Analytic calculation of post-Newtonian expansions of gauge invariants

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We now have a long list of local quantities to calculate and compare

$$\Delta U = \frac{r_0}{r_0 - 3M}(h_{tt} + \Omega h_{t\phi} + \Omega^2 h_{\phi\phi})$$

$$\begin{aligned}\Delta\psi =& \frac{1}{2r_0\Omega}\sqrt{\frac{r_0 - 3M}{r_0}}\Big[h_{tr,\phi} - h_{t\phi,r} + \Omega(h_{r\phi,\phi} - h_{\phi\phi,r} + f r_0 h_{rr})\Big] \\ &+ \frac{1}{2Mr_0 f}\sqrt{\frac{M}{r_0 - 3M}}\Big[\Omega(Mr_0^2 h_{tt} + r_0 f^2 h_{\phi\phi}) + 2Mf h_{t\phi})\Big],\end{aligned}$$

$$\begin{aligned}\Delta\lambda_1^E =& \frac{\Omega^2 f(2r_0 - 3M)}{r_0 - 3M}h_{rr} - \frac{\Omega^2(2r_0^2 - 6Mr_0 + 3M^2)}{f(r_0 - 3M)^2}h_{tt} - \frac{6M\Omega f h_{t\phi}}{r_0(r_0 - 3M)^2} \\ &- \frac{\Omega^2(r_0^2 - 3Mr_0 + 3M^2)h_{\phi\phi}}{r_0^2(r_0 - 3M)^2} - \frac{r_0 - 2M}{2(r_0 - 3M)}\Big[h_{tt,rr} + 2\Omega h_{t\phi,rr} + \Omega^2 h_{\phi\phi,rr}\Big] \\ &- \frac{\Omega^2 h_{r\phi,\phi} + \Omega[h_{tr,\phi} + h_{t\phi,r}] + h_{tt,r}}{r_0}\end{aligned}$$

etc

Look at the odd parity perturbations, r<r0

$$G_{lm}(r,r') = \frac{X^{\rm in}_{\ell m}(r) X^{\rm up}_{\ell m}(r')}{W} \qquad\qquad T(r) = s_1 \delta(r - r_0) + s_2 \delta'(r - r_0)$$

$$\begin{aligned} R_{\rm inhom}(r) &= \int dr' f(r')^{-1} G(r,r') T(r') \\ &\sim \alpha(r) \frac{X^{\rm in}_{\ell m}(r) X^{\rm up}_{\ell m}(r_0)}{W} + \beta(r) \frac{X^{\rm in}_{\ell m}(r) X^{\rm up}_{\ell m}{}'(r_0)}{W} \end{aligned}$$

Metric perturbation is a combination of the inhomogeneous solution and its derivative.

$$\begin{aligned} h_{t\phi} = -4\pi \sqrt{\frac{M}{r_0 - 3M}} \frac{f(r)^2}{\lambda(\lambda-1) r_0} \Bigg(& r r_0 \frac{X^{\rm in}_{\ell m}{}'(r) X^{\rm up}_{\ell m}{}'(r_0)}{W} + r_0 \frac{X^{\rm in}_{\ell m}(r) X^{\rm up}_{\ell m}{}'(r_0)}{W} \\ & + r \frac{X^{\rm in}_{\ell m}{}'(r) X^{\rm up}_{\ell m}(r_0)}{W} + \frac{X^{\rm in}_{\ell m}(r) X^{\rm up}_{\ell m}(r_0)}{W} \Bigg) \times \sin(\theta) Y^*_{lm,\theta}(\theta_0, \phi_0) Y_{lm,\theta}(\theta,\phi) \end{aligned}$$

For even parity, we can get away with using the odd parity solutions!

$$X_{lm}^{\text{even}\pm} = \frac{1}{\lambda + \lambda^2 \pm 3i\omega M}\left[\left(\lambda + \lambda^2 + \frac{9M^2(r-2M)}{r^2(r\lambda + 3M)}\right) X_{lm}^{\text{odd}\pm} + 3Mf\frac{dX_{lm}^{\text{odd}\pm}}{dr}\right]$$

Any time we get a second derivative, use the RW equation.

$$\Delta U = \frac{r_0}{r_0 - 3M}(h_{tt} + \Omega h_{t\phi} + \Omega^2 h_{\phi\phi})$$

(summing over l,m and regularising)

$$\frac{X_{\ell m}^{\text{in}} X_{\ell m}^{\text{up}}}{W} \quad \frac{X_{\ell m}^{\text{in}\,\prime} X_{\ell m}^{\text{up}}}{W}$$
$$\frac{X_{\ell m}^{\text{in}} X_{\ell m}^{\text{up}\,\prime}}{W} \quad \frac{X_{\ell m}^{\text{in}\,\prime} X_{\ell m}^{\text{up}\,\prime}}{W}$$

\+ Regularisation Parameters

$\longrightarrow$ Redshift, spin precession, tidal invariants, octupoles …

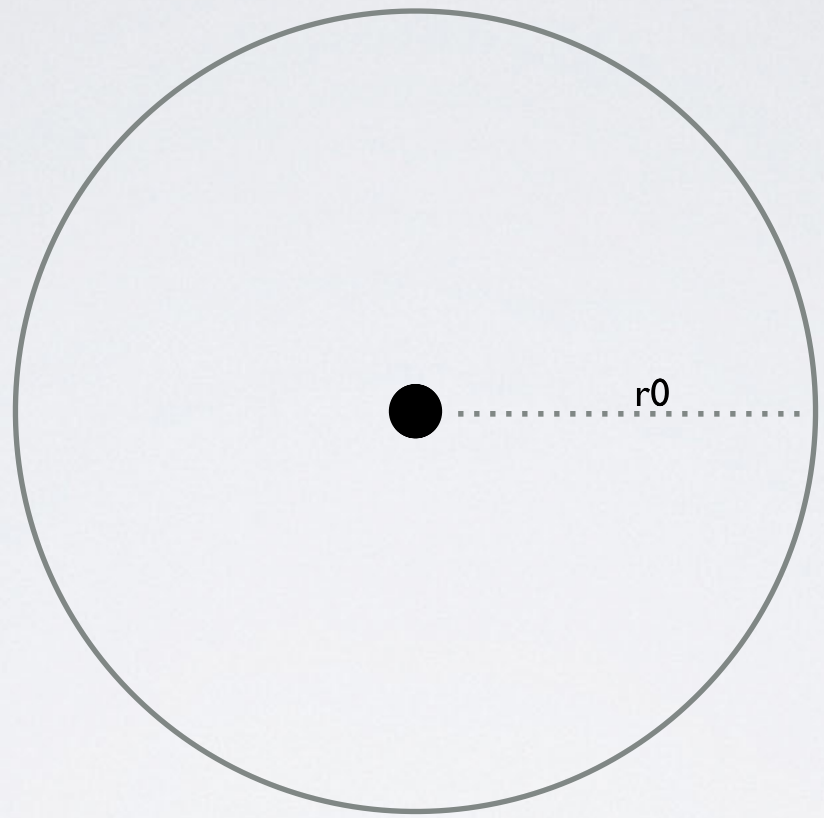
The infinite l sum is split into three sections

$l = 0, 1$ Analytically derived by Zerilli

$l = 2, \ldots, l_{\max}$ Special function series solutions derived by Mano, Suzuki & Takasugi

$l = l_{\max} + 1, \ldots$ Ansatz based on the form of the low l modes.

