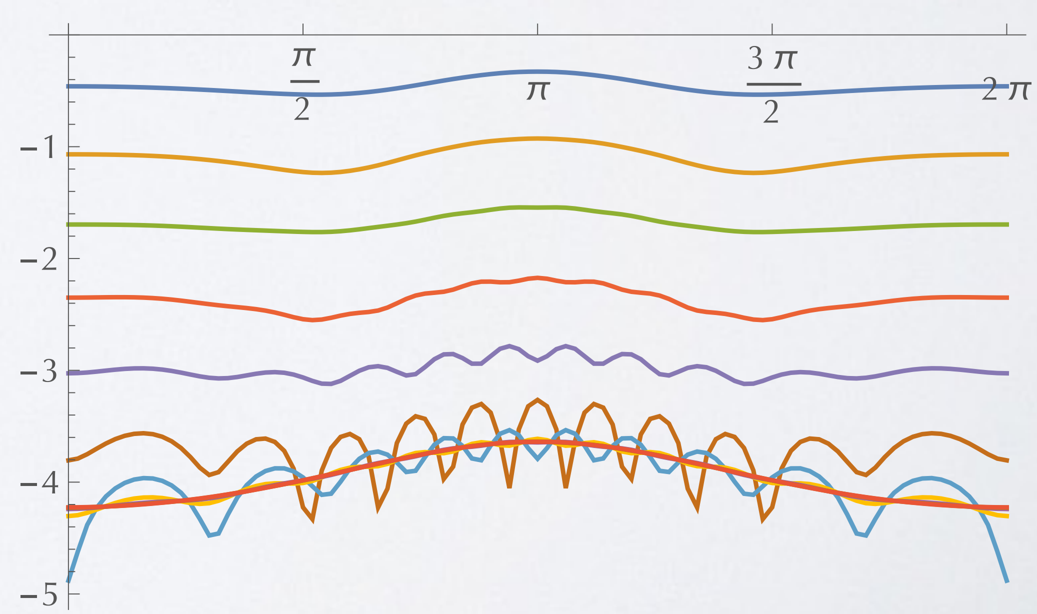
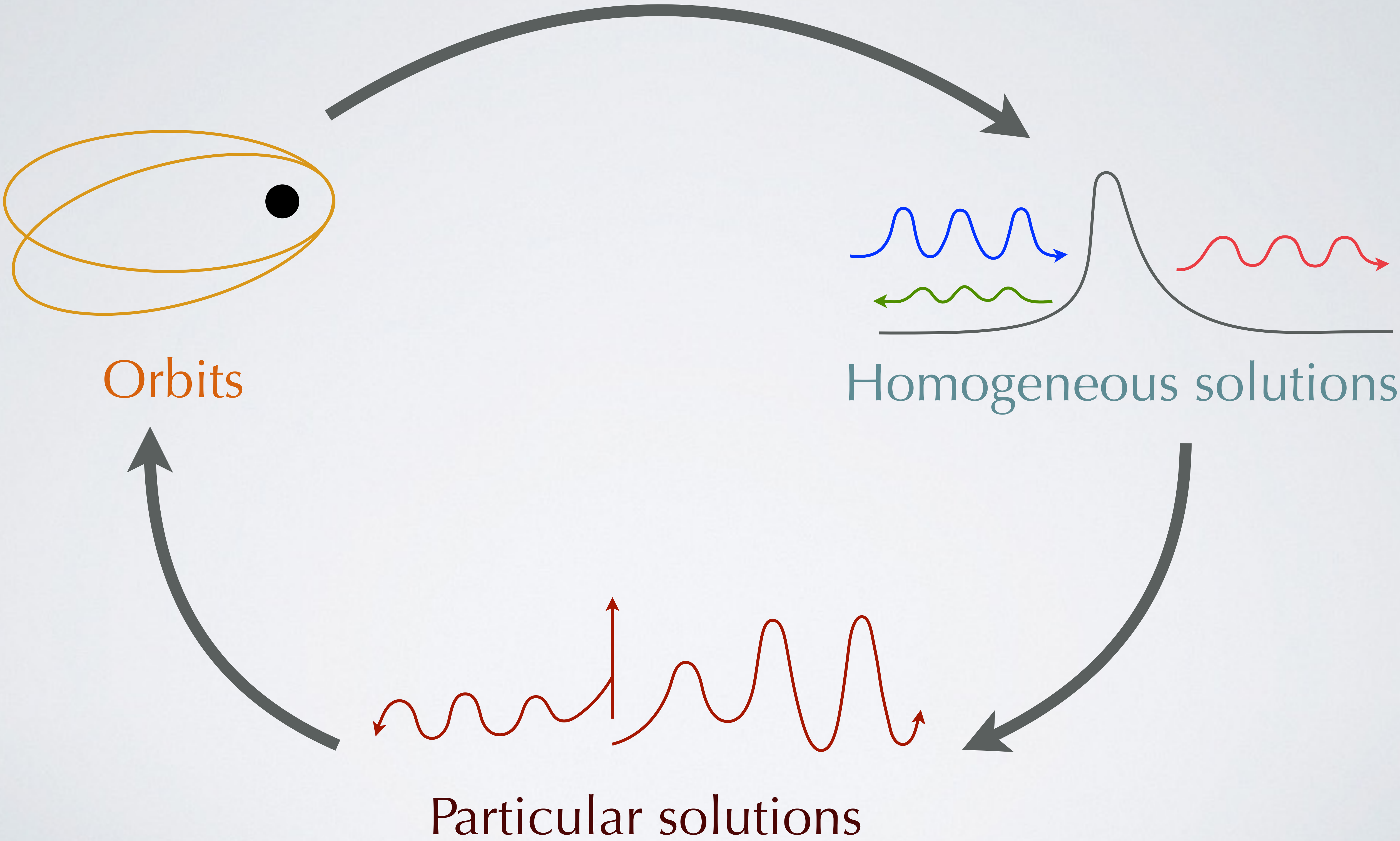






Outline - Black Hole Perturbation Theory



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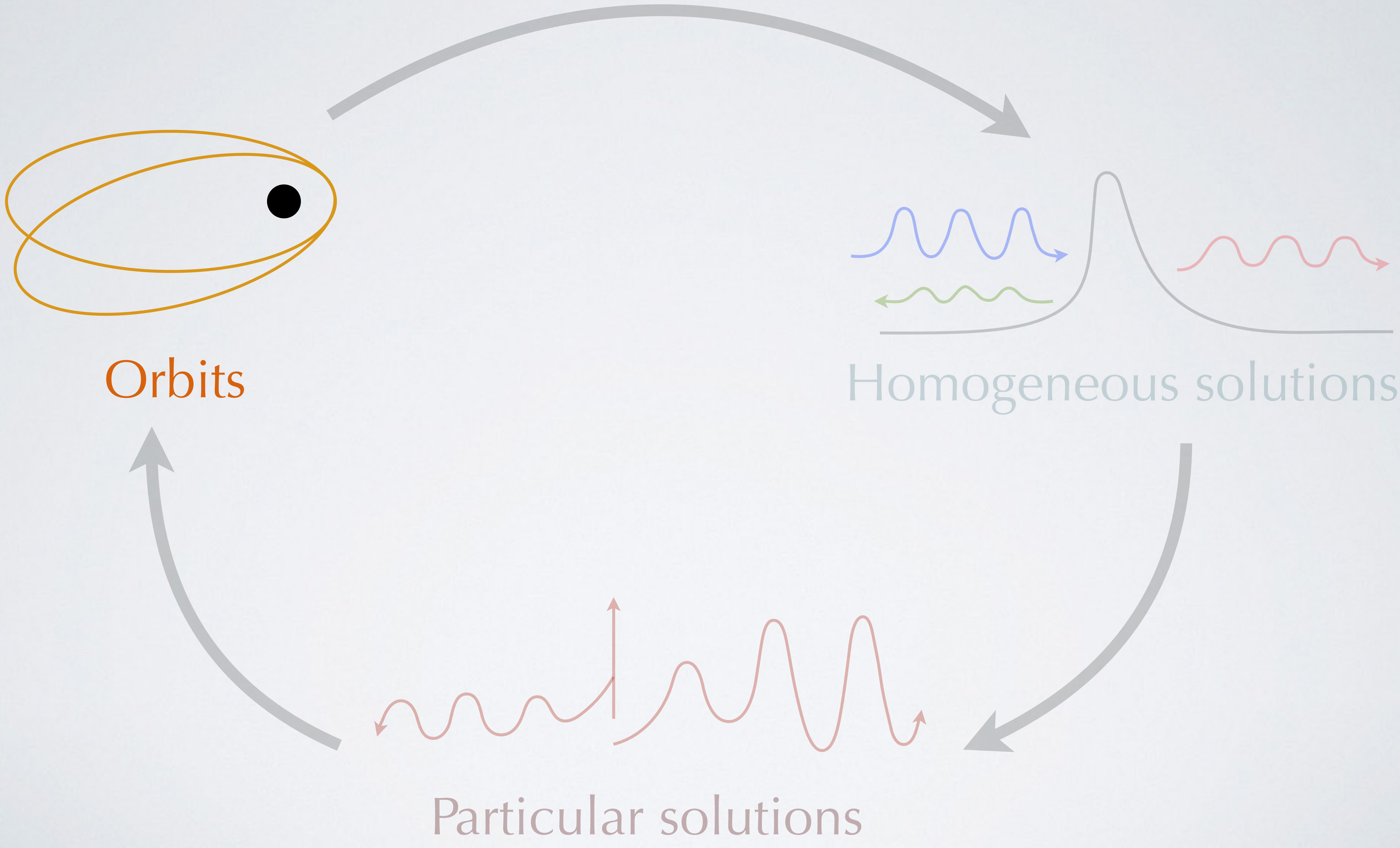
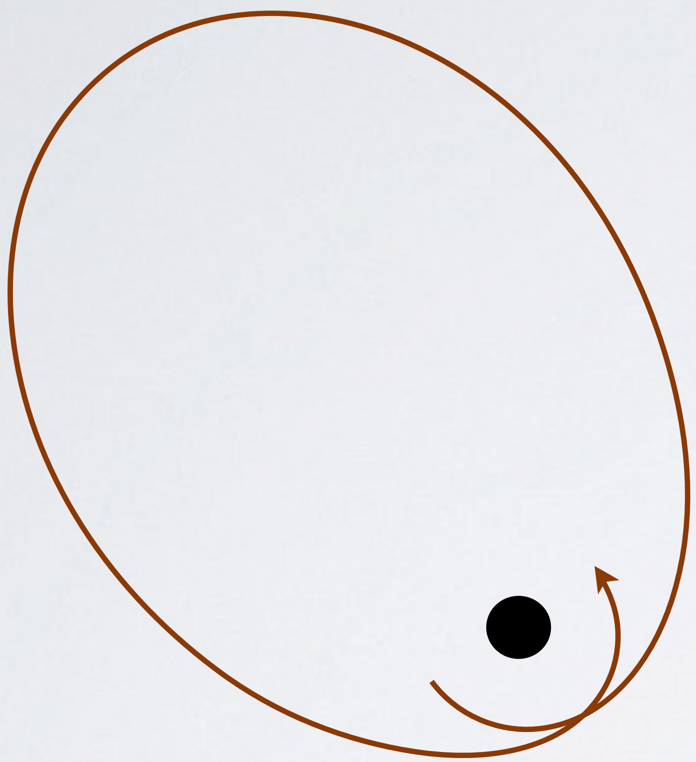
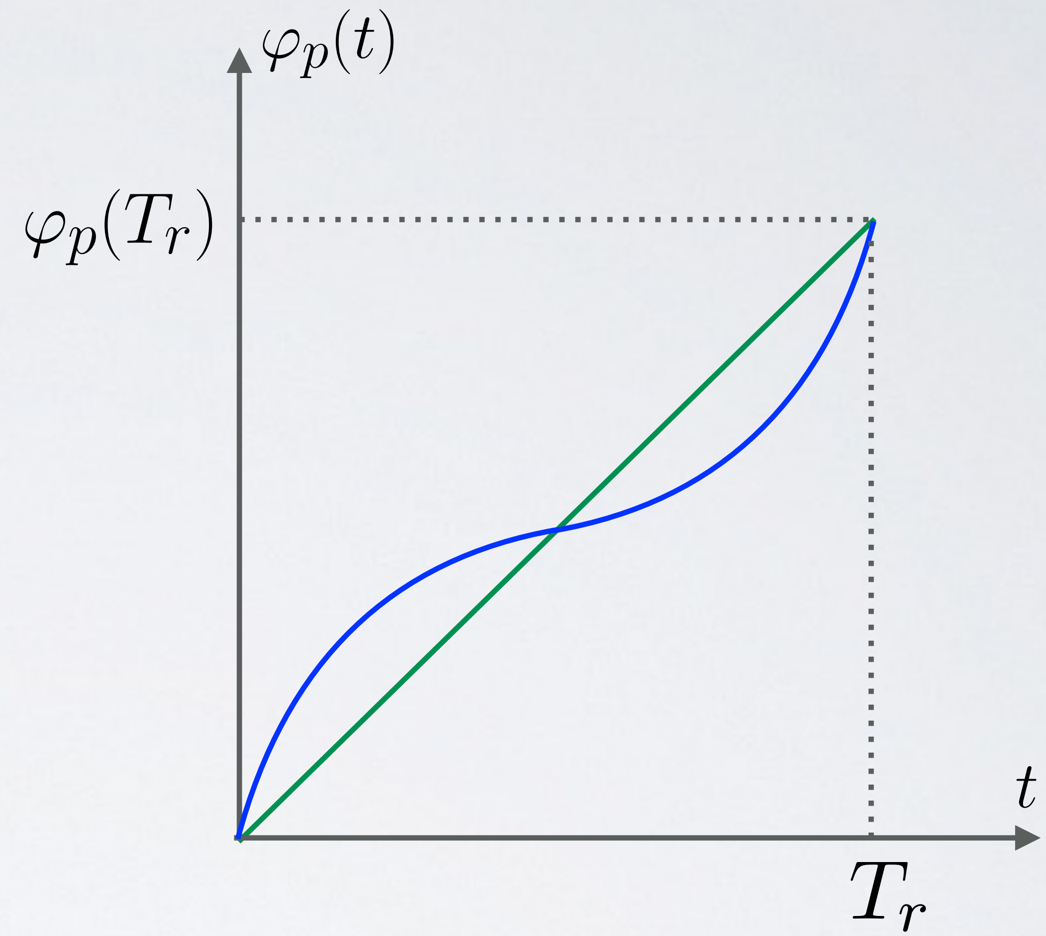


Eccentric orbits on Schwarzschild have two frequencies





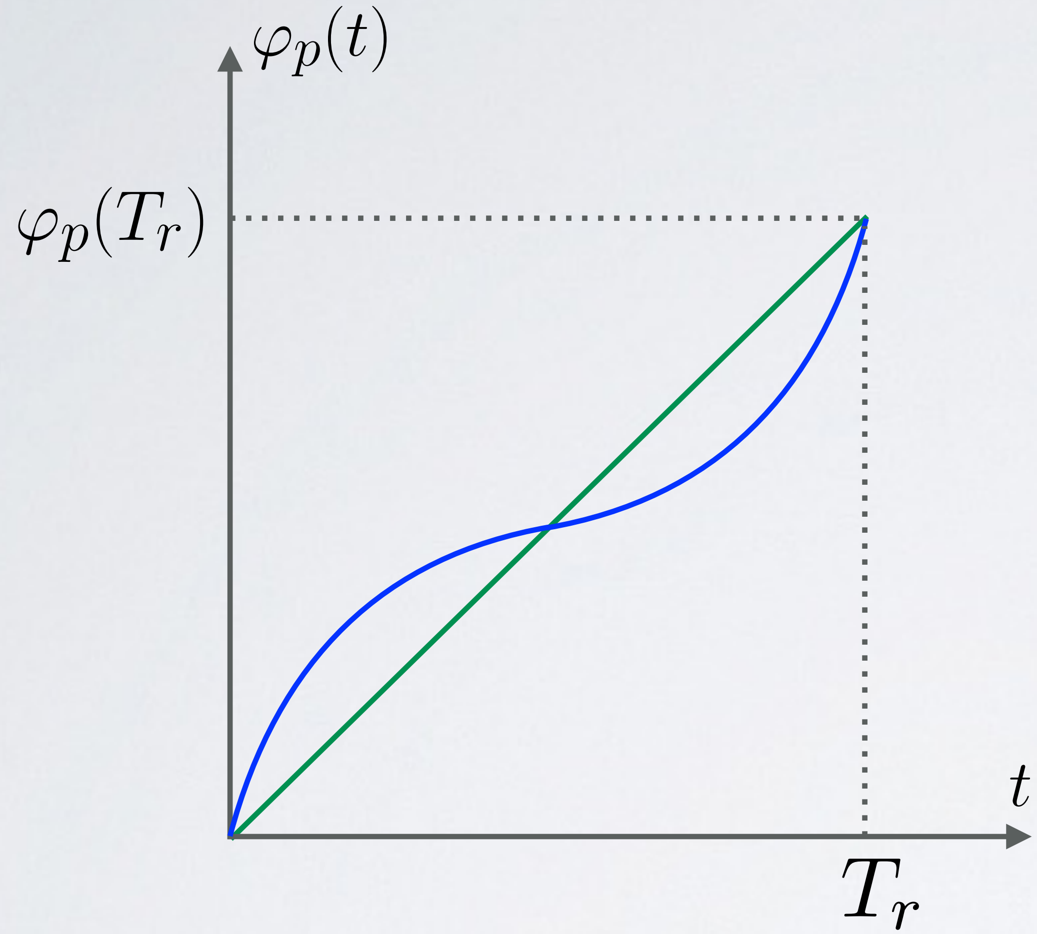
$$r_{\min} \to r_{\min}$$

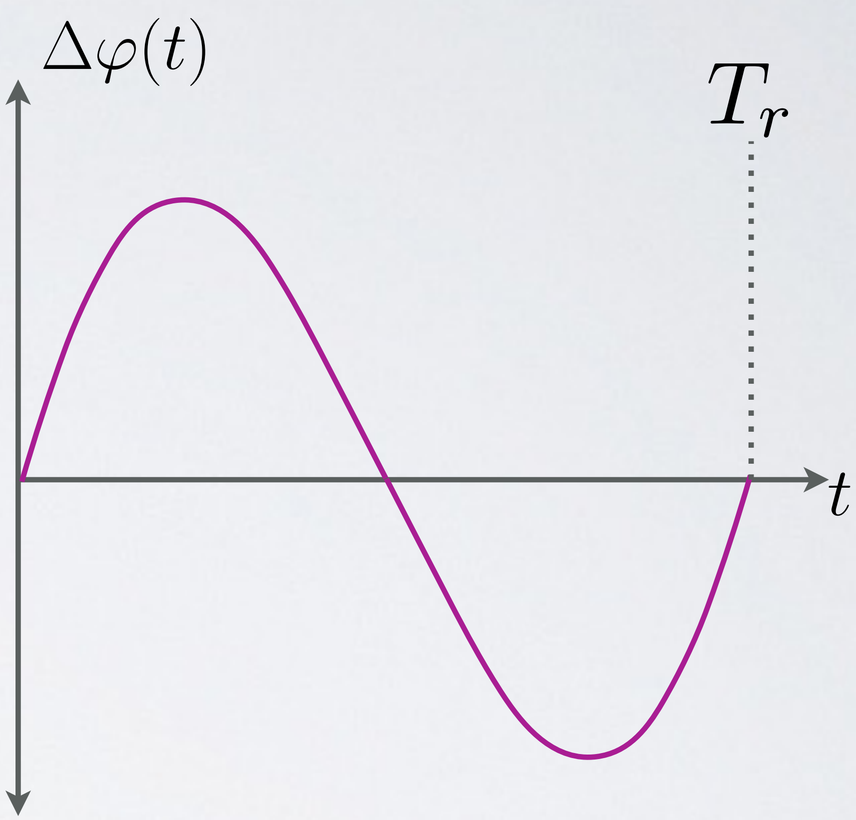
$$0 \leq t \leq T_r$$

$$\Omega_r = \frac{2\pi}{T_r}$$

$$\Omega_\varphi = \frac{\varphi_p(T_r)}{T_r}$$

The radial frequency is oscillatory, but the azimuthal frequency is not



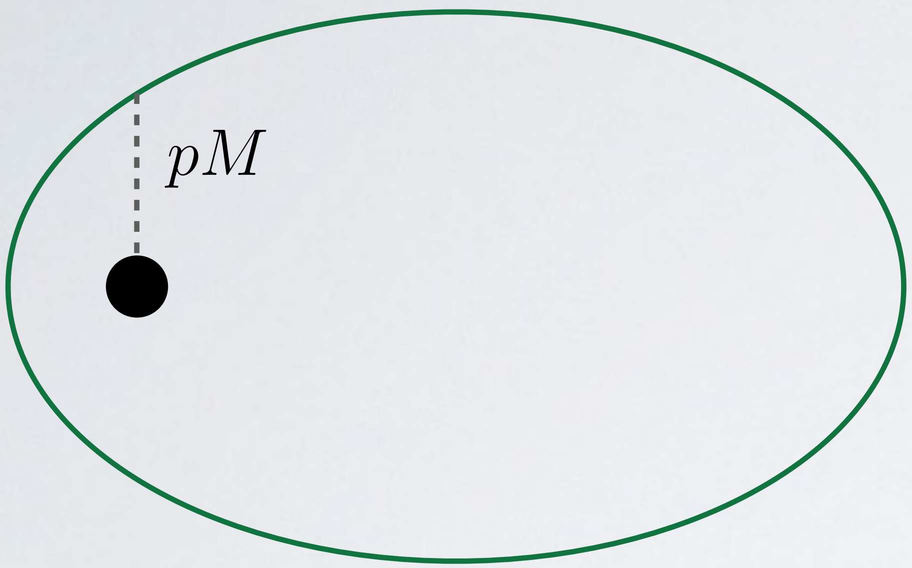


$$\Omega_r = \frac{2\pi}{T_r}$$

$$\Omega_\varphi = \frac{\varphi_p(T_r)}{T_r}$$

$$\Delta\varphi(t) = \varphi_p(t) - \Omega_\varphi t$$

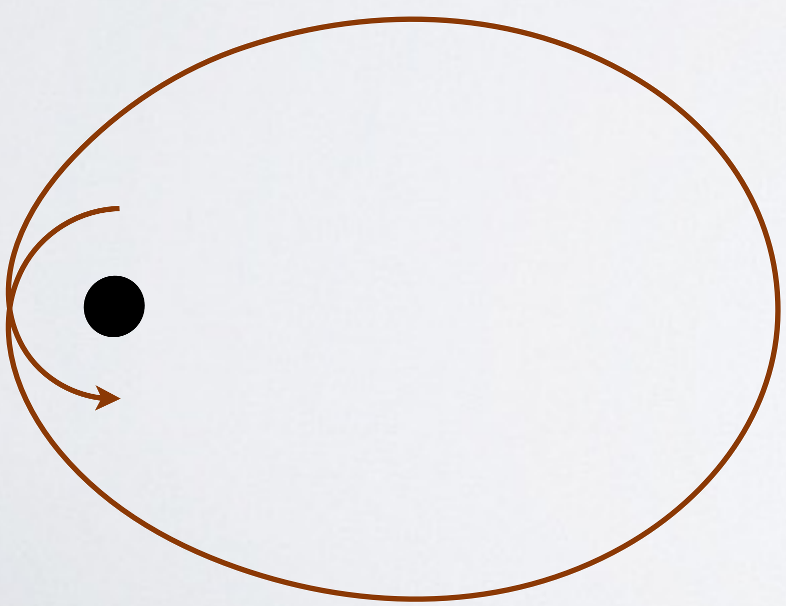
Darwin's parametrization puts the orbit into a "Newtonian-like" form



$$r_p(\chi) = \frac{pM}{1 + e\cos\chi}$$

$$0 \leq \chi \leq 2\pi \iff 0 \leq t \leq T_r$$

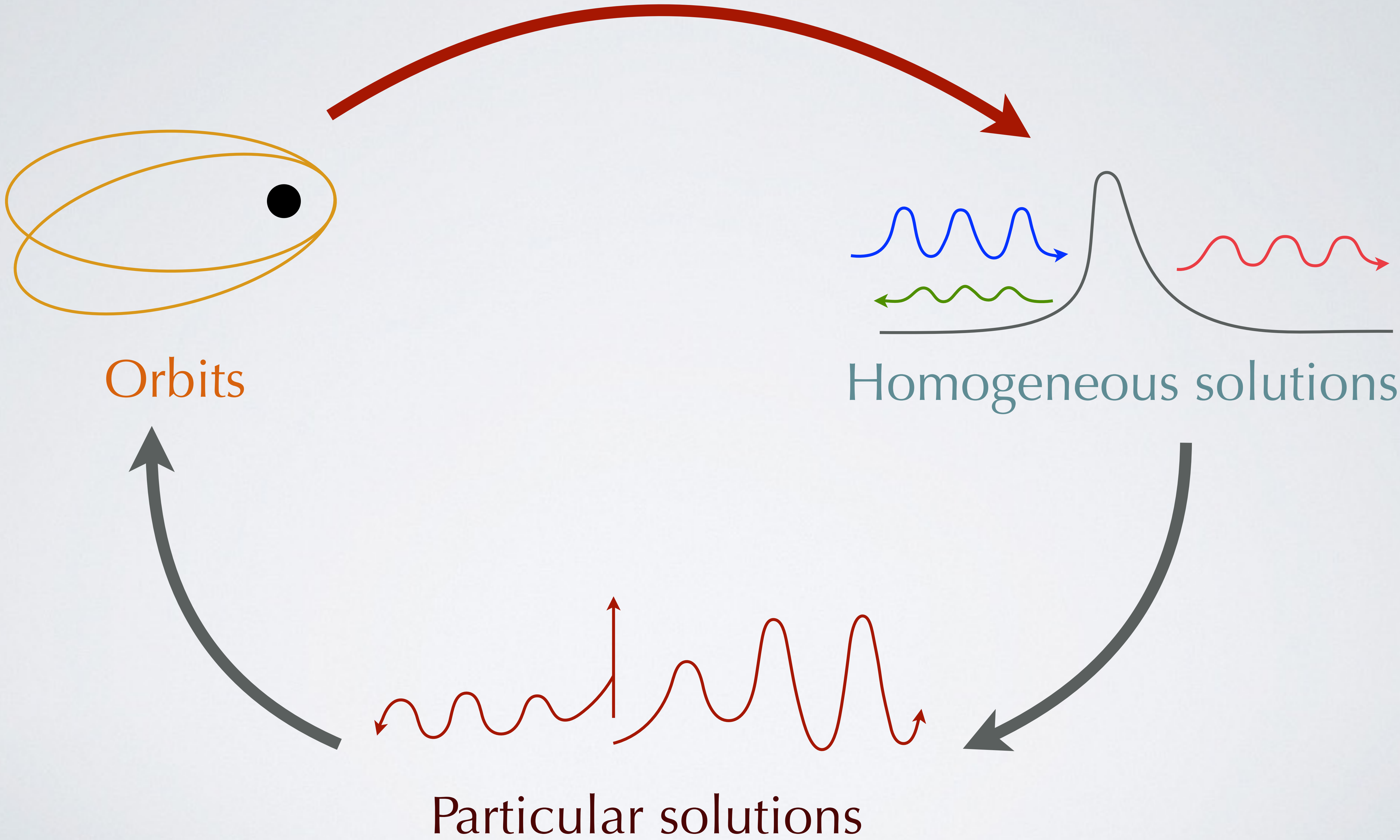
One radial period



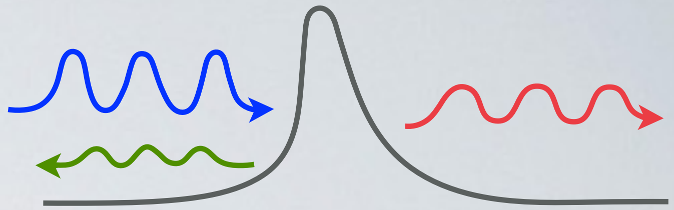
$$\varphi_p(\chi) = \left(\frac{4p}{p-6-2e}\right)^{1/2} F\left(\frac{\chi}{2}\,\middle|\, -\frac{4e}{p-6-2e}\right)$$

$$t_p(\chi) = \int_0^{\chi} \frac{p^2 M}{(p-2-2e\cos\chi')(1+e\cos\chi')^2}\left[\frac{(p-2)^2-4e^2}{p-6-2e\cos\chi'}\right]^{1/2} d\chi'$$

Outline - Black Hole Perturbation Theory



The metric perturbation in Regge-Wheeler gauge is found by solving a wave equation



TD master equation $\longrightarrow$

$$\left[-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial r_*^2} - V_\ell(r)\right] \Psi_{\ell m}(t,r) = S_{\ell m}(t,r)$$