RW equation on circular orbits: large radius, small frequency double expansions



$$\omega \xrightarrow{r \to r_0} m\Omega$$

$$\Omega = \sqrt{\frac{GM}{r_0^3}}$$

$$\begin{cases} X_1 = \dfrac{GM}{r} \\ X_2 = \omega^2 r^2 \end{cases}$$

Units $\dfrac{1}{c^2}$

Expand in 1/c to keep track of omega and r. Note, every two orders in 1/c is one pN order.

Bini & Damour 2013

How do we truncate our series solutions?

$$l = 2, \ldots, l_{\max}$$

$$X^{\text{in}}_{\ell\omega}(r) = C^{\nu}_{(\text{in})}(x) \sum_{n=-\infty}^{\infty} a^{\nu}_{n} \times \overline{F}(n+\nu-1-i\epsilon, -n-\nu-2-i\epsilon, 1-2i\epsilon, x)$$

$$X^{\text{up}}_{\ell\omega}(r) = C^{\nu}_{(\text{up})}(z) \sum_{n=-\infty}^{\infty} a^{\nu}_{n}(-2iz)^{n} \times \overline{U}(n+\nu+1-i\epsilon, 2n+2\nu+2, -2iz)$$

$$\epsilon = \frac{2GM\omega}{c^3} \qquad x = 1 - \frac{c^2 r}{2GM} \qquad z = \frac{\omega r}{c}$$

MST series solution: Inner solution examination

$$l = 2, \ldots, l_{\max}$$

$$X^{\text{in}}_{\ell m}(r) = C^{\nu}_{(\text{in})}(x) \sum_{n=-\infty}^{\infty} a^{\nu}_{n} \times \overline{F}(n+\nu-1-i\epsilon, -n-\nu-2-i\epsilon, 1-2i\epsilon, x)$$

Series coefficients satisfy three-term recurrence relations

$$\alpha^{\nu}_{n} a_{n+1} + \beta^{\nu}_{n} a_{n} + \gamma^{\nu}_{n} a_{n-1} = 0$$

$l = 2$

a_5	0	0	0	0	0	ϵ^5		
a_4	0	0	0	0	ϵ^4			
a_3	0	0	0	ϵ^3				
a_2	0	0	ϵ^2					
a_1	0	ϵ						
a_0	1							
a_{-1}	0	0	0	ϵ^3				
a_{-2}	0	0	0	0	ϵ^4			
a_{-3}	0	0	0	0	ϵ^4			
a_{-4}	0	0	0	0	0	ϵ^5		
a_{-5}	0	0	0	0	ϵ^4			
a_{-6}	0	0	0	0	0	ϵ^5		
a_{-7}	0	0	0	0	0	0	ϵ^6	
a_{-8}	0	0	0	0	0	0	0	ϵ^7

$l = 3$

a_5	0	0	0	0	0	ϵ^5		
a_4	0	0	0	0	ϵ^4			
a_3	0	0	0	ϵ^3				
a_2	0	0	ϵ^2					
a_1	0	ϵ						
a_0	1							
a_{-1}	0	ϵ						
a_{-2}	0	0	0	0	ϵ^4			
a_{-3}	0	0	0	0	0	ϵ^5		
a_{-4}	0	0	0	0	0	ϵ^5		
a_{-5}	0	0	0	0	0	0	ϵ^6	
a_{-6}	0	0	0	0	0	0	0	ϵ^7
a_{-7}	0	0	0	0	0	0	ϵ^6	
a_{-8}	0	0	0	0	0	0	0	ϵ^7

$l = 5$

a_5	0	0	0	0	0	ϵ^5				
a_4	0	0	0	0	ϵ^4					
a_3	0	0	0	ϵ^3						
a_2	0	0	ϵ^2							
a_1	0	ϵ								
a_0	1									
a_{-1}	0	ϵ								
a_{-2}	0	0	ϵ^2							
a_{-3}	0	0	0	ϵ^3						
a_{-4}	0	0	0	0	0	ϵ^6				
a_{-5}	0	0	0	0	0	0	0	ϵ^7		
a_{-6}	0	0	0	0	0	0	0	ϵ^7		
a_{-7}	0	0	0	0	0	0	0	0	ϵ^8	
a_{-8}	0	0	0	0	0	0	0	0	0	ϵ^9

MST series solution: Inner solution examination

$$l = 2, \ldots, l_{\max}$$

$$\overline{F}(n+\nu-1-i\epsilon, -n-\nu-2-i\epsilon, 1-2i\epsilon, x)$$

$$\overline{F}\,(a,b,c,x) = \underbrace{\frac{\Gamma(a)\Gamma(b-a)}{\Gamma(c-a)}(1-x)^{-a}{}_2F_1(a, c-b, a-b+1, \tfrac{1}{1-x})}_{\overline{F}_1} + \underbrace{\frac{\Gamma(b)\Gamma(a-b)}{\Gamma(c-b)}(1-x)^{-b}{}_2F_1(c-a, b, b-a+1, \tfrac{1}{1-x})}_{\overline{F}_2}$$

$$\frac{1}{1-x} = \frac{2X_1}{c^2} \qquad \nu = l + O(1/c^6) \qquad \epsilon = \frac{2GM\omega}{c^3}$$

a,b,c change character with changing n

$$\Gamma(k+\delta) \sim \begin{cases} O(1), & k > 0 \\ O(1/\delta), & k \leq 0 \end{cases}$$

jump of three orders

Order counting minimises the workload

$l = 2, \ldots, l_{\max}$

$\eta = 1/c$

$l = 2$

$a_n^\nu \overline{F}_1$

n	8	7	6	5	4	3	2	1	0	-1	-2	-3	-4	-5	-6	-7	-8
	η^{15}	η^{14}	η^{13}	η^{12}	η^{17}	η^{12}	η^{12}	η^{11}	η^{10}	η^{15}	η^{20}	η^{25}	η^{30}	η^{35}	η^{40}	η^{45}	η^{50}

$a_n^\nu \overline{F}_2$

n	8	7	6	5	4	3	2	1	0	-1	-2	-3	-4	-5	-6	-7	-8
	η^{8}	η^{7}	η^{6}	η^{5}	η^{4}	η^{3}	η^{2}	η	1	η^{11}	η^{12}	η^{12}	η^{17}	η^{22}	η^{27}	η^{32}	η^{37}

Then to get 30 correct orders we need

$$X_{\ell m}^{\rm in} = C_{(\rm in)}^{\nu}(x)\left(\sum_{n=-4}^{23} a_n^\nu \overline{F}_1 + \sum_{n=-6}^{30} a_n^\nu \overline{F}_2\right)$$

In[42]:= **Xin**

Out[42]=

$$\frac{i}{384\,\mathrm{X1}^4\,\mathrm{X2H}\,\eta^9}-\frac{i\,\mathrm{X2H}}{5376\,\mathrm{X1}^4\,\eta^7}+\frac{47}{6720\,\mathrm{X1}^3\,\eta^6}+\frac{-\frac{13\,i\,\mathrm{X2H}}{8064\,\mathrm{X1}^3}+\frac{i\,\mathrm{X2H}^3}{193\,536\,\mathrm{X1}^4}}{\eta^5}-\frac{47\,\mathrm{X2H}^2}{94\,080\,\mathrm{X1}^3\,\eta^4}+\frac{\frac{i\,\mathrm{X2H}^3}{10\,368\,\mathrm{X1}^3}-\frac{i\,\mathrm{X2H}^5}{12\,773\,376\,\mathrm{X1}^4}+\frac{\mathrm{X2H}\left(-\frac{41\,869\,i}{1\,881\,600}+\frac{107\,i\,\mathrm{Log}\left[2\,\mathrm{X1}\,\eta^2\right]}{20\,160}\right)}{\mathrm{X1}^2}}{\eta^3}+$$