Fourier Series

$$\Psi_{\ell m}(t,r) = \sum_{n=-\infty}^{\infty} R_{\ell mn}(r)\, e^{-i\omega t}$$

$$S_{\ell m}(t,r) = \sum_{n=-\infty}^{\infty} Z_{\ell mn}(r)\, e^{-i\omega t}$$

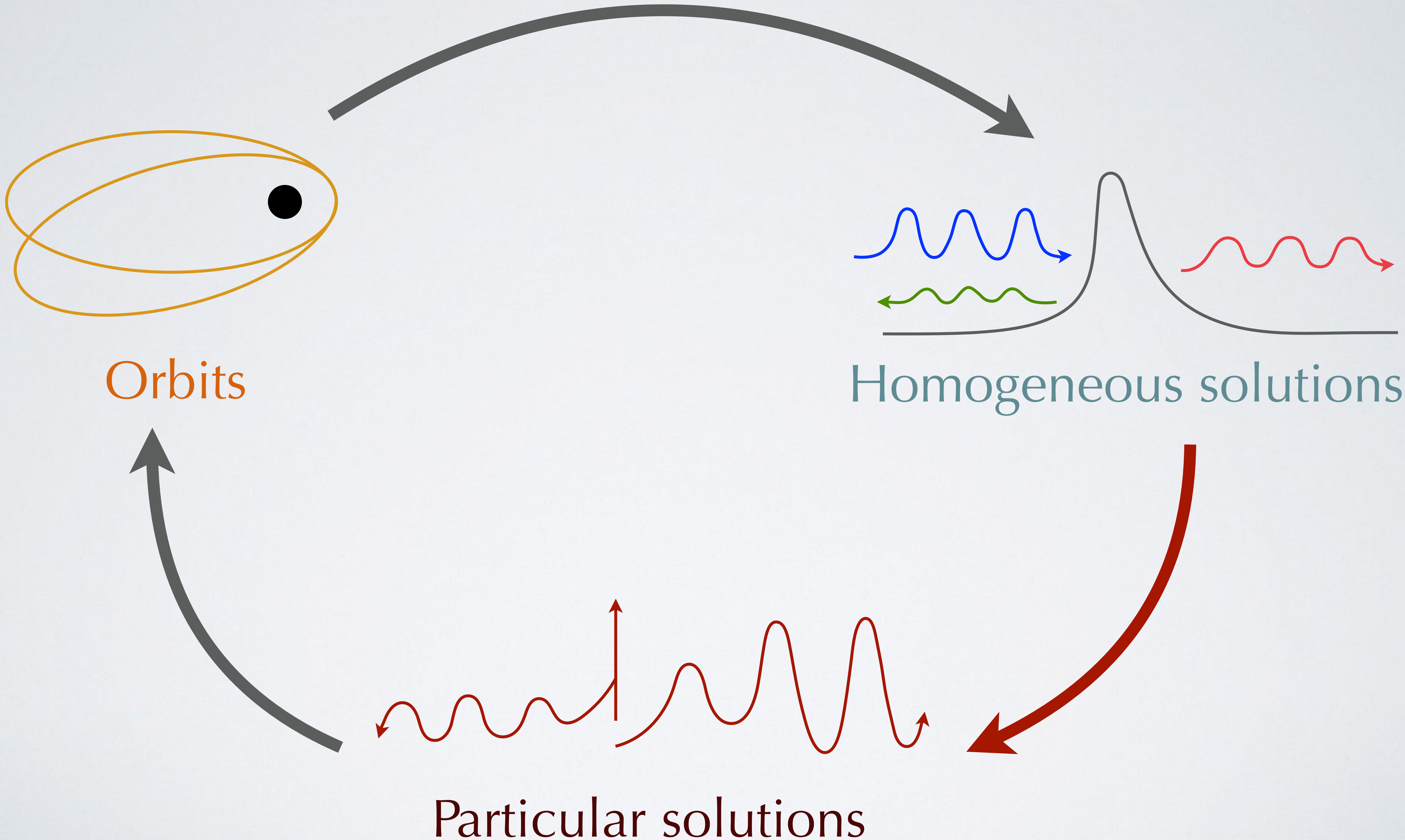
$$\omega \equiv \omega_{mn} = m\Omega_\varphi + n\Omega_r$$

Doubly periodic

FD master equation $\longrightarrow$

$$\left[\frac{d^2}{dr_*^2} + \omega^2 - V_\ell(r)\right] R_{\ell mn}(r) = Z_{\ell mn}(r)$$

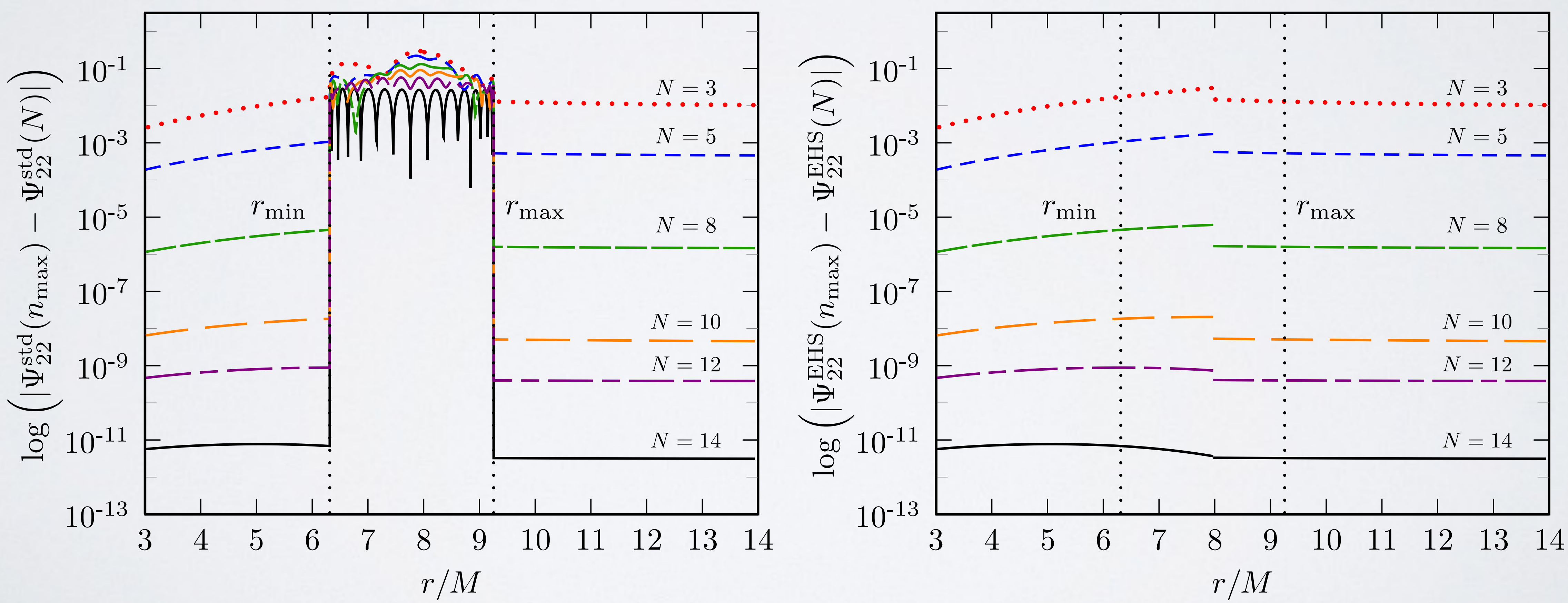
Outline - Black Hole Perturbation Theory



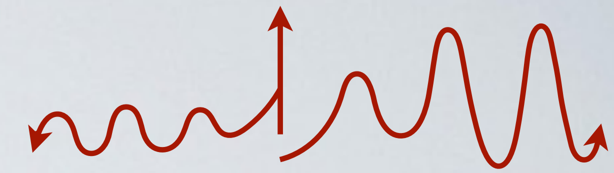
Exponential convergence is achieved through extended homogeneous solutions

Time domain EHS ⟶ $$\Psi^{\pm}_{\ell m}(t,r) = \sum_n C^{\pm}_{\ell mn} \hat{R}^{\pm}_{\ell mn}(r)\, e^{-i\omega t} \qquad r > 2M$$

$$\Psi_{\ell m}(t,r) = \Psi^{+}_{\ell m}(t,r)\theta\left[r - r_p(t)\right] + \Psi^{-}_{\ell m}(t,r)\theta\left[r_p(t) - r\right]$$



Normalization coefficients are found by integrating a radially-periodic function



Homogeneous solutions

$$C^{\pm}_{\ell mn} = \frac{1}{W_{\ell mn} T_r} \int_0^{T_r} \left[\frac{1}{f_p} \hat{R}^{\mp}_{\ell mn}(r_p) G_{\ell m}(t) + \left(\frac{2M}{r_p^2 f_p^2} \hat{R}^{\mp}_{\ell mn}(r_p) - \frac{1}{f_p} \frac{d\hat{R}^{\mp}_{\ell mn}(r_p)}{dr} \right) F_{\ell m}(t) \right] e^{i\omega t}\, dt$$

Master equation source term

$$S_{\ell m}(t, r) = G_{\ell m}(t)\, \delta[r - r_p(t)] + F_{\ell m}(t)\, \delta'[r - r_p(t)]$$

$$\bar{G}_{\ell m}(t) \equiv G_{\ell m}(t)\, e^{im\Omega_\varphi t}, \qquad \bar{F}_{\ell m}(t) \equiv F_{\ell m}(t)\, e^{im\Omega_\varphi t}$$

Radially periodic

$$\omega = m\Omega_\varphi + n\Omega_r$$

$$C^{\pm}_{\ell mn} = \frac{1}{W_{\ell mn} T_r} \int_0^{T_r} \left[\frac{1}{f_p} \hat{R}^{\mp}_{\ell mn}(r_p) \bar{G}_{\ell m}(t) + \left(\frac{2M}{r_p^2 f_p^2} \hat{R}^{\mp}_{\ell mn}(r_p) - \frac{1}{f_p} \frac{d\hat{R}^{\mp}_{\ell mn}(r_p)}{dr} \right) \bar{F}_{\ell m}(t) \right] e^{in\Omega_r t}\, dt$$

Manifestly radially periodic

Removing the azimuthal frequency dependence from the master function makes it periodic

$$\Psi^{\pm}_{\ell m}(t,r) = \sum_n R^{\pm}_{\ell mn}(r)\, e^{-i\omega t}$$

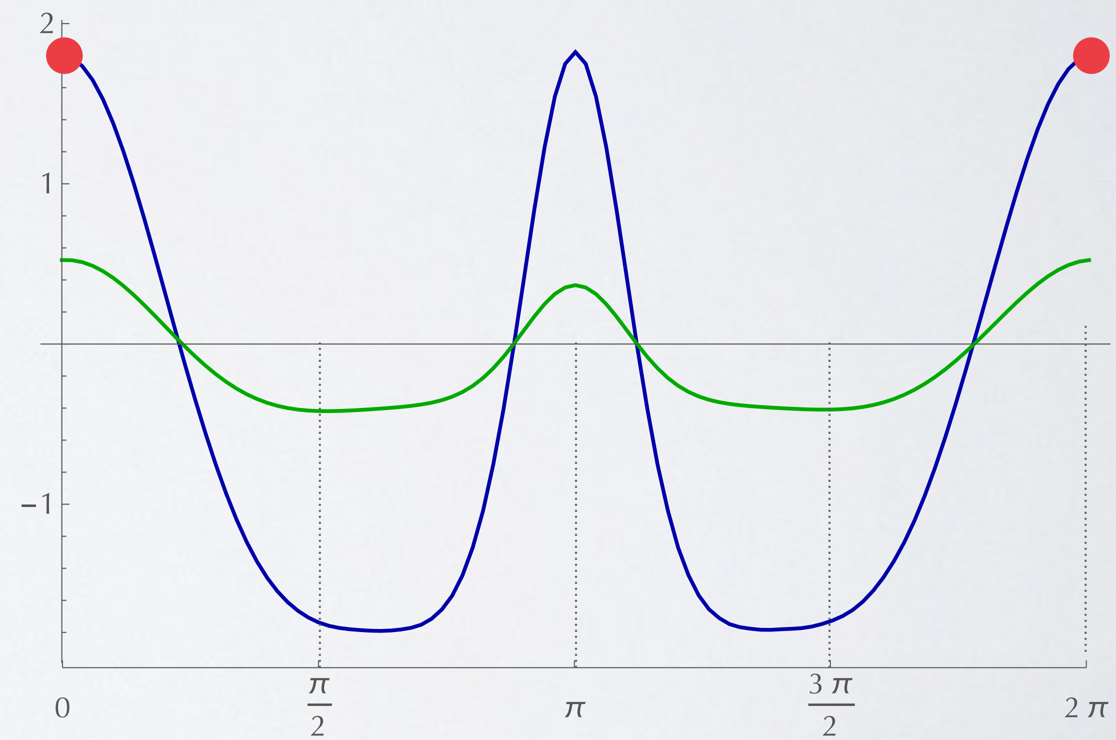
$$\bar{\Psi}^{\pm}_{\ell m}(t,r) \equiv \Psi^{\pm}_{\ell m}(t,r) e^{im\Omega_\varphi t} = \sum_n R^{\pm}_{\ell mn}(r)\, e^{-in\Omega_r t}$$

$\Psi^{\pm}_{22}(\chi)$

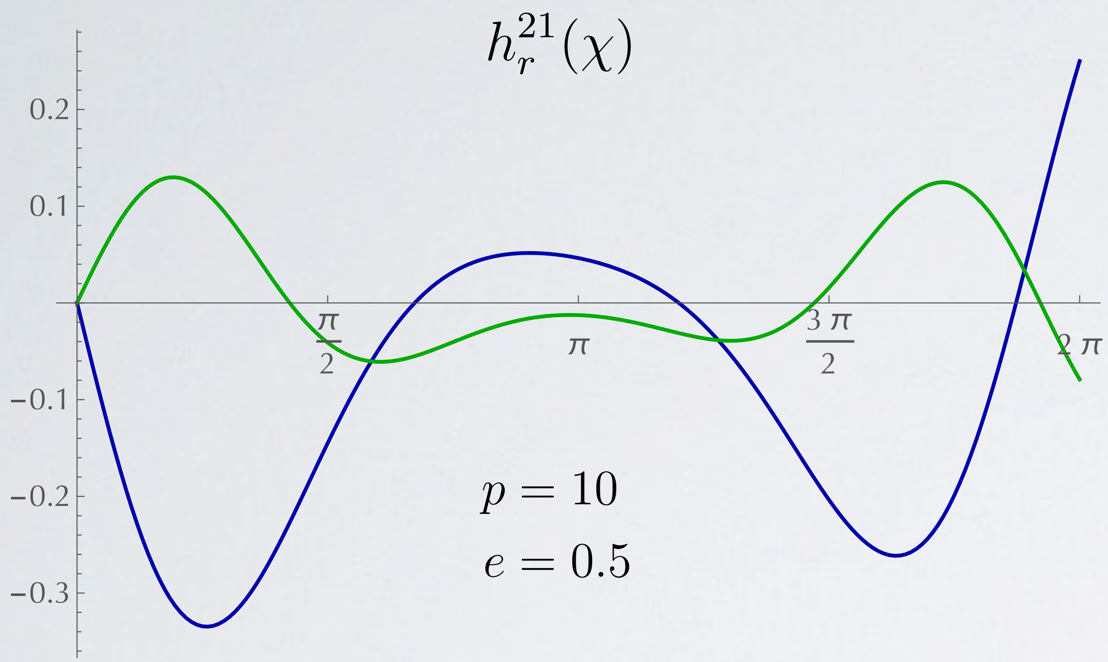
$p = 10$

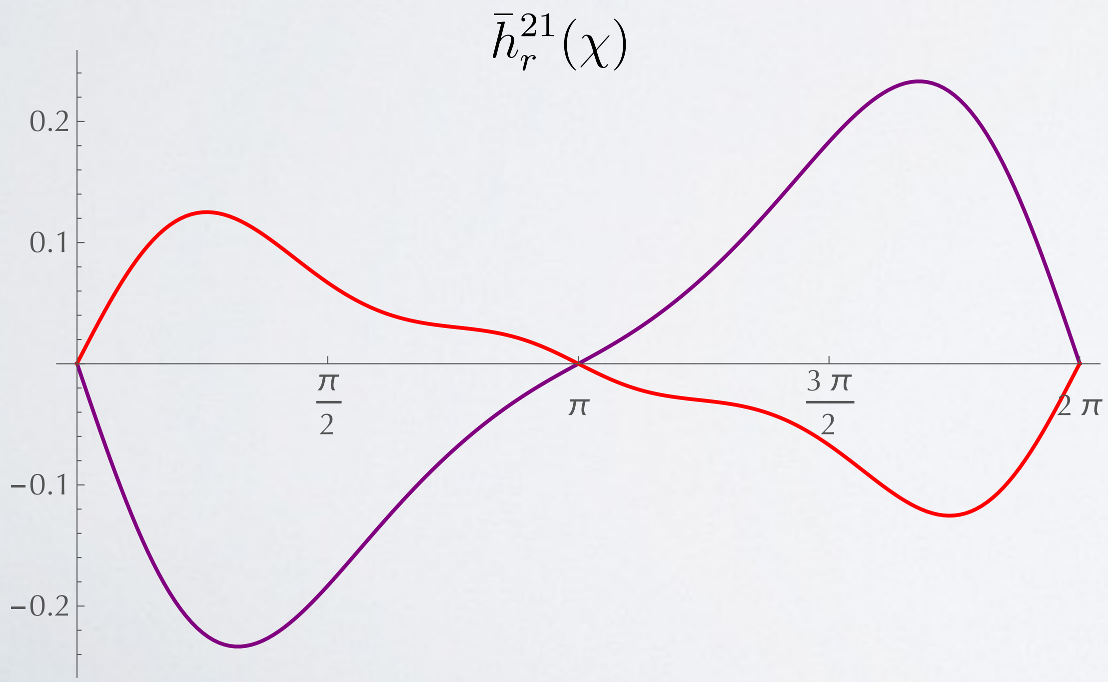
$e = 0.5$

$\bar{\Psi}^{\pm}_{22}(\chi)$



"Barred" versions of the metric perturbation amplitudes pick up counter terms



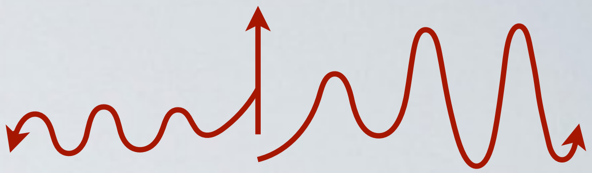


$$h_t^{\ell m,\pm} = \frac{f}{2}\partial_r\left(r\Psi^{\pm}_{\ell m}\right)$$
$$h_r^{\ell m,\pm} = \frac{r}{2f}\left(\partial_t\Psi^{\pm}_{\ell m}\right)$$

$$\bar{\Psi}^{\pm}_{\ell m}(t,r) = \Psi^{\pm}_{\ell m}(t,r)e^{im\Omega_\varphi t}$$

$$\bar{h}_t^{\ell m,\pm} = \frac{f}{2}\partial_r\left(r\bar{\Psi}^{\pm}_{\ell m}\right)$$
$$\bar{h}_r^{\ell m,\pm} = \frac{r}{2f}\left(\partial_t\bar{\Psi}^{\pm}_{\ell m} - im\Omega_\varphi\bar{\Psi}^{\pm}_{\ell m}\right)$$

Quantities evaluated on the worldline are strictly periodic in the radial motion



Dependent on four variables →

$$p_{tt}(t,r,\theta,\varphi) = \sum_{\ell,m} h_{tt}^{\ell m}(t,r) Y^{\ell m}(\theta,\varphi)$$

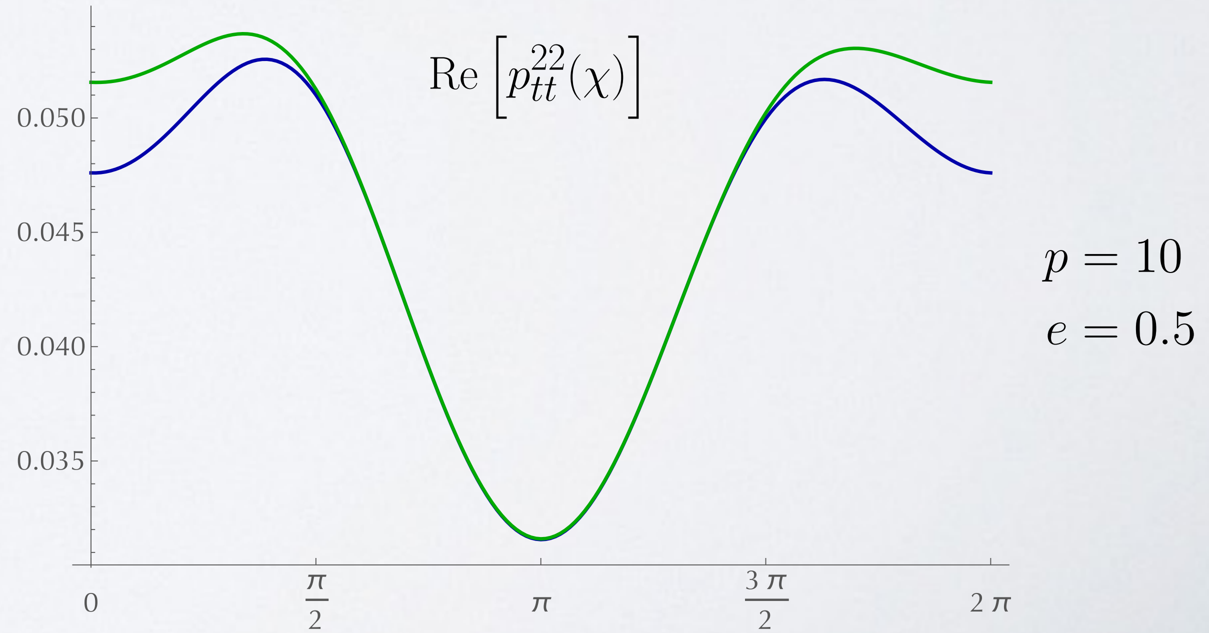
Evaluated at the particle

$$p_{tt}(\chi) = \sum_{\ell,m} h_{tt}^{\ell m}(t_p,r_p) Y^{\ell m}(\theta_p,\varphi_p)$$

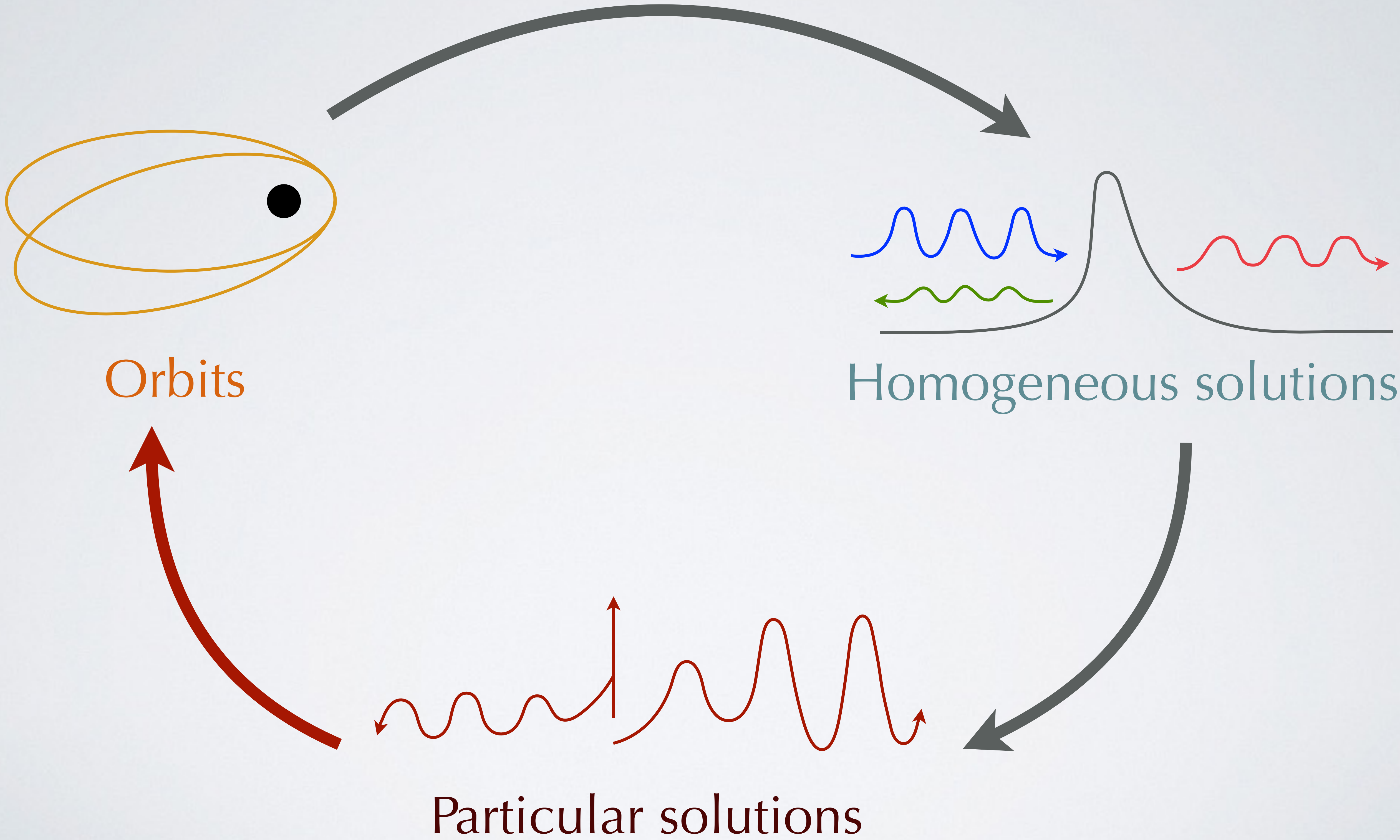
$$p_{tt}(\chi) = \sum_{\ell,m} \left[\bar{h}_{tt}^{\ell m}(t_p,r_p) e^{-im\Omega_\varphi t}\right] \cdot \left[Y^{\ell m}(\theta_p,0) e^{im\Delta\varphi(t)} e^{im\Omega_\varphi t_p}\right]$$

$$p_{tt}(\chi) = \sum_{\ell,m} \bar{h}_{tt}^{\ell m}(t,r_p) \bar{Y}^{\ell m}(\theta_p,\varphi_p)$$

$$\bar{Y}^{\ell m}(\theta_p,\varphi_p) = Y^{\ell m}(\theta_p,\varphi_p) e^{-im\Omega_\varphi t}$$



Outline - BHPT + Post-Newtonian Theory



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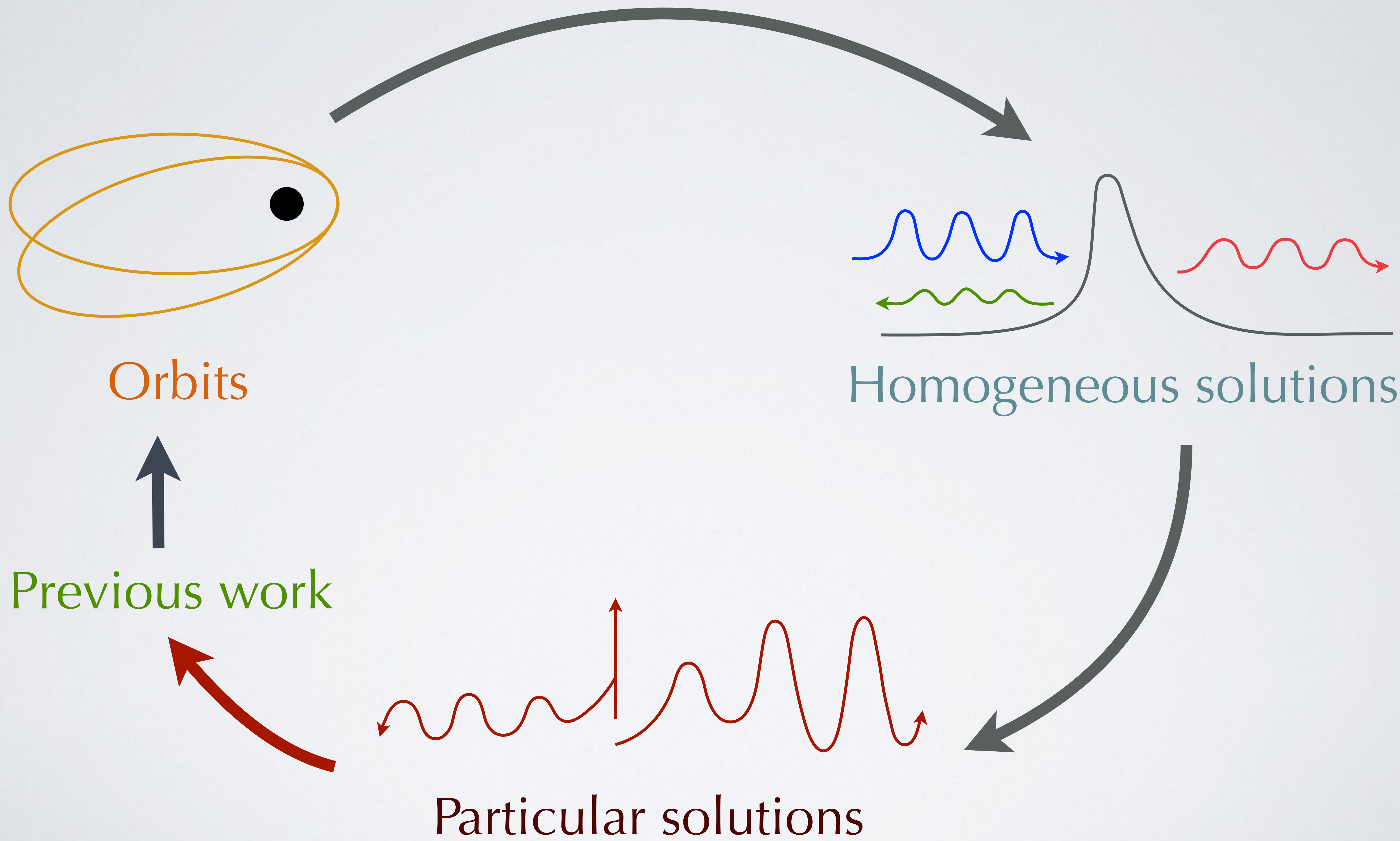