$$\frac{-\frac{611\,\mathrm{X2H}^2}{141\,120\,\mathrm{X1}^2}+\frac{47\,\mathrm{X2H}^4}{3\,386\,880\,\mathrm{X1}^3}}{\eta^2}+\frac{\frac{319\,i\,\mathrm{X2H}}{20\,160\,\mathrm{X1}}-\frac{53\,i\,\mathrm{X2H}^5}{22\,809\,600\,\mathrm{X1}^3}+\frac{i\,\mathrm{X2H}^7}{1\,328\,431\,104\,\mathrm{X1}^4}+\frac{\mathrm{X2H}^3\left(\frac{454\,841\,i}{237\,081\,600}-\frac{107\,i\,\mathrm{Log}\left[2\,\mathrm{X1}\,\eta^2\right]}{282\,240}\right)}{\mathrm{X1}^2}}{\eta}+\left(\frac{47\,\mathrm{X2H}^4}{181\,440\,\mathrm{X1}^2}-\frac{47\,\mathrm{X2H}^6}{223\,534\,080\,\mathrm{X1}^3}+\frac{\mathrm{X2H}^2\left(\frac{-2\,105\,693+209\,720\,\pi^2}{59\,270\,400}+\frac{5029\,\mathrm{Log}\left[2\,\mathrm{X1}\,\eta^2\right]}{352\,800}\right)}{\mathrm{X1}}\right)+$$

$$\left(\frac{51\,i\,\mathrm{X2H}}{2240}+\frac{227\,i\,\mathrm{X2H}^7}{7\,264\,857\,600\,\mathrm{X1}^3}-\frac{i\,\mathrm{X2H}^9}{199\,264\,665\,600\,\mathrm{X1}^4}+\frac{\mathrm{X2H}^5\left(-\frac{17\,245\,693\,i}{281\,652\,940\,800}+\frac{107\,i\,\mathrm{Log}\left[2\,\mathrm{X1}\,\eta^2\right]}{10\,160\,640}\right)}{\mathrm{X1}^2}+\frac{\mathrm{X2H}^3\left(\frac{3\,895\,673\,i}{355\,622\,400}-\frac{1391\,i\,\mathrm{Log}\left[2\,\mathrm{X1}\,\eta^2\right]}{423\,360}\right)}{\mathrm{X1}}\right)\eta+$$

$$\left(\frac{14\,993\,\mathrm{X2H}^2}{352\,800}-\frac{2491\,\mathrm{X2H}^6}{399\,168\,000\,\mathrm{X1}^2}+\frac{47\,\mathrm{X2H}^8}{23\,247\,544\,320\,\mathrm{X1}^3}+\frac{\mathrm{X2H}^4\left(\frac{405\,583-29\,960\,\pi^2}{118\,540\,800}-\frac{5029\,\mathrm{Log}\left[2\,\mathrm{X1}\,\eta^2\right]}{4\,939\,200}\right)}{\mathrm{X1}}\right)\eta^2+\left(\frac{223\,i\,\mathrm{X1}\,\mathrm{X2H}}{5040}-\frac{11\,911\,i\,\mathrm{X2H}^9}{43\,937\,858\,764\,800\,\mathrm{X1}^3}+\frac{i\,\mathrm{X2H}^{11}}{40\,649\,991\,782\,400\,\mathrm{X1}^4}+\right.$$

$$\left.\frac{\mathrm{X2H}^7\left(\frac{50\,008\,649\,i}{48\,331\,644\,641\,280}-\frac{107\,i\,\mathrm{Log}\left[2\,\mathrm{X1}\,\eta^2\right]}{670\,602\,240}\right)}{\mathrm{X1}^2}+\frac{\mathrm{X2H}^5\left(-\frac{10\,830\,943\,i}{15\,088\,550\,400}+\frac{107\,i\,\mathrm{Log}\left[2\,\mathrm{X1}\,\eta^2\right]}{544\,320}\right)}{\mathrm{X1}}+\mathrm{X2H}^3\left(\frac{129\,666\,772\,886\,101\,i}{8\,550\,193\,800\,960\,000}-\frac{107\,i\,\pi^2}{18\,144}-\frac{6\,707\,783\,i\,\mathrm{Log}\left[2\,\mathrm{X1}\,\eta^2\right]}{177\,811\,200}+\frac{11\,449\,i\,\mathrm{Log}\left[2\,\mathrm{X1}\,\eta^2\right]^2}{2\,116\,800}+\frac{107\,i\,\mathrm{Zeta}[3]}{2520}\right)\right)\eta^3+$$

$$\left(-\frac{\mathrm{X1}^2}{30}+\frac{2397\,\mathrm{X1}\,\mathrm{X2H}^2}{39\,200}+\frac{10\,669\,\mathrm{X2H}^8}{127\,135\,008\,000\,\mathrm{X1}^2}-\frac{47\,\mathrm{X2H}^{10}}{3\,487\,131\,648\,000\,\mathrm{X1}^3}+\mathrm{X2H}^4\left(\frac{366\,241-55\,640\,\pi^2}{25\,401\,600}-\frac{65\,377\,\mathrm{Log}\left[2\,\mathrm{X1}\,\eta^2\right]}{7\,408\,800}\right)+\frac{\mathrm{X2H}^6\left(\frac{-16\,386\,787+988\,680\,\pi^2}{140\,826\,470\,400}+\frac{5029\,\mathrm{Log}\left[2\,\mathrm{X1}\,\eta^2\right]}{177\,811\,200}\right)}{\mathrm{X1}}\right)\eta^4+$$