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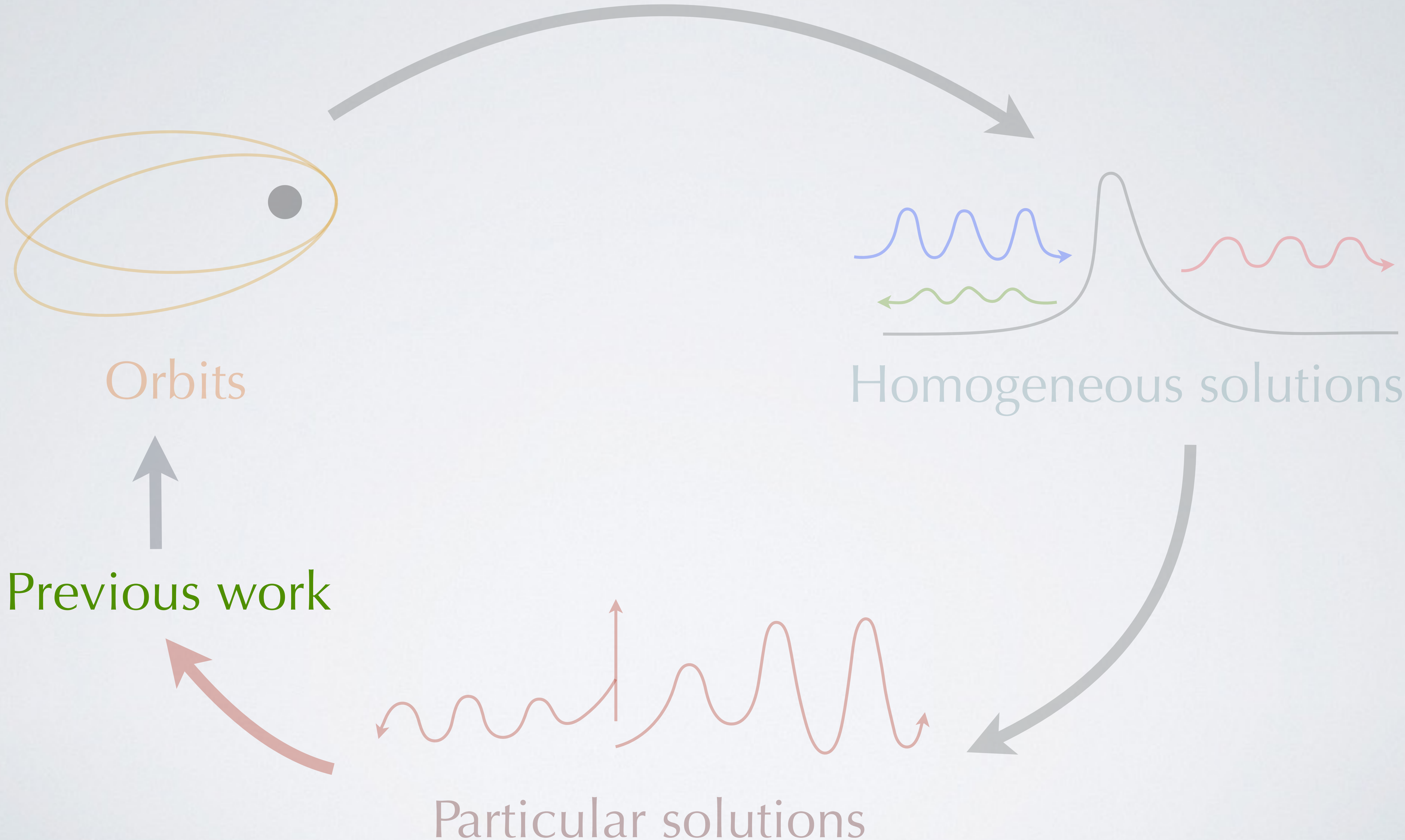
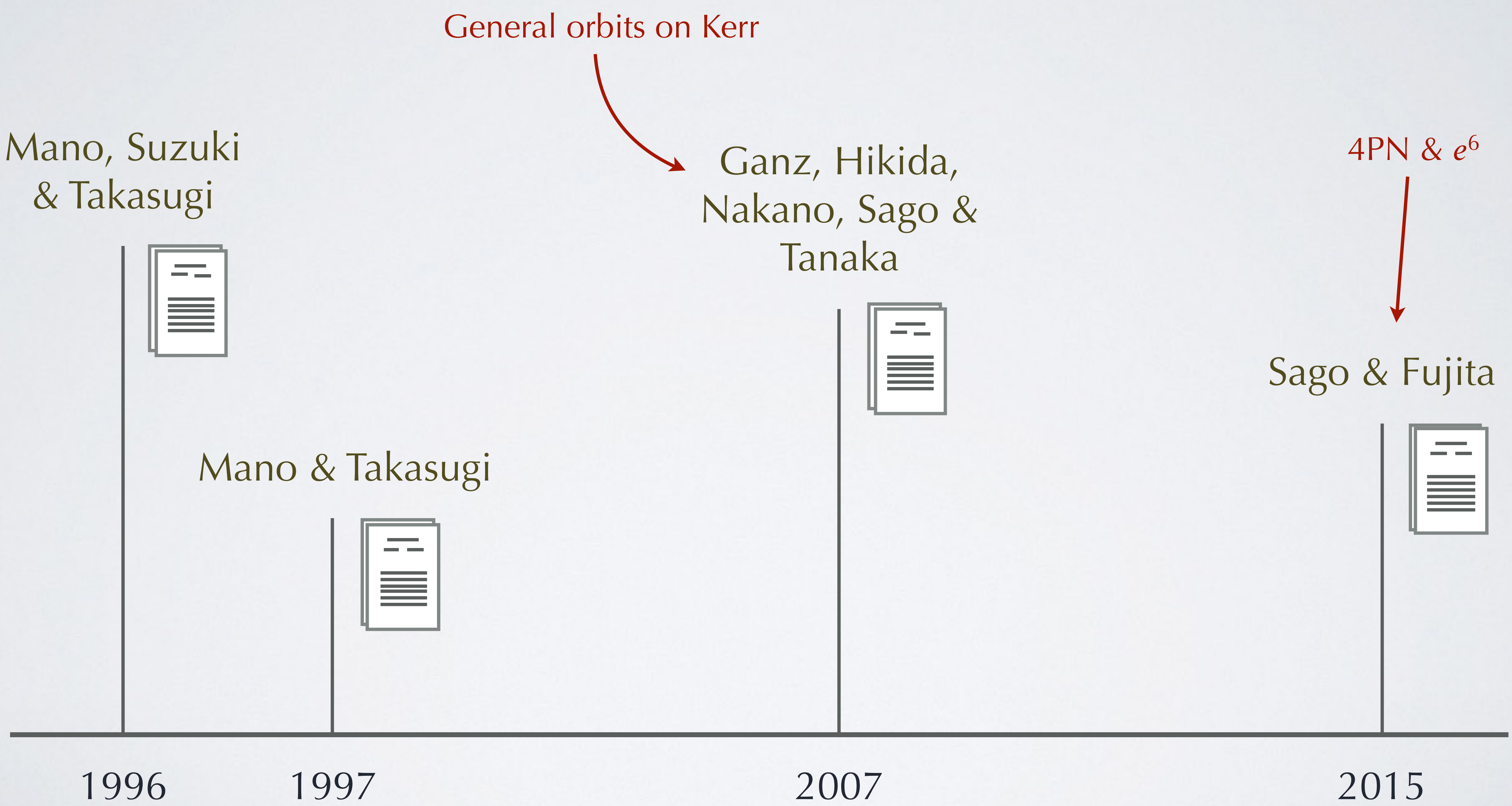
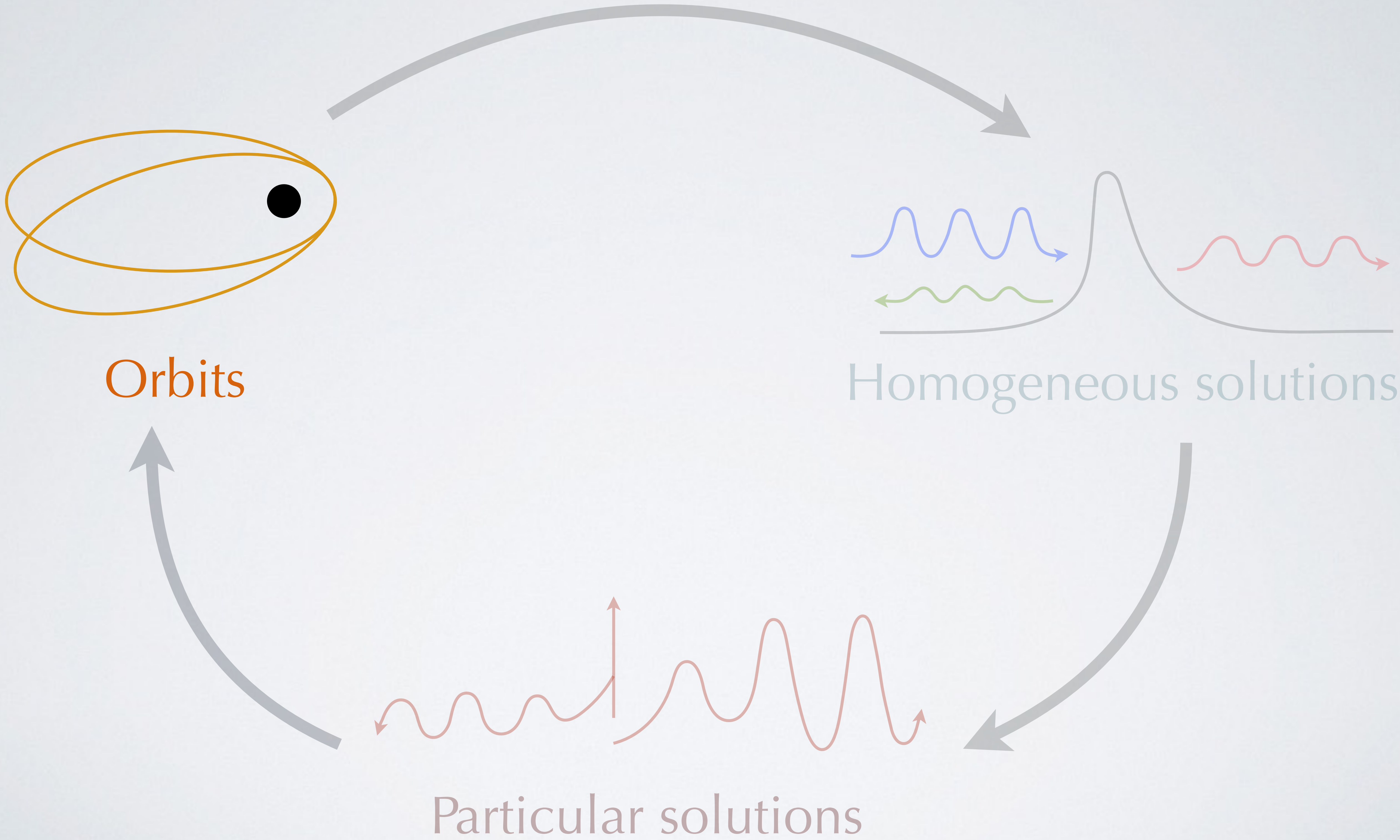


There has been a lot of previous work, but here are a couple highlights



Outline - BHPT + Post-Newtonian Theory



The fundamental frequencies can be expanded in the gauge invariant PN parameter x

Integrate order-by-order $\longrightarrow$ $\displaystyle\int_0^{2\pi} \frac{dt_p}{d\chi} d\chi$ $\longrightarrow$ $\displaystyle T_r = 2\pi M \left(\frac{p}{1-e^2}\right)^{3/2} \left[1 + \frac{3\left(1-e^2\right)}{p} + \mathcal{O}\left(p^{-2}\right)\right]$

x as an expansion in $1/p$

$$\Omega_\varphi = \frac{1}{M}\left(\frac{1-e^2}{p}\right)^{3/2}\left[1 + 3\frac{e^2}{p} + \mathcal{O}\left(p^{-2}\right)\right]$$

$$x = \left(M\Omega_\varphi\right)^{2/3}$$

Invert the expansion $\longrightarrow$ $\displaystyle p = \frac{1-e^2}{x} + 2e^2 + \mathcal{O}\left(x^1\right)$

$$\Omega_r = \frac{x^{3/2}}{M}\left[1 - \frac{3}{1-e^2}x + \mathcal{O}(x^2)\right] \qquad \Omega_\varphi = \frac{x^{3/2}}{M}$$

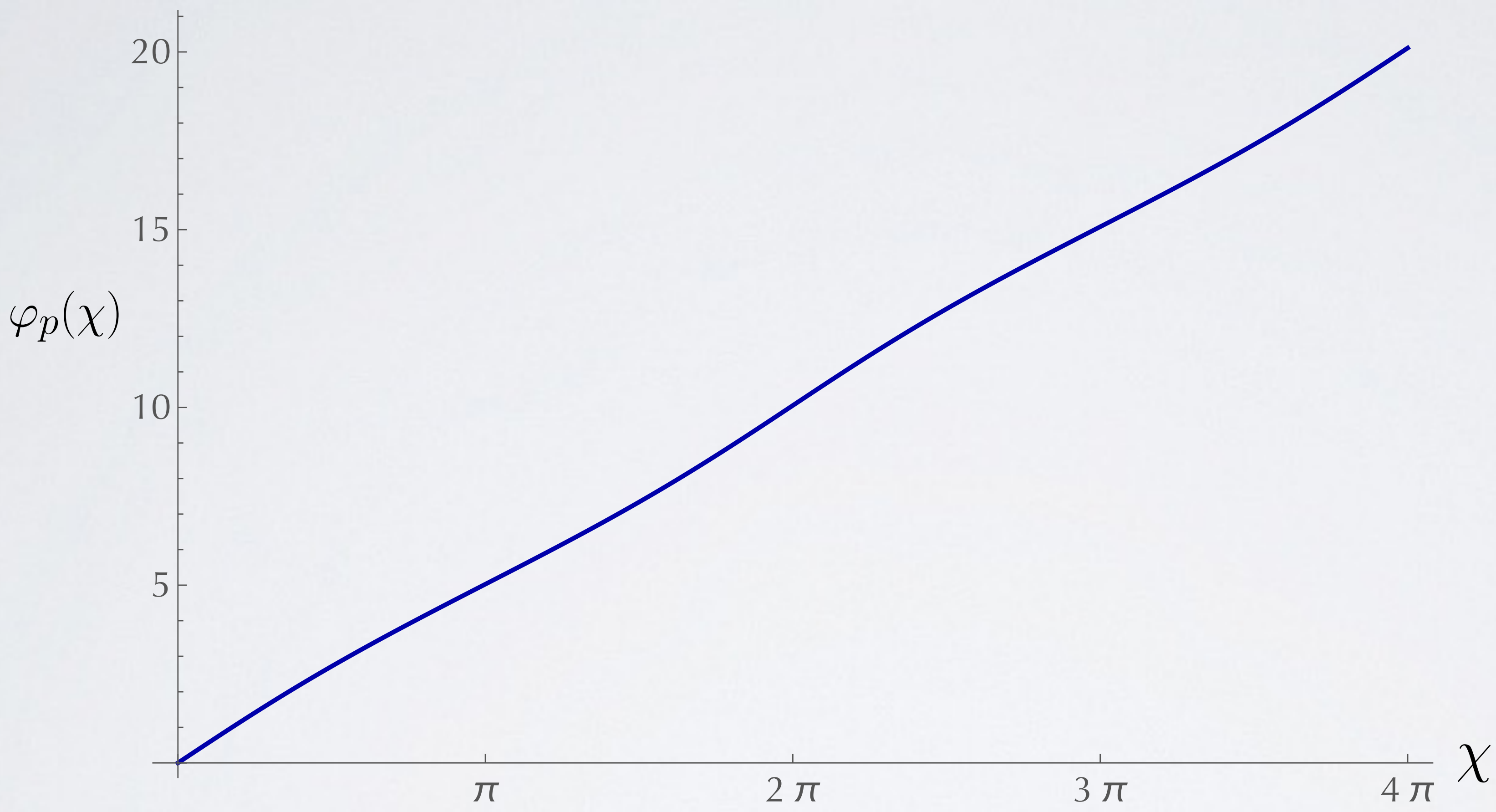
Fundamental frequencies

Radial position is described by a cosine series at each PN order

$p = 10$
$e = 0.5$
$x = 0.08$

$$r_p(\chi) = \left[M - eM\cos(\chi) + \mathcal{O}\left(e^2\right)\right]x^{-1} + \left[2e^2M - 2e^3M\cos(\chi) + \mathcal{O}\left(e^4\right)\right]x^0 + \mathcal{O}\left(x^1\right)$$