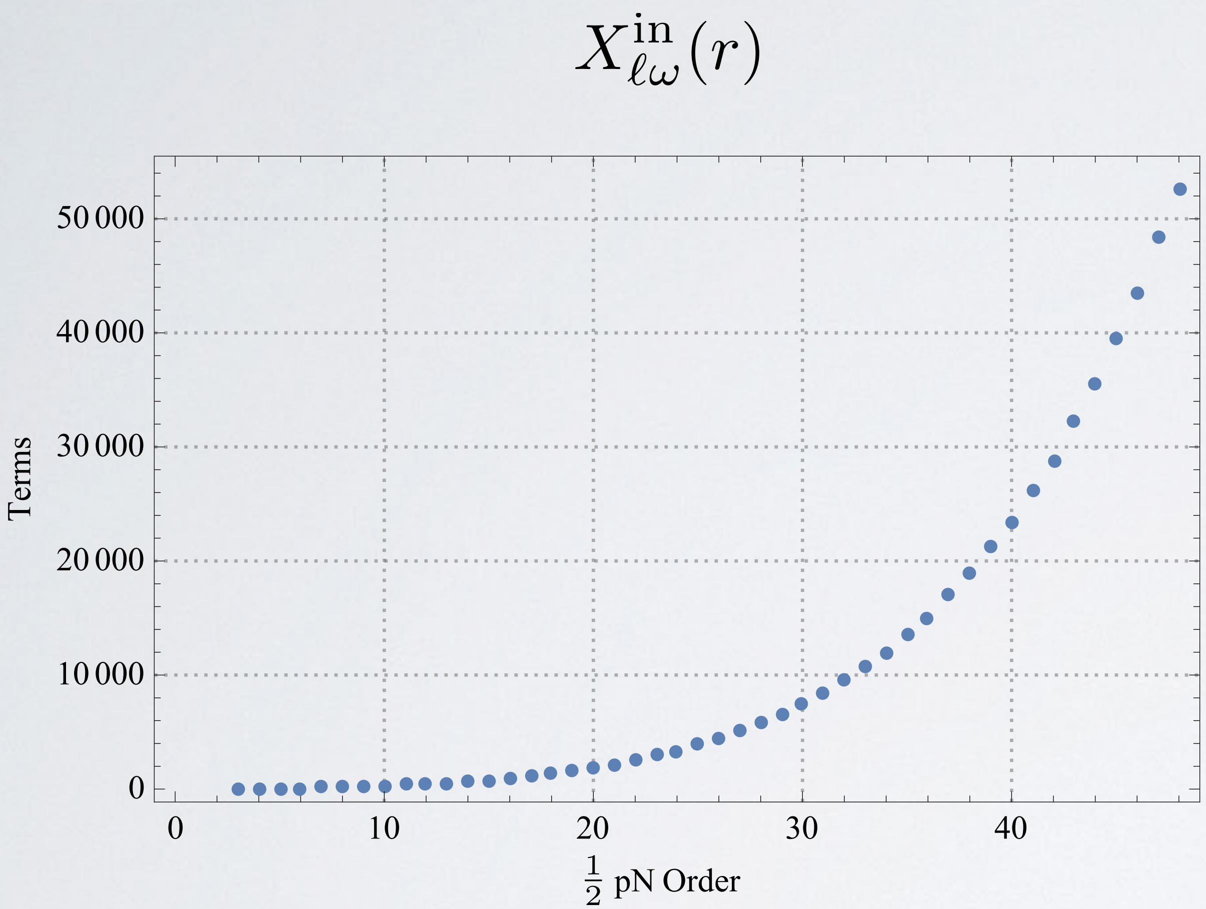
$$\left(\frac{121\,i\,\mathrm{X1}^2\,\mathrm{X2H}}{1120}+\frac{2713\,i\,\mathrm{X2H}^{11}}{1\,643\,275\,917\,803\,520\,\mathrm{X1}^3}-\frac{i\,\mathrm{X2H}^{13}}{10\,812\,897\,814\,118\,400\,\mathrm{X1}^4}+\frac{\mathrm{X2H}^9\left(-\frac{39\,059\,479\,i}{3\,590\,350\,744\,780\,800}+\frac{107\,i\,\mathrm{Log}\left[2\,\mathrm{X1}\,\eta^2\right]}{69\,742\,632\,960}\right)}{\mathrm{X1}^2}+\frac{\mathrm{X2H}^7\left(\frac{1\,568\,087\,173\,i}{86\,306\,508\,288\,000}-\frac{5671\,i\,\mathrm{Log}\left[2\,\mathrm{X1}\,\eta^2\right]}{1\,197\,504\,000}\right)}{\mathrm{X1}}+\right.$$

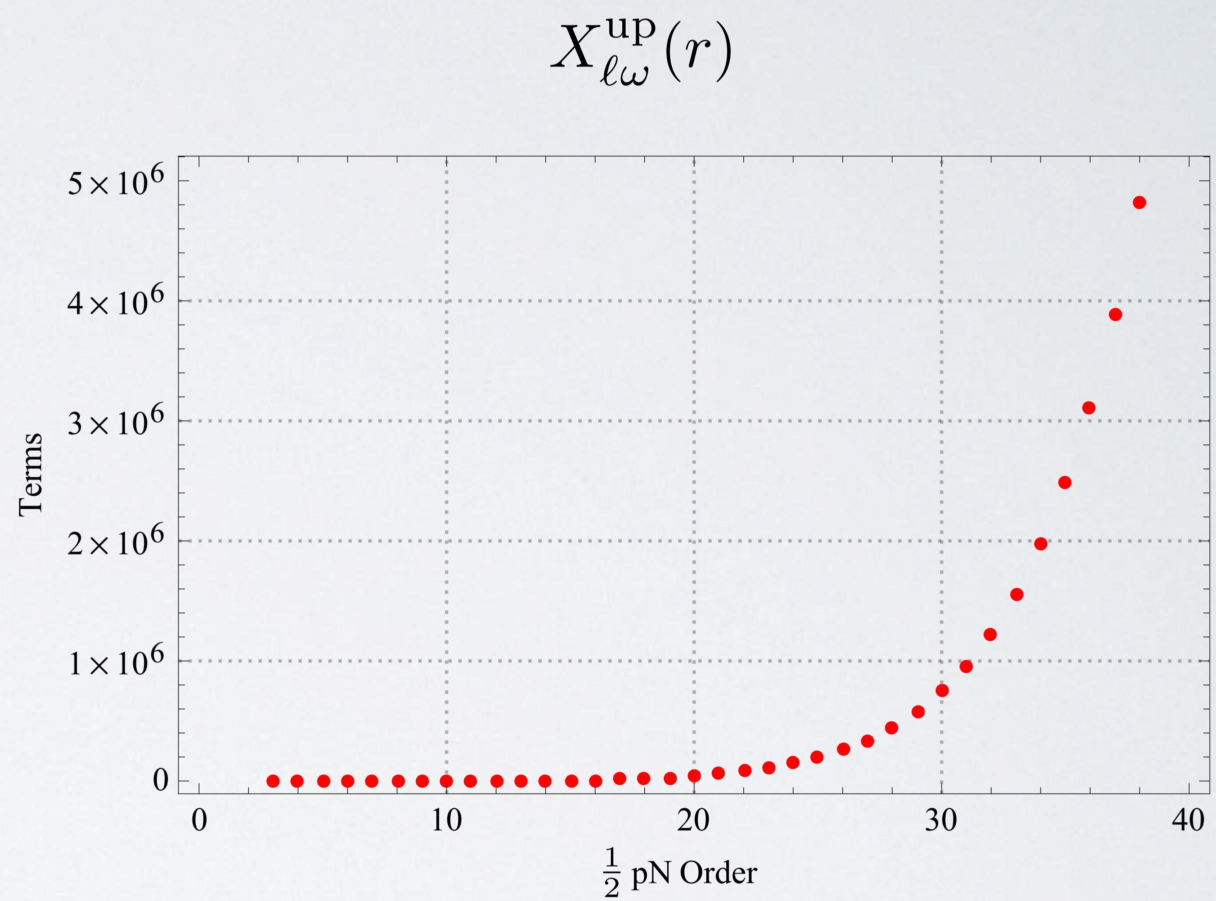
$$\left.\mathrm{X1}\,\mathrm{X2H}^3\left(-\frac{21\,197\,851\,i}{177\,811\,200}+\frac{34\,133\,i\,\mathrm{Log}\left[2\,\mathrm{X1}\,\eta^2\right]}{1\,058\,400}\right)+\mathrm{X2H}^5\left(-\frac{13\,719\,168\,130\,527\,833\,i}{3\,950\,189\,536\,043\,520\,000}+\frac{107\,i\,\pi^2}{254\,016}+\frac{1\,196\,773\,i\,\mathrm{Log}\left[2\,\mathrm{X1}\,\eta^2\right]}{355\,622\,400}-\frac{11\,449\,i\,\mathrm{Log}\left[2\,\mathrm{X1}\,\eta^2\right]^2}{29\,635\,200}-\frac{107\,i\,\mathrm{Zeta}[3]}{35\,280}\right)\right)\eta^5+$$

$$\left(\mathrm{X1}^3\quad 9991\,\mathrm{X1}^2\,\mathrm{X2H}^2\quad 559\,817\,\mathrm{X2H}^{10}\quad 47\,\mathrm{X2H}^{12}\quad \mathrm{X2H}^8\left(\frac{1\,238\,700\,311-64\,264\,200\,\pi^2}{604\,145\,558\,016\,000}-\frac{5029\,\mathrm{Log}\left[2\,\mathrm{X1}\,\eta^2\right]}{11\,735\,539\,200}\right)\quad \left(-54\,408\,859+6\,920\,760\,\pi^2\quad 5029\,\mathrm{Log}\left[2\,\mathrm{X1}\,\eta^2\right]\right)\right.$$