Azimuthal motion is dominated by linear growth

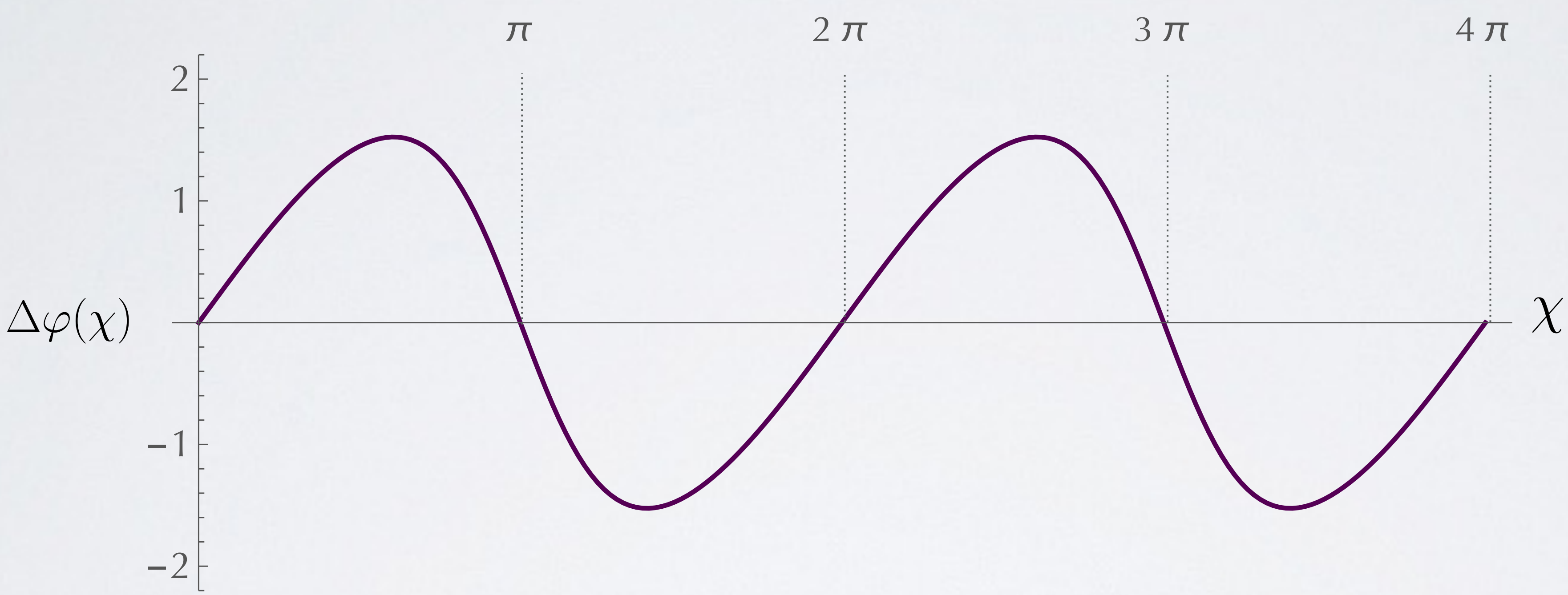


$p = 10$
$e = 0.5$
$x = 0.08$

$$\varphi_p(\chi) = \chi + \left[3\chi + \sin(\chi)e + 3\chi e^2 + \mathcal{O}\left(e^3\right)\right] x + \mathcal{O}\left(x^2\right)$$

$\Delta\varphi$ is described by a sine series at each PN order

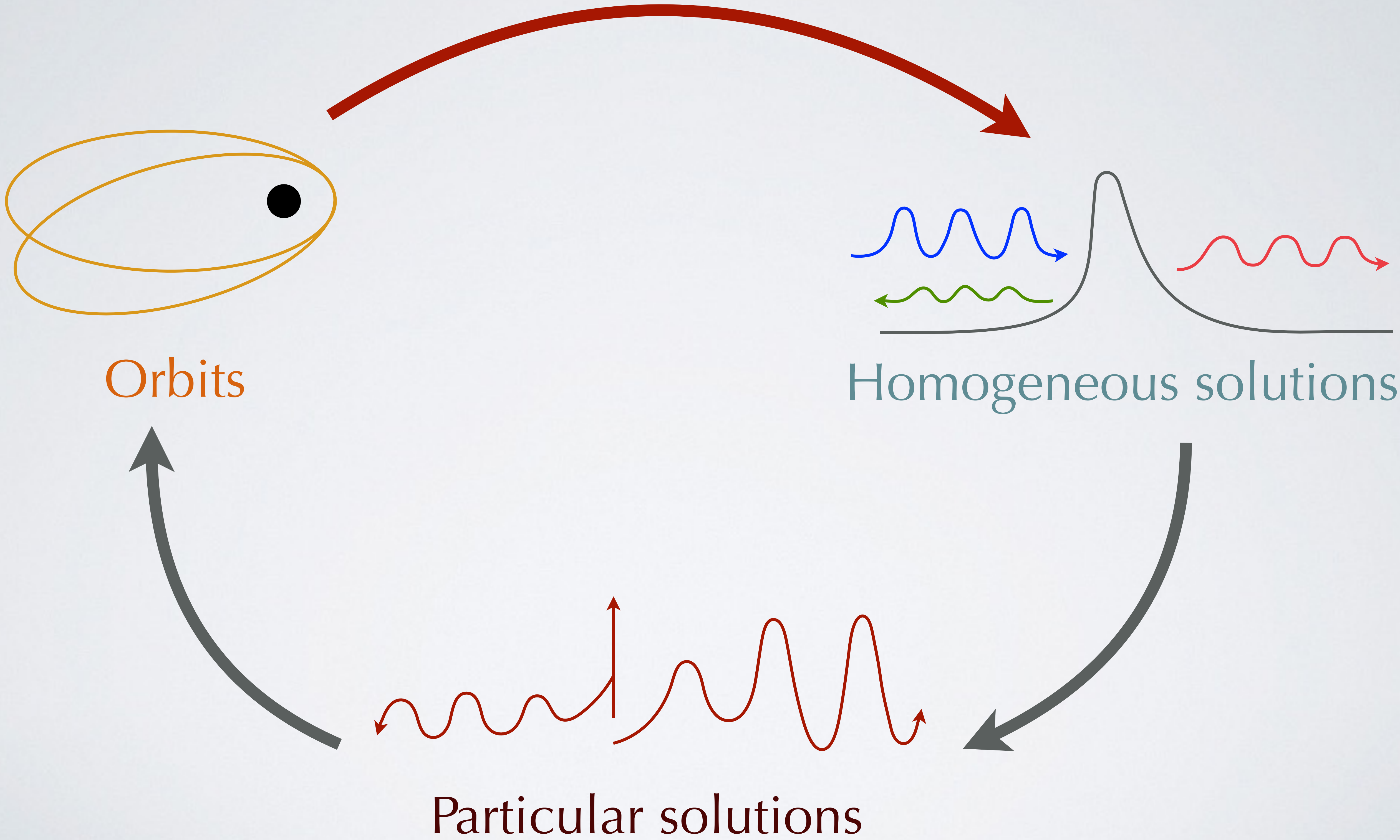
$$\Delta\varphi(\chi) = \varphi_p(\chi) - \Omega_\varphi t_p(\chi)$$



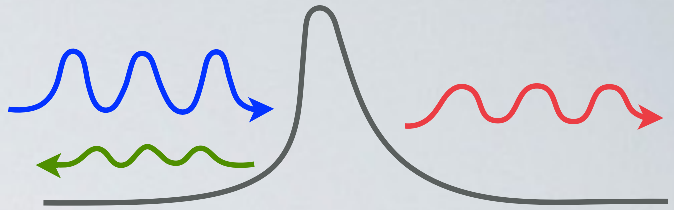
$p = 10$
$e = 0.5$
$x = 0.08$

$$\Delta\varphi(\chi) = \left[2e\sin(\chi) - \frac{3}{4}e^2\sin(2\chi) + \mathcal{O}\left(e^3\right)\right]x^0 + \left[4e\sin(\chi) - \frac{3}{4}e^2\sin(2\chi) + \mathcal{O}\left(e^3\right)\right]x^1 + \mathcal{O}\left(x^2\right)$$

Outline - BHPT + Post-Newtonian Theory



Homogeneous solutions are found by expanding in two separate limits



Odd parity, homogenous equation

$$\left[\frac{d^2}{dr_*^2} + \omega^2 - V_\ell(r)\right] R_{\ell\omega}(r) = 0$$

Expansion variables

$$X_1 = \frac{M}{r} \qquad X_2 = (\omega r)^2$$

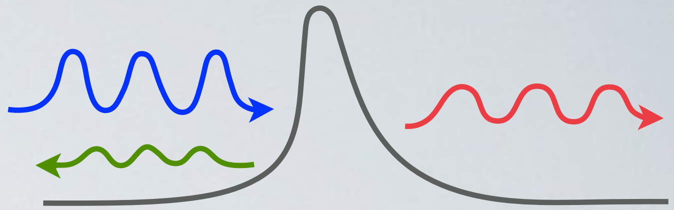
$$X_1 \sim X_2 \ll 1$$

Example, *l* = 2, through 1PN

$$R^+ = -\frac{i}{16 X_2} c^2 - \frac{i(10 X_1 + X_2)}{96 X_2} c^0 + \mathcal{O}\left(c^{-2}\right)$$

$$R^- = \frac{i}{384 X_1^4 \sqrt{X_2}} c^9 - \frac{i\sqrt{X_2}}{5376 X_1^4} c^7 + \mathcal{O}\left(c^5\right)$$

The homogeneous solutions are evaluated at the location of the particle



Generic X_1 and X_2

$$R^+ = -\frac{i}{16X_2}c^2 - \frac{i(10X_1 + X_2)}{96X_2}c^0 + \mathcal{O}\left(c^{-2}\right)$$

$$R^- = \frac{i}{384X_1^4\sqrt{X_2}}c^9 - \frac{i\sqrt{X_2}}{5376X_1^4}c^7 + \mathcal{O}\left(c^5\right)$$

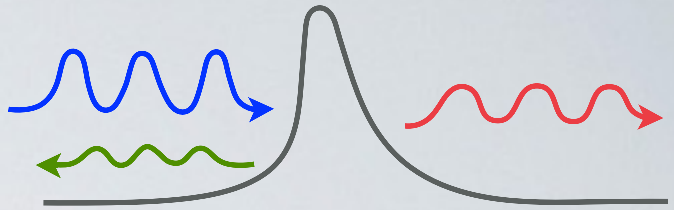
Evaluated at the particle, frequency unspecified

$$X_1 \to \frac{M}{r_p} \qquad X_2 \to (\omega r_p)^2$$

$$R^+ = -\frac{i}{16\omega^2 r_p^2}c^2 - \frac{1}{96}i\left(\frac{10M}{\omega^2 r_p^3} + 1\right)c^0 + \mathcal{O}\left(c^{-2}\right)$$

$$R^- = \frac{i r_p^3}{384M^4\omega}c^9 - \frac{i\omega r_p^5}{5376M^4}c^7 + \mathcal{O}\left(c^5\right)$$

Use expressions for the position and the frequency, expanded in both *x* and *e*



Spectrum from source's Fourier series

$$r_p(\chi) = \left[M - eM\cos(\chi) + \mathcal{O}\left(e^2\right)\right]x^{-1} + \left[2e^2M - 2e^3M\cos(\chi) + \mathcal{O}\left(e^4\right)\right]x^0 + \mathcal{O}\left(x^1\right)$$

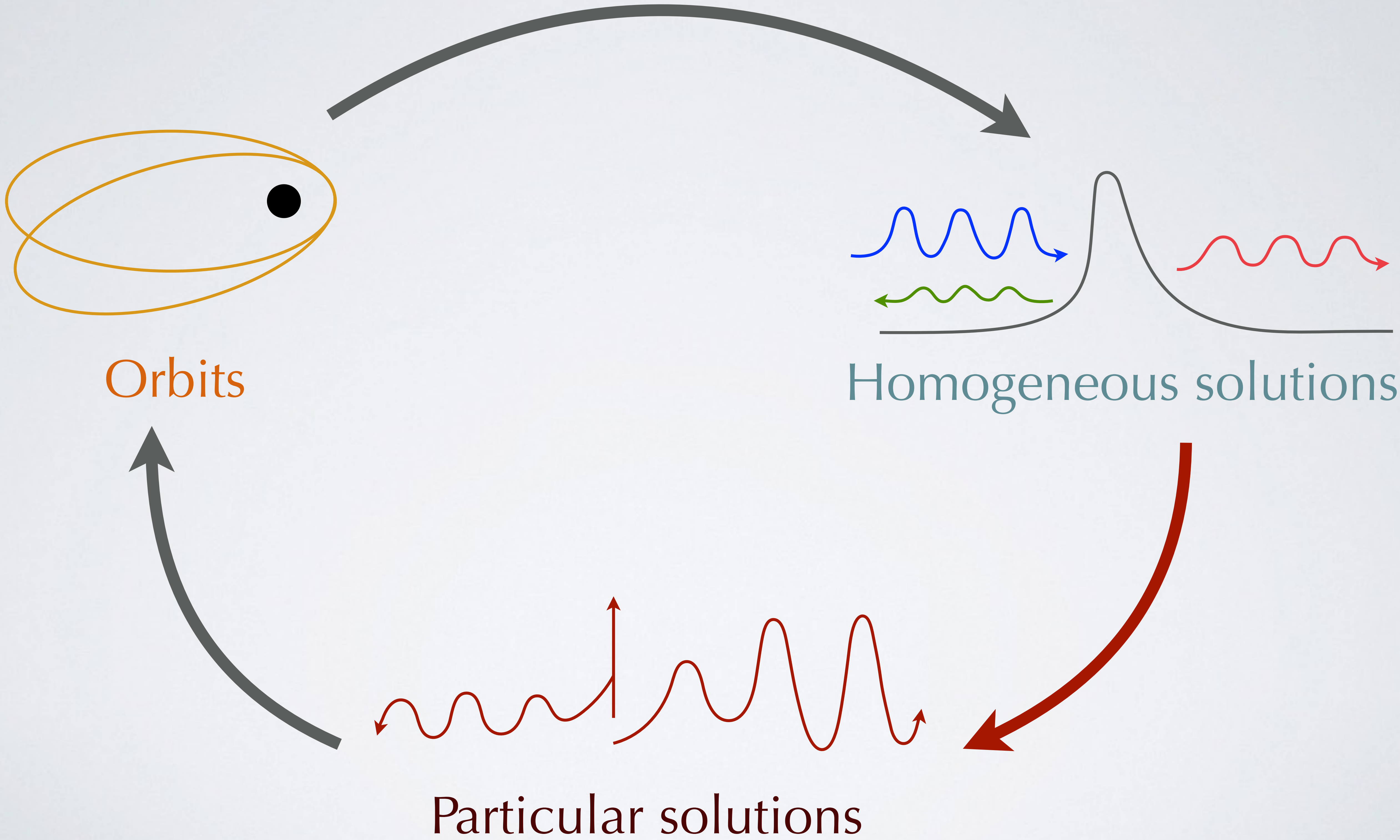
$$\omega = \frac{(m+n)}{M}x^{3/2} + \left[-\frac{3n}{M} - \frac{3ne^2}{M} + \mathcal{O}\left(e^3\right)\right]x^{5/2} + \mathcal{O}\left(x^3\right)$$

$$R^+ = \left[-\frac{i}{16(m+n)^2} - \frac{i\cos(\chi)e}{8(m+n)^2} + \mathcal{O}\left(e^2\right)\right]x^{-1} + \left[-\frac{i\left(m^3+3m^2n+3mn^2+10m+n^3+46n\right)}{96(m+n)^3} - \frac{ie(5m+17n)\cos(\chi)}{16(m+n)^3} + \mathcal{O}\left(e^2\right)\right]x^0 + \mathcal{O}\left(x^1\right)$$

$$R^- = \left[\frac{i}{384(m+n)} - \frac{i\cos(\chi)e}{128(m+n)} + \mathcal{O}\left(e^2\right)\right]x^{-9/2} + \left[-i\frac{(m+n)^3 - 42n}{5376(m+n)^2} + \frac{i\left(5m^3+15nm^2+15n^2m+5n^3-126n\right)\cos(\chi)e}{5376(m+n)^2} + \mathcal{O}\left(e^2\right)\right]x^{-7/2} + \mathcal{O}\left(x^{-5/2}\right)$$

Homogeneous solutions, odd parity, *l*=2, generic *m*,*n*

Outline - BHPT + Post-Newtonian Theory



Expansions of orbit quantities feed into G and F for generic *l* and *m*

$$C^{\pm}_{\ell mn} = \frac{1}{W_{\ell mn} T_r} \int_0^{2\pi} \left[\frac{1}{f_p} \hat{R}^{\mp}_{\ell mn}(\chi) \bar{G}_{\ell m}(\chi) + \left(\frac{2M}{r_p^2 f_p^2} \hat{R}^{\mp}_{\ell mn}(\chi) - \frac{1}{f_p} \frac{d\hat{R}^{\mp}_{\ell mn}(\chi)}{dr} \right) \bar{F}_{\ell m}(\chi) \right] e^{in\Omega_r t_p(\chi)} \frac{dt_p}{d\chi} d\chi$$

$$\bar{G}_{\ell m}(\chi) = 8\sqrt{\pi} \frac{\mu}{M} \frac{(\ell - m + 1)}{(\ell-1)\ell(\ell+1)(\ell+2)} \sqrt{\frac{(2\ell+1)(\ell-m)!}{(m+\ell)!}} P^m_{\ell+1}$$
$$\times \left\{ \left[-2 + \left(2e^{i\chi}(m-1) - 2e^{-i\chi}(m+1) \right) e + \mathcal{O}\left(e^2\right) \right] x^{3/2} \right.$$
$$\left. + \left[1 + \left(e^{-i\chi}(4-2m) + 2e^{i\chi}(m+2) \right) e + \mathcal{O}\left(e^2\right) \right] x^{5/2} + \mathcal{O}\left(x^{7/2}\right) \right\}$$

Generic *l*,*m*, no *n* dependence

Normalized frequency domain solutions are found for generic *m* and *n*

$$C^{+}_{2mn} = \mu\sqrt{\frac{(2-m)!}{(2+m)!}}\frac{\sin(n\pi)}{\sqrt{5\pi}}\left\{\left[-\frac{128i(m-3)(m+n)^2P_3^m}{3n}+\frac{128i(m-3)(2m-n)(m+n)^2P_3^m e}{3\left(n^2-1\right)}+\mathcal{O}\left(e^2\right)\right]x^{3/2}\right.$$
$$+\left[\frac{32i(m-3)\left(3m^4+12nm^3+2\left(9n^2+7\right)m^2+4n\left(3n^2+49\right)m+n^2\left(3n^2+182\right)\right)P_3^m}{21n}\right.$$
$$\left.\left.-\frac{32i(m-3)(m+n)\left(6m^4+9nm^3-\left(9n^2+77\right)m^2-7n\left(3n^2-44\right)m-n^2\left(9n^2+119\right)\right)P_3^m e}{21\left(n^2-1\right)}+\mathcal{O}\left(e^2\right)\right]x^{5/2}+\mathcal{O}\left(x^3\right)\right\}$$

$$R^{\pm}_{\ell mn}(\chi) = C^{\pm}_{\ell mn}\hat{R}^{\pm}_{\ell mn}(\chi)$$

$$R^{+}_{2mn} = \mu\sqrt{\frac{(2-m)!}{(2+m)!}}\frac{\sin(n\pi)}{\sqrt{5\pi}}\left\{\left[-\frac{8\left((m-3)P_3^m\right)}{3n}+\left(\frac{8(m-3)(2m-n)}{3\left(n^2-1\right)}-\frac{16(m-3)\cos(\chi)}{3n}\right)P_3^m e+\mathcal{O}\left(e^2\right)\right]\sqrt{x}\right.$$
$$+\left[-\frac{2\left((m-3)\left(5m^2+10nm+5n^2+98\right)P_3^m\right)}{63n}+\left(\frac{2(m-3)\left(10m^3+33nm^2+36n^2m+511m+13n^3-287n\right)}{63\left(n^2-1\right)}\right.\right.$$
$$\left.\left.\left.+\frac{4(m-3)\left(3m^2+6nm+3n^2-56\right)\cos(\chi)}{21n}\right)P_3^m e+\mathcal{O}\left(e^2\right)\right]x^{3/2}+\mathcal{O}\left(x^2\right)\right\}$$

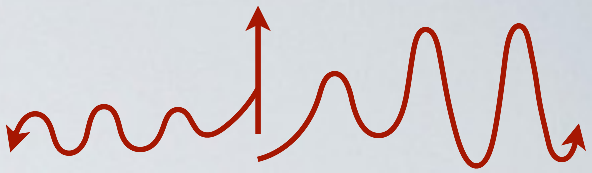
FD extended homogeneous solutions, odd parity, *l*=2, generic *m*,*n*

The finite-order *e* expansion truncates the sum over harmonics

$$\bar{\Psi}^{\pm}_{\ell m}(\chi) = \sum_{n} R^{\pm}_{\ell m n}(\chi)\, e^{-in\Omega_r t_p(\chi)}$$

n	$R^{+}_{2mn}(\chi)$
-3	0
-2	$-\frac{\mu}{3}\sqrt{\frac{\pi(2-m)!}{5(2+m)!}}(m-3)\left(4m^2-m-2\right)P_3^m e^2$
-1	$-\frac{4\mu}{3}\sqrt{\frac{\pi(2-m)!}{5(2+m)!}}(m-3)(-2m-1)P_3^m\left[e+2e^2\cos(\chi)\right]$
0	$-\frac{8\mu}{3}\sqrt{\frac{\pi(2-m)!}{5(2+m)!}}(m-3)P_3^m\left\{1+2e\cos(\chi)+e^2\left[2-m^2+\cos^2(\chi)\right]\right\}$
1	$-\frac{4\mu}{3}\sqrt{\frac{\pi(2-m)!}{5(2+m)!}}(m-3)(2m-1)P_3^m\left[e+2e^2\cos(\chi)\right]$
2	$-\frac{\mu}{3}\sqrt{\frac{\pi(2-m)!}{5(2+m)!}}(m-3)\left(4m^2+m-2\right)P_3^m e^2$
3	0

The retarded metric perturbation is computed only once for each l mode



Ω_φ, Ω_r

$\mathcal{E}, \mathcal{J}$

$r_p(\chi),\ \dot{r}_p(\chi), t_p(\chi),\ \Delta\varphi(\chi)$

Orbit quantities

$\hat{R}^{e/o,\pm}_{\ell mn}(\chi)$

$\partial_r \hat{R}^{e/o,\pm}_{\ell mn}(\chi)$

$\partial^2_r \hat{R}^{e/o,\pm}_{\ell mn}(\chi)$

Homog. sols.

$\bar{G}^{e/o}_{\ell m}(\chi),\ \bar{F}^{e/o}_{\ell m}(\chi)$

Source terms

$C^{e/o,\pm}_{\ell mn}$

Norm. coeffs.

$R^{e/o,\pm}_{\ell mn}(\chi)$

$\partial_r R^{e/o,\pm}_{\ell mn}(\chi)$

$\partial^2_r R^{e/o,\pm}_{\ell mn}(\chi)$

FD particular sols.

$\bar{\Psi}^{e/o,\pm}_{\ell m}(\chi)$

$\partial_t \bar{\Psi}^{e/o,\pm}_{\ell m}(\chi), \partial^2_t \bar{\Psi}^{e/o,\pm}_{\ell m}(\chi)$

$\partial_r \bar{\Psi}^{e/o,\pm}_{\ell m}(\chi), \partial^2_r \bar{\Psi}^{e/o,\pm}_{\ell m}(\chi)$

$\vdots$

TD particular sols.

$p^{\ell m,\pm}_{\mu\nu}(\chi)$

$\partial_t p^{\ell m,\pm}_{\mu\nu}(\chi)$

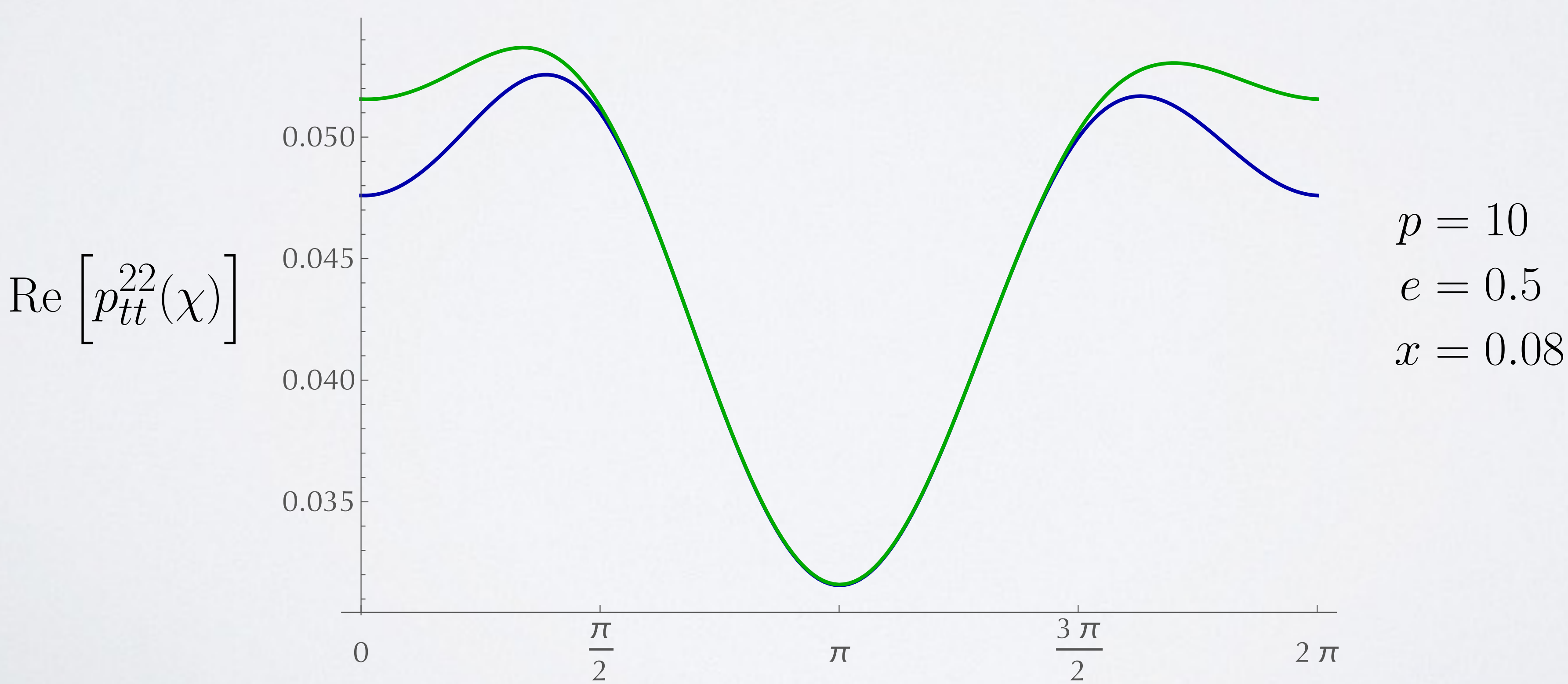
$\partial_r p^{\ell m,\pm}_{\mu\nu}(\chi)$

Metric perturbation

Metric perturbation expressions are remarkably simple

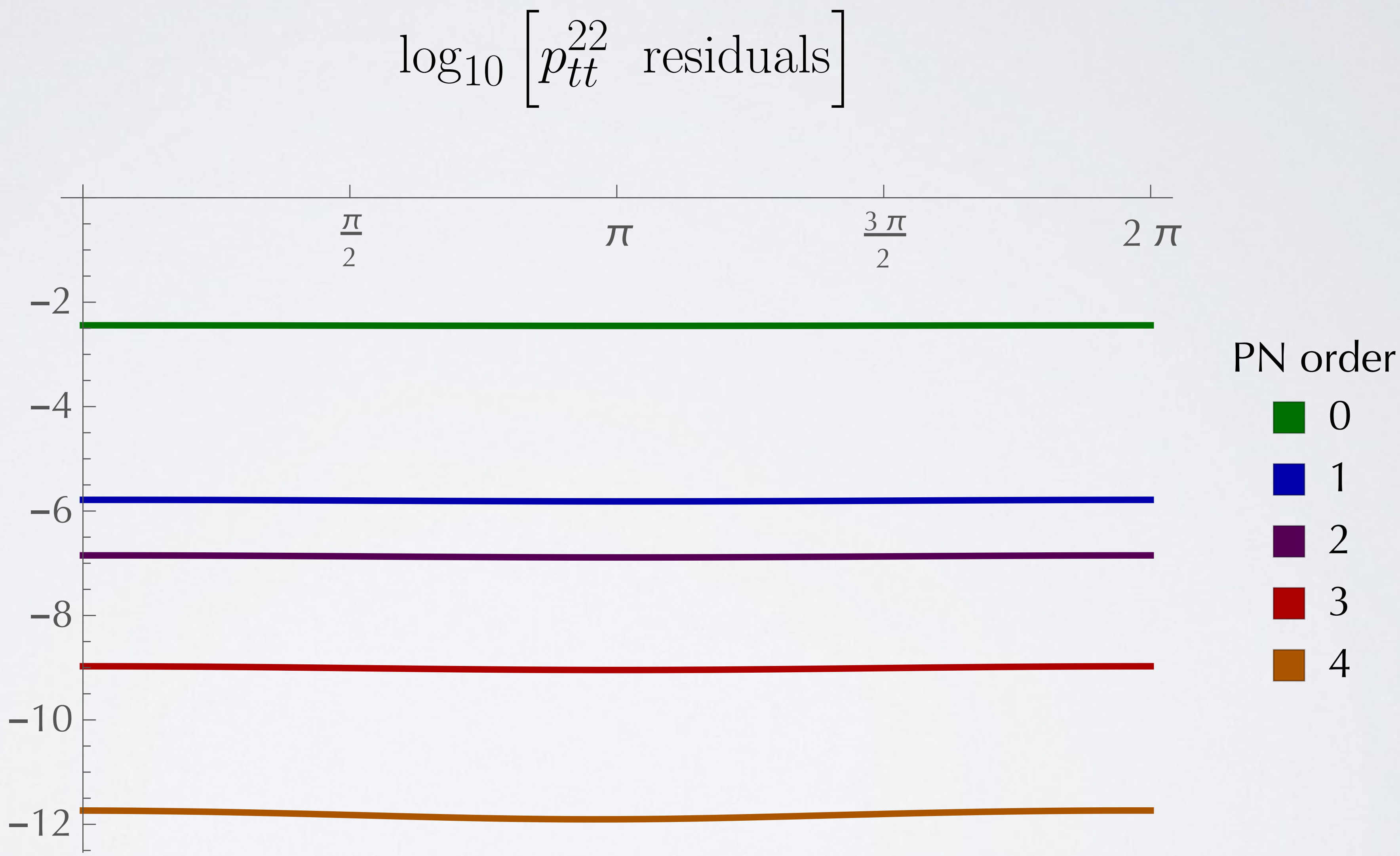
$$p_{tt}^{22,+} = \frac{\mu}{M}\left[\frac{3}{4} + \left(\frac{3e^{-i\chi}}{8} + \frac{3e^{i\chi}}{8}\right) e + \mathcal{O}\left(e^2\right)\right] x + \frac{\mu}{M}\left[-\frac{149}{56} + \left(-\frac{229}{112}e^{-i\chi} - \frac{437e^{i\chi}}{112}\right) e + \mathcal{O}\left(e^2\right)\right] x^2 + \mathcal{O}\left(x^3\right)$$

$$p_{tt}^{22,-} = \frac{\mu}{M}\left[\frac{3}{4} + \left(\frac{3e^{-i\chi}}{8} + \frac{3e^{i\chi}}{8}\right) e + \mathcal{O}\left(e^2\right)\right] x + \frac{\mu}{M}\left[-\frac{149}{56} + \left(-\frac{509}{112}e^{-i\chi} - \frac{157e^{i\chi}}{112}\right) e + \mathcal{O}\left(e^2\right)\right] x^2 + \mathcal{O}\left(x^3\right)$$