Things get bad very quickly…

$$l = 2, \ldots, l_{\max}$$





Extracting phase terms gives a massive simplification

$$l = 2, \ldots, l_{\max}$$

$$X_1 X_2^{1/2} = \frac{GM}{r}\omega r = GM\omega \quad \longrightarrow \quad \text{constant in r}$$

Looking at Xin for l=2,

$$X^{\mathrm{in}}_{\ell m} \sim 1 - \frac{X_2}{14}\eta^2 - \frac{94}{35} i X_1 {X_2}^{1/2}\eta^3 + \left(-\frac{12}{21}X_1X_2 + \frac{X_2^2}{504}\right)\eta^4 + \frac{47}{245} i X_1 X_2^{3/2}\eta^5 + \left(\frac{X_1X_2^2}{27} - \frac{X_2^3}{33264} - 2X_1^2X_2\left(\frac{41869}{9800} - \frac{107}{105}\log\left(2X_1\eta^2\right)\right)\right)\eta^6$$

Take out the log terms along with the r independence

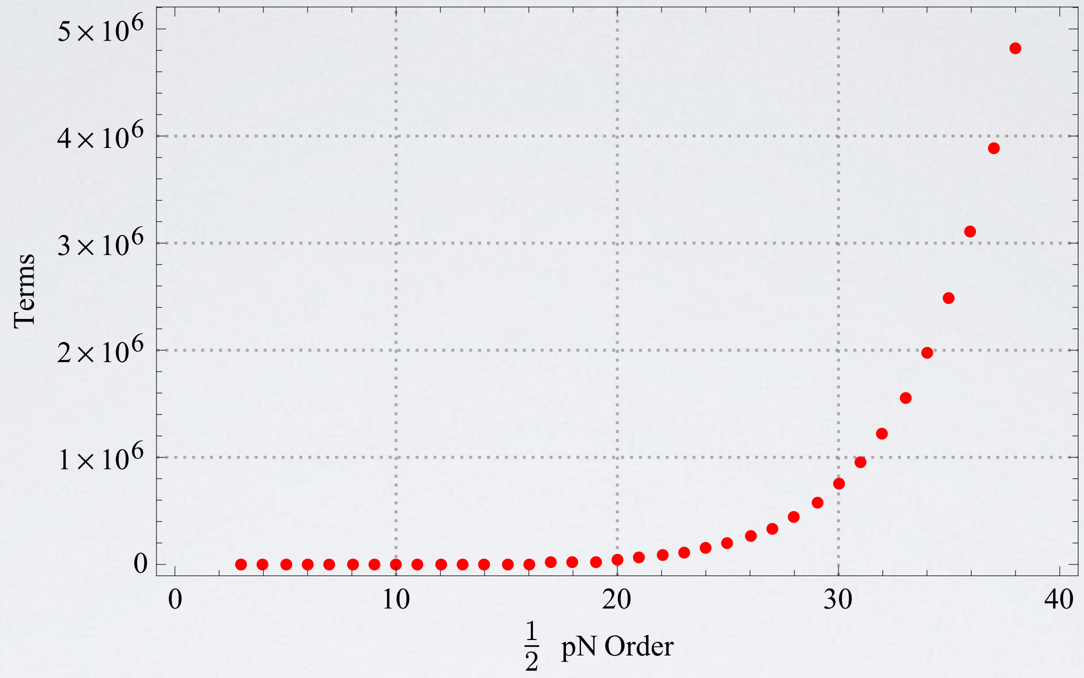
$$X^{\mathrm{in}}_{\ell m} = e^{\Psi^{\mathrm{in}}}(384X1^4{X_2}^{1/2}\eta^9)^{-1}\left(1 - \frac{1}{14}X_2\eta^2 + \frac{1}{504}(X_2^2 - 312X_1X_2)\eta^4 + \frac{1}{33264}(1232X1X_2^2 - X_2^3)\eta^6 + O(\eta^8)\right)$$

$$\Psi^{\mathrm{in}} = \left(\frac{47i}{35}2X_1{X_2}^{1/2}\eta^3 + \frac{24197}{19600}(2X_1{X_2}^{1/2}\eta^3)^2 + \ldots + \left(\frac{107}{210}(2X_1{X_2}^{1/2}\eta^3)^2 + \frac{1695233}{9261000}(2X_1{X_2}^{1/2}\eta^3)^4 + \ldots\right)\log(2X_1\eta^2)\right)$$