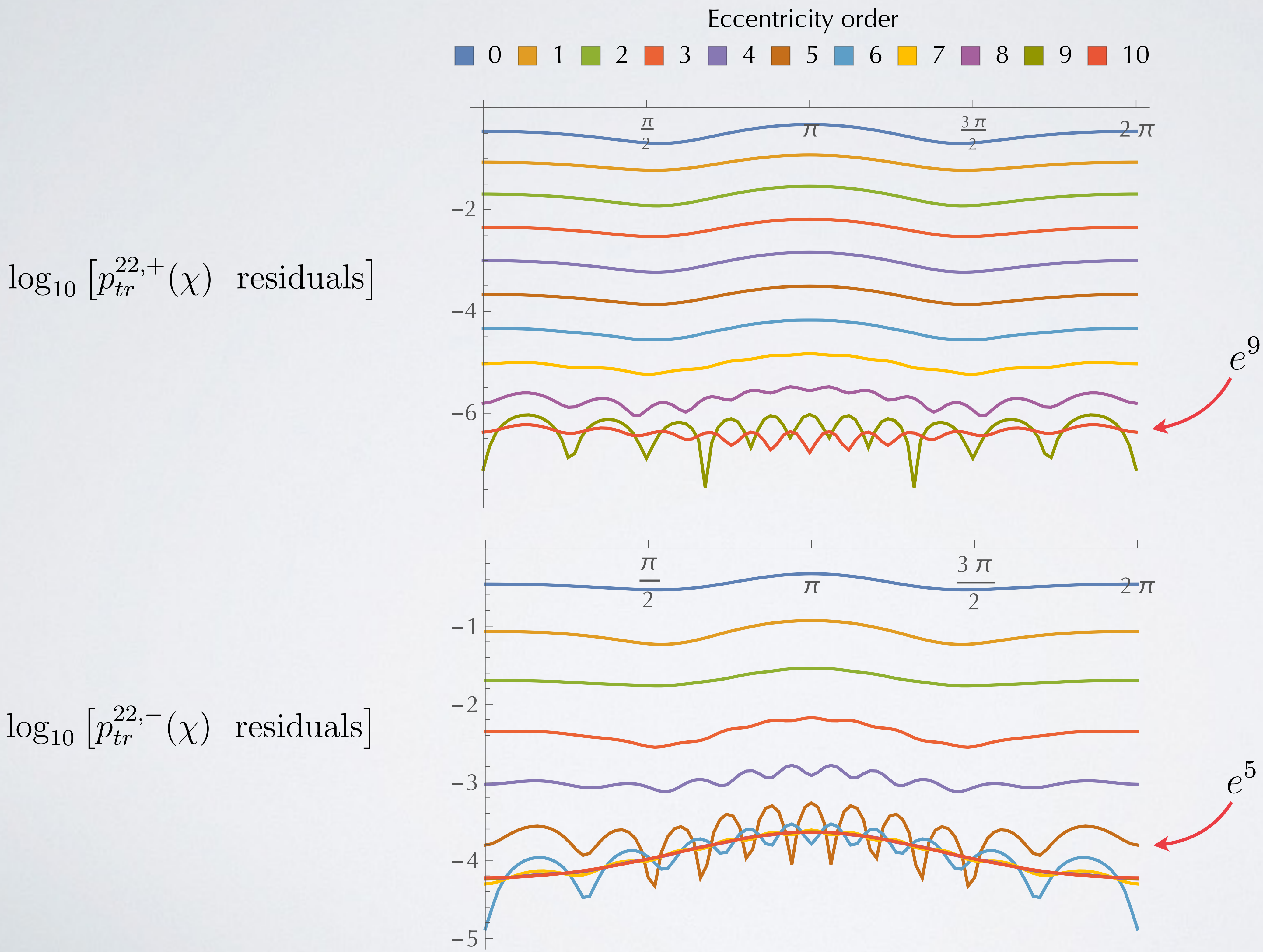


We see convergence in x for small values of e

$p = 1000$
$e = 0.01$
$x \approx 0.001$



Sometimes the convergence with e stalls



$p = 10^6$

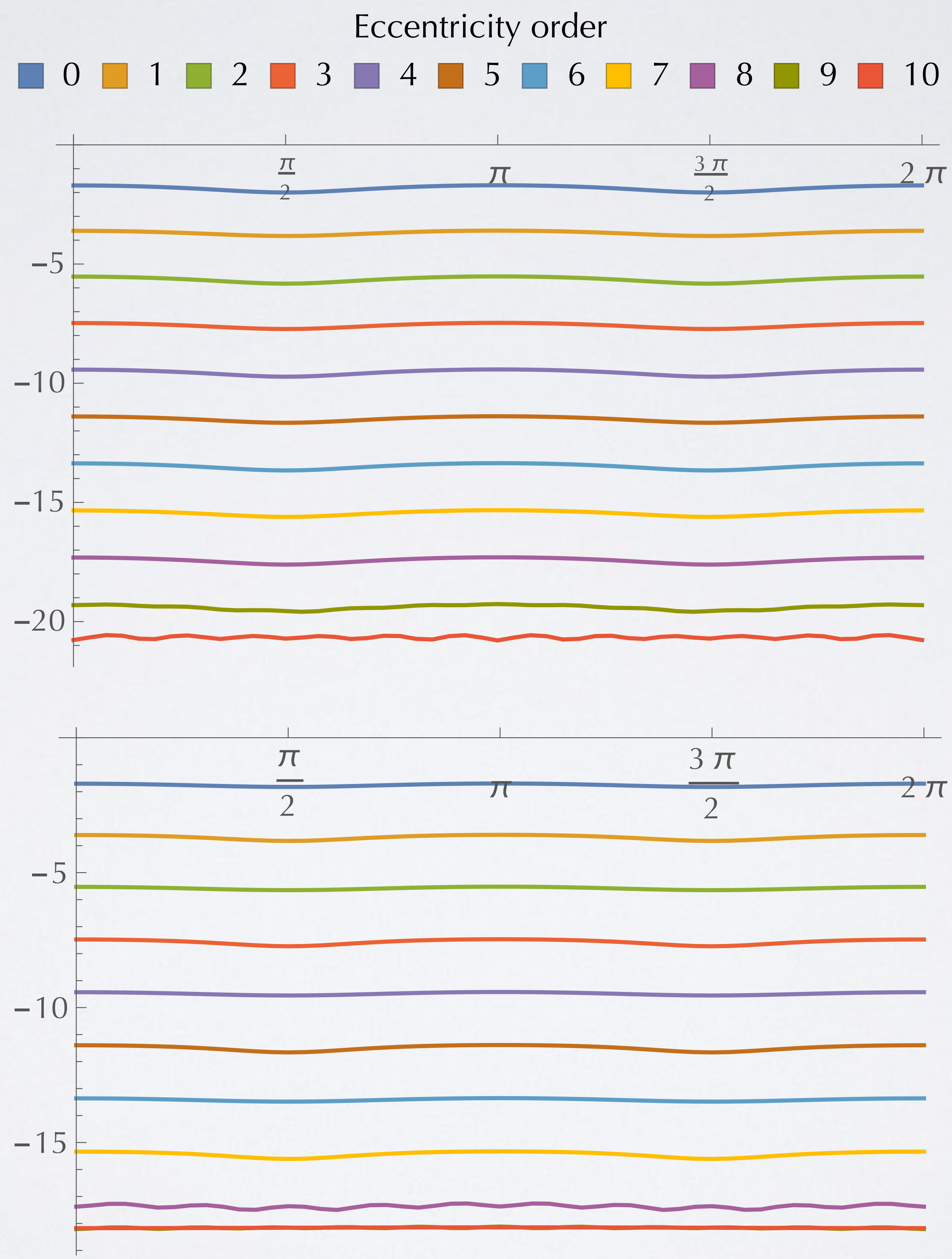
$e = 0.2$

$x = 0.96 \times 10^{-6}$

For small values of e the convergence continues

$\log_{10}\left[p_{tr}^{22,+}(\chi)\quad \text{residuals}\right]$

$\log_{10}\left[p_{tr}^{22,-}(\chi)\quad \text{residuals}\right]$



$p = 10^6$

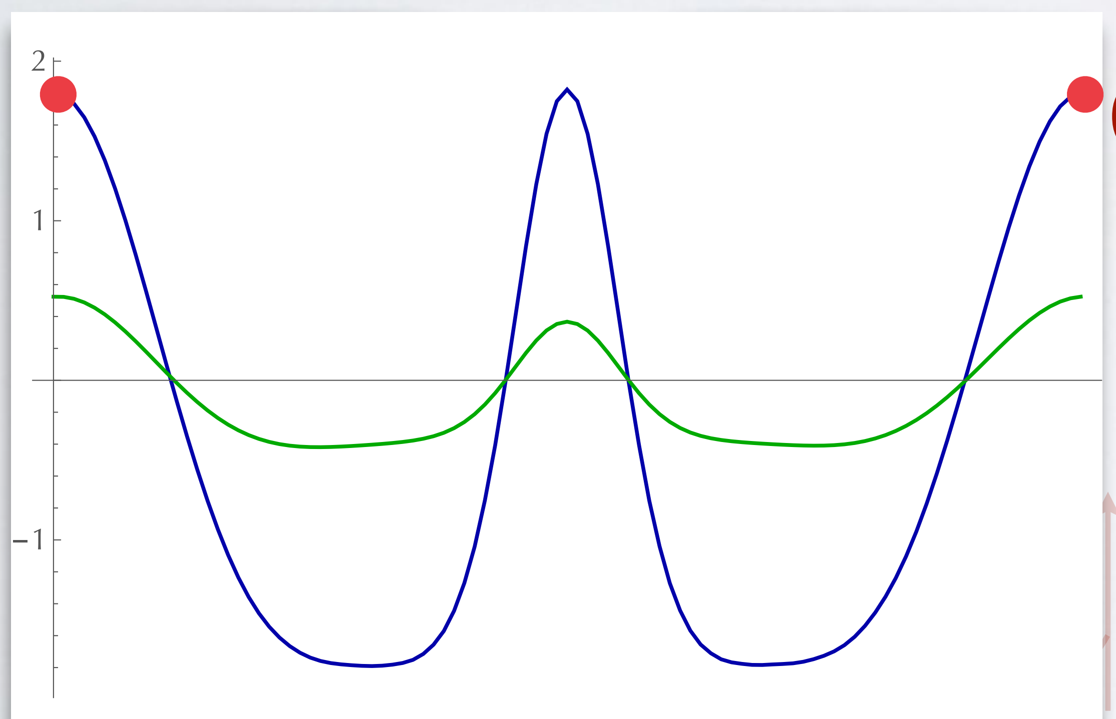
$e = 0.01$

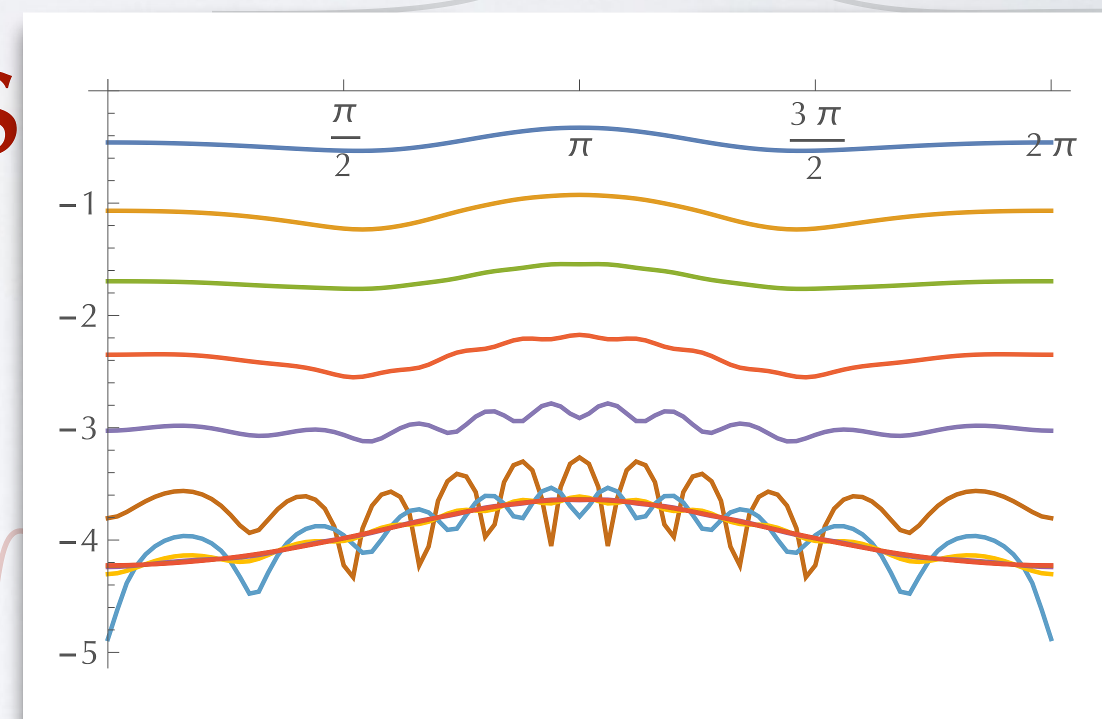
$x = 10^{-6}$

These are the main points

$$p_{\mu\nu}(\chi), \frac{\partial p_{\mu\nu}}{\partial t}(\chi), \frac{\partial p_{\mu\nu}}{\partial r}(\chi) \qquad 0 \leq \ell \leq 10 \qquad 4\ \mathrm{PN}, e^{10}$$

Hofstadter's Law:
It always takes longer than you expect, even when you take into account Hofstadter's Law.
- Douglas Hofstadter





Questions?