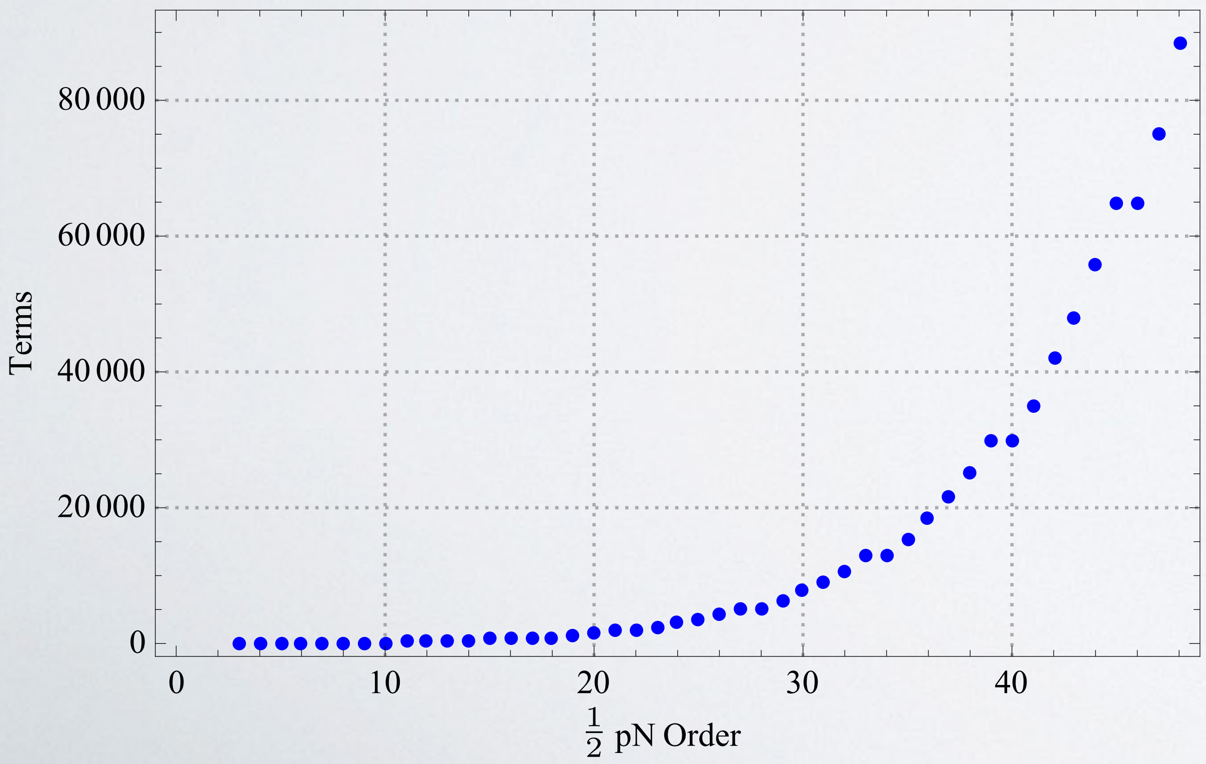
From millions to thousands…

$$l = 2, \ldots, l_{\max}$$

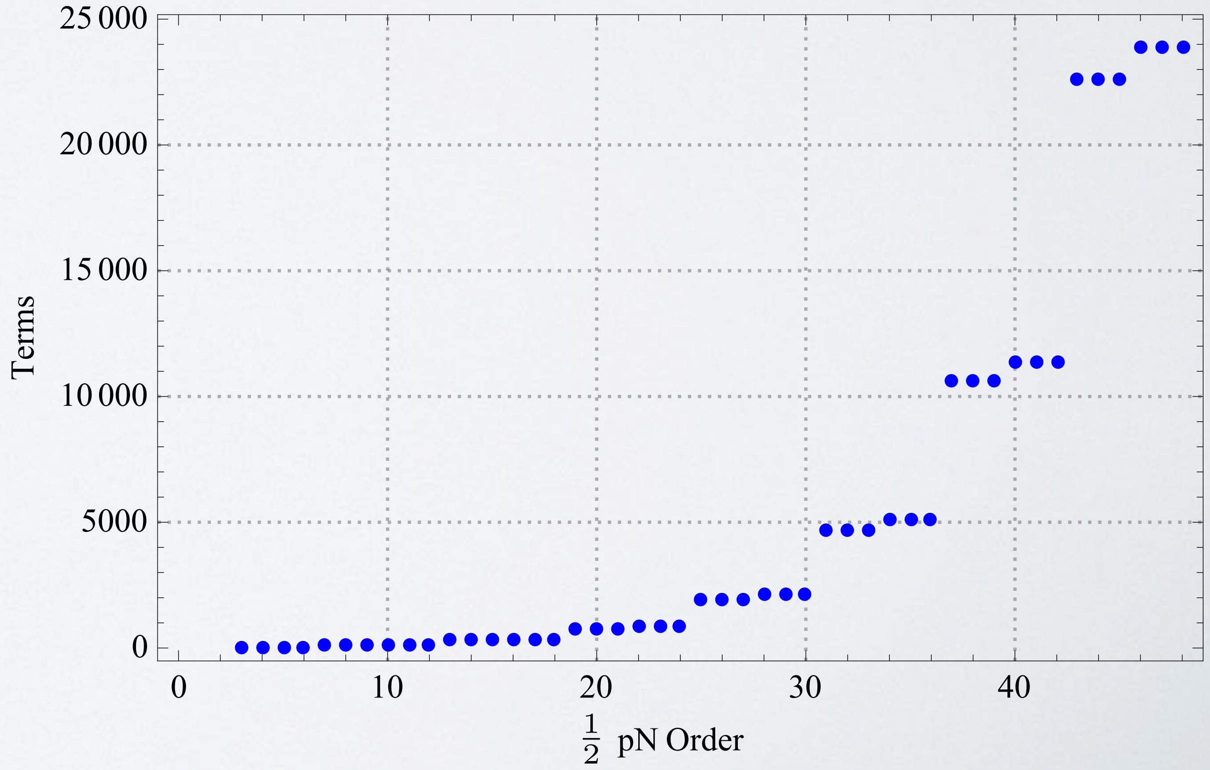
Before



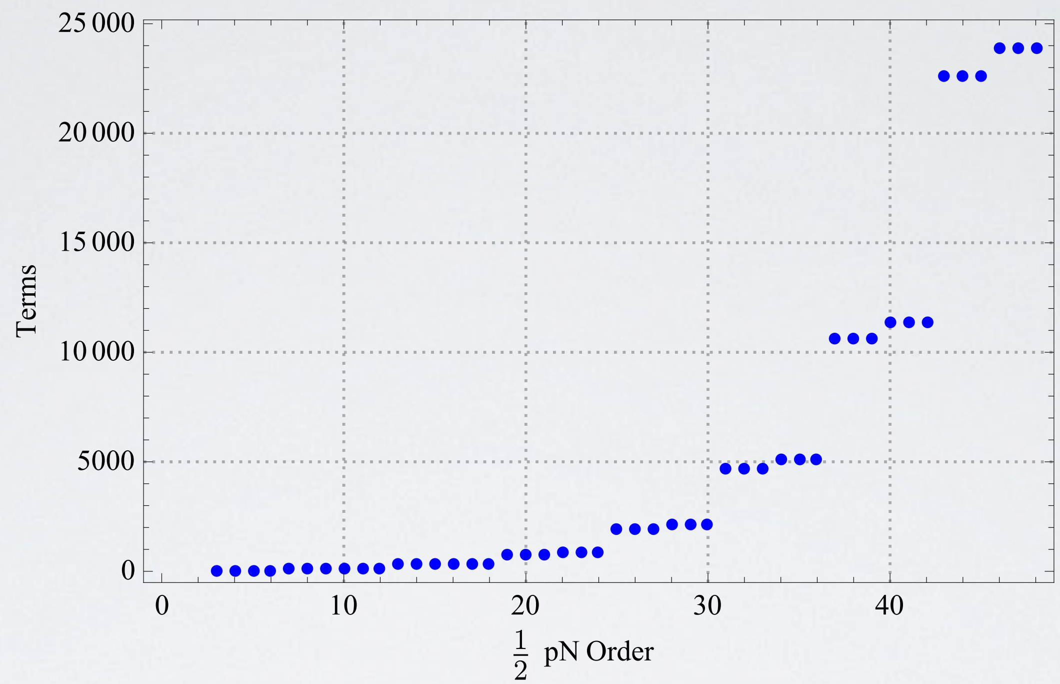
After



+



$$l = 2, \ldots, l_{\max}$$



$$\Psi^{\text{in}}_{\log} = \left(\frac{13}{42}(2X_1{X_2}^{1/2}\eta^3)^2 + \frac{10921}{271656}(2X_1{X_2}^{1/2}\eta^3)^4 + \frac{95353832269}{7709149047600}(2X_1{X_2}^{1/2}\eta^3)^6 + O(\eta^{24})\right)\log(2X_1\eta^2)$$

$$\Psi^{\text{up}}_{\log} = \left(-\frac{13}{42}(2X_1{X_2}^{1/2}\eta^3)^2 - \frac{10921}{271656}(2X_1{X_2}^{1/2}\eta^3)^4 - \frac{95353832269}{7709149047600}(2X_1{X_2}^{1/2}\eta^3)^6 + O(\eta^{24})\right)\log(2{X_2}^{1/2}\eta)$$

$$\nu = 3 - \frac{13}{42}(2X_1{X_2}^{1/2}\eta^3)^2 - \frac{10921}{271656}(2X_1{X_2}^{1/2}\eta^3)^4 - \frac{95353832269}{7709149047600}(2X_1{X_2}^{1/2}\eta^3)^6 + O(\eta^{24})$$

Log terms give an ansatz for the large l behaviour-new way to get nu!

$$l = l_{\max} + 1, \ldots$$

$$X^{\text{in}}_{\ell m} = X_1^{-\ell-1-\sum_{j=1}^{\infty} a_{(6j,2j)}(2X_1 {X_2}^{1/2}\eta^3)^{2j}} \left[1 + \eta^2 A^{\ell}_2 + \eta^4 A^{\ell}_4 + \eta^6 A^{\ell}_6 + \ldots\right]$$

$$\eta = 1/c$$

Where the A's are purely polynomials of X_1, X_2.

$$A^{\ell}_2 = a_{(4,0)} X_2^2 + a_{(4,2)} X_2 X_1 + a_{(4,4)} X_1^2$$

And we solve for the a's by demanding this satisfy the RW equation.
As a bonus the $a_{(6j,2j)}$ give us the coefficients for nu.

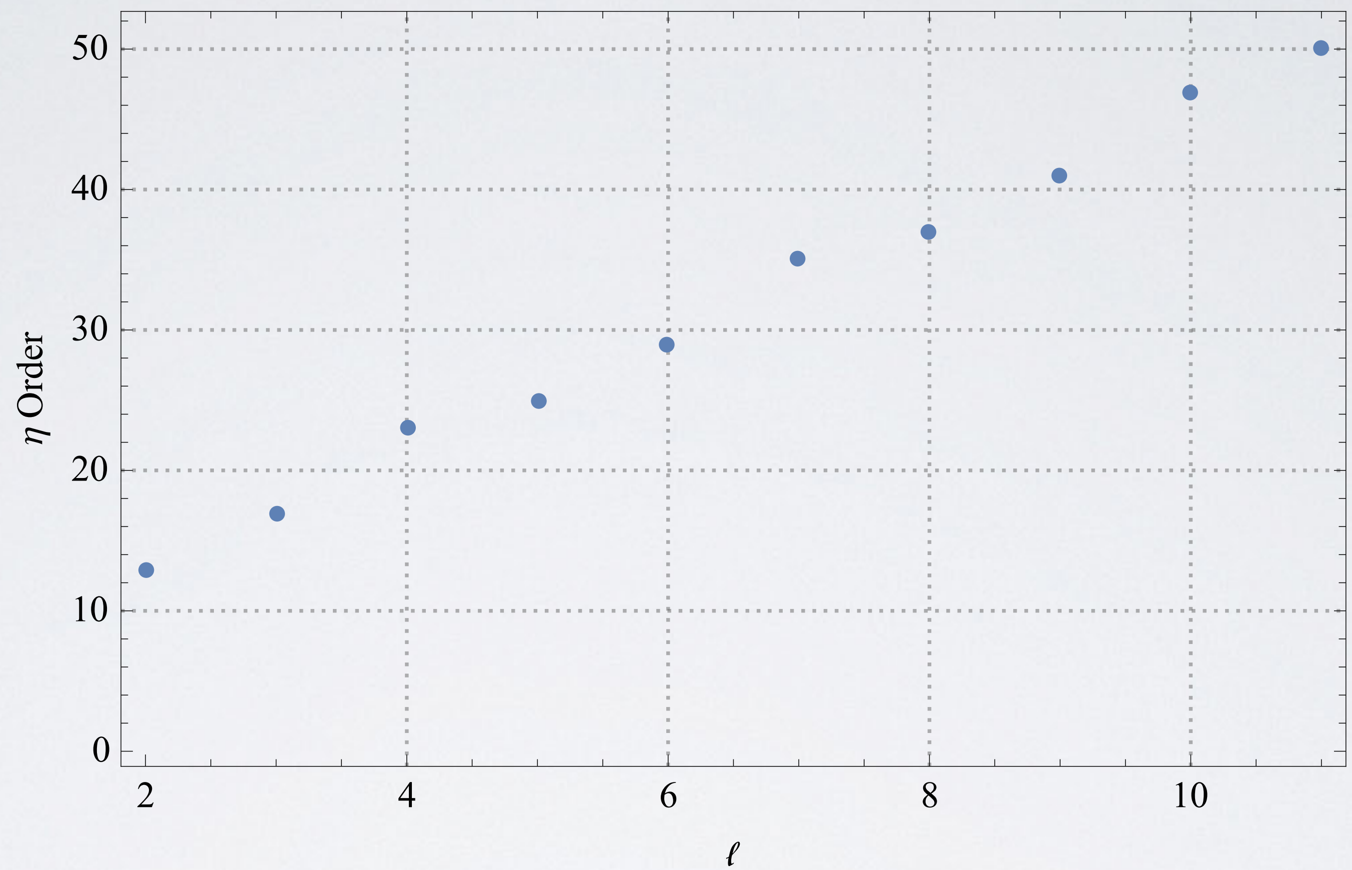
The outer solution seems to be a bit more complicated for lower values of l. But for large values we can
obtain all the orders we'd want by a similar ansatz.

$$X^{\text{up}}_{\ell m} = ({X_2}^{1/2})^{-l-\sum_{j=1}^{\infty} b_{(6j,2j)}(2X_1 {X_2}^{1/2}\eta^3)^{2j}} \left[1 + \eta^2 B^{\ell}_2 + \eta^4 B^{\ell}_4 + \ldots\right]$$

General *l* is only valid for 'large-*l*'

$$l = l_{\max} + 1, \ldots$$

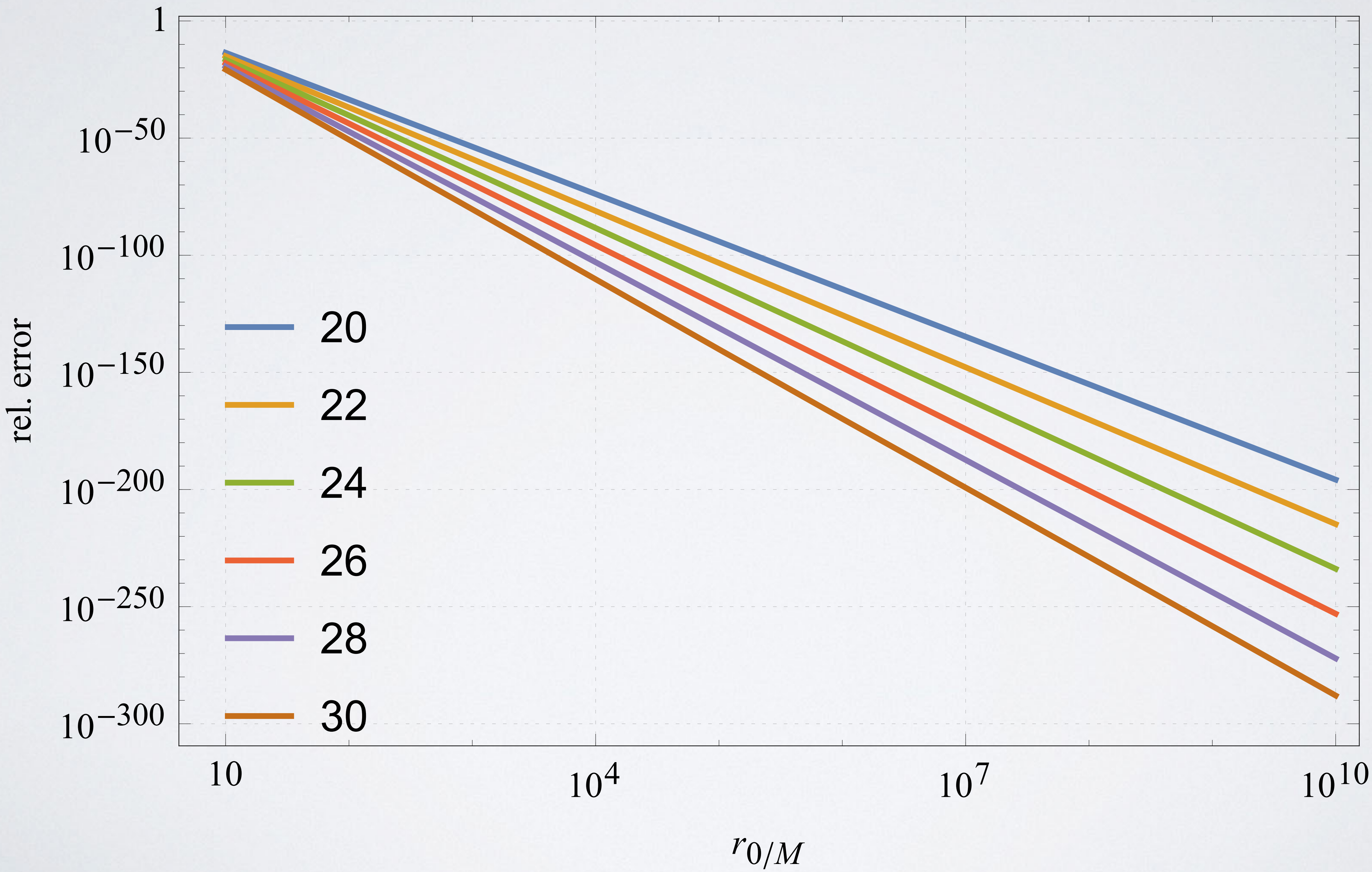
$$X^{\text{in}}_{\text{MST}} - X^{\text{in}}_{\text{Ansatz}}$$



Outer solution generally blows up at $O(\eta^{2l+4})$, and is valid until then. So at the very least we chose $l_{\max}$ by this.

Checking some 30pN data

$$\frac{X^{\mathrm{in}}_{\ell m} X^{\mathrm{up}}_{\ell m}}{W}$$



Bringing everything together involves an infinite *l* sum

For the low *l's* we just add up directly-no issue there

With the large *l*, at every pN order we have a contribution from the infinite sum. For instance after regularisation, at y^4 we have

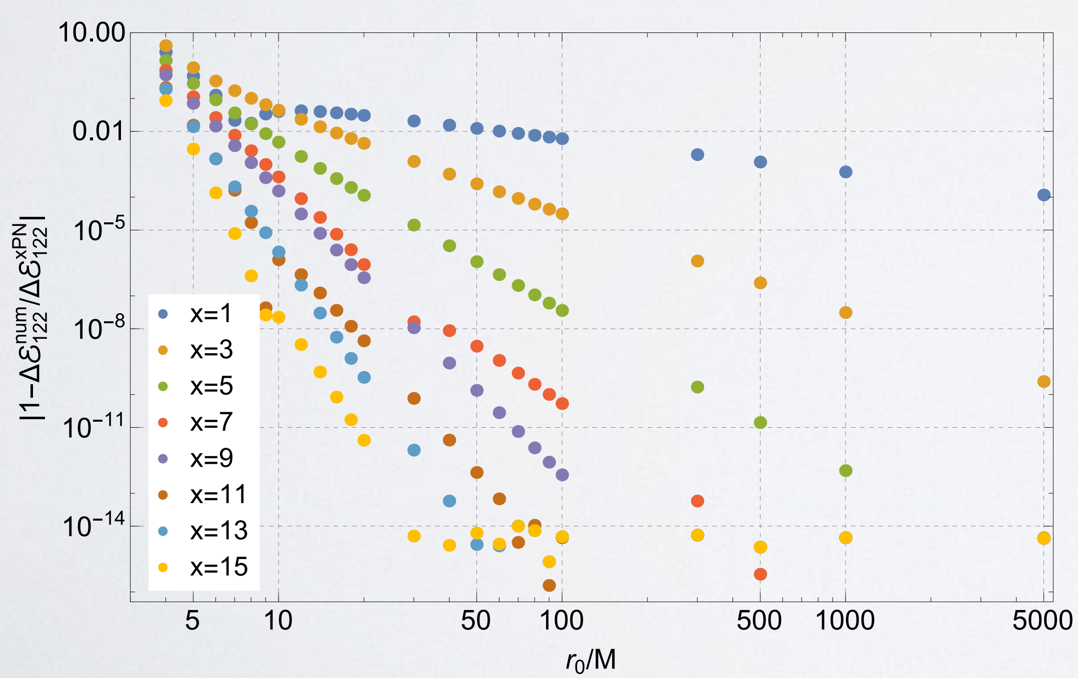
$$\Delta\psi_{largel} = -\frac{3\left(45008l^8 + 180032l^7 - 1860476l^6 - 6211540l^5 + 77739287l^4 + 26041178l^3 - 4741099l^2 - 18826950l - 13977600\right)}{1024(l-1)l(l+1)(l+2)(2l-5)(2l-3)(2l-1)(2l+3)(2l+5)(2l+7)}$$

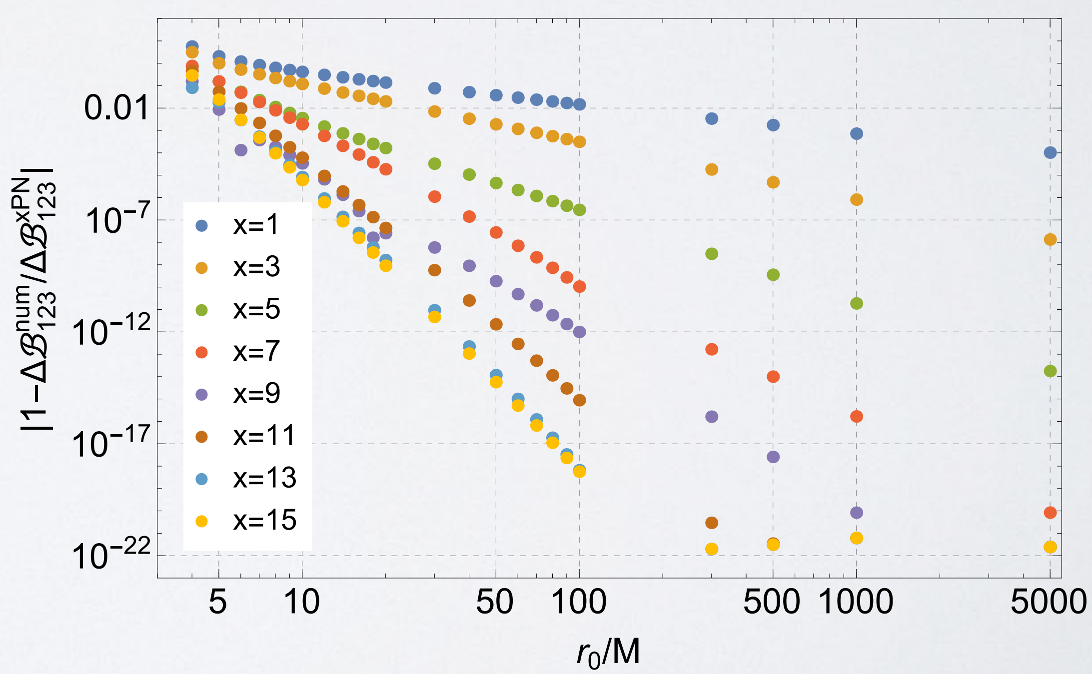
Mathematica can do these sums analytically, but for higher pN orders the polynomials becomes massive, and the calculation just takes too long.

Instead, split the denominator, extract manifestly convergent modes, bring everything else together and 'hope' it's manageable!

Put everything together to get some invariants

ΔU	21.5pN	Confirmed by Shah et al
$\Delta\psi$	21.5pN	
$\Delta\lambda_i$	18.5pN	
$\Delta E_i, \Delta B_i$	15pN	Compared with Paddy's numerics





A lot of this holds up well for Kerr.

Doing some low order expansions, phase extraction still gives good simplifications, however we no longer have a purely even series-slightly different ansatz

$$R^{\rm in}_{lm} = X_1^{-\ell+s-\sum_{j=1}^{\infty} a_{(3j,j)}(2X_1 {X_2}^{1/2}\eta^3)^j} \left[1+\eta A^{\ell}_1+\eta^2 A^{\ell}_2+\eta^3 A^{\ell}_3+\eta^4 A^{\ell}_4+\eta^5 A^{\ell}_5+\eta^6 A^{\ell}_6+\ldots\right]$$

Have Rin, Rup for some low *l*'s, large *l* for about 6pN (12 in 1/c) as a test.

All we need to do is metric reconstruction. -Cesar, Abhay, Martin.

How about eccentric orbits?

-Seth!