Experimental mathematics meets gravitational self-force

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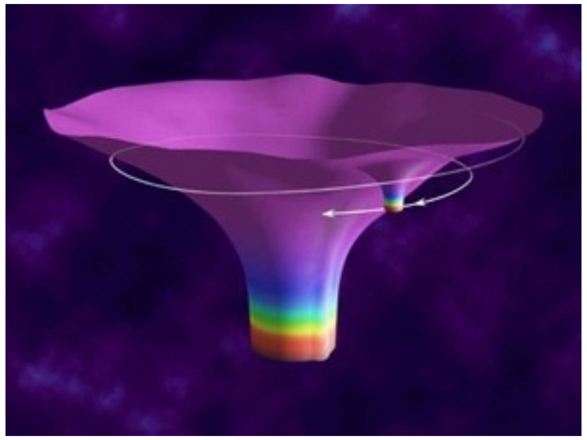
$$\begin{aligned}
&\Bigg[\frac{38142291450400809104702462420710 97}{1379871188591686265475 0000} - \frac{49750898616691548782381025 7447}{366017907748818124800 00}\pi^2 + \frac{121345100669686907714672417 3}{223434836512918732800 0}\pi^4 \\
&- \frac{527694089856719318 9}{25975962206208}\pi^6 - \frac{2759468242424}{19864845}\zeta(3) + \frac{3283328}{105}\zeta(5) - \frac{175005952}{55125}\log(2/R) + \frac{54784}{525}\log^2(2/R) \\
&+ \left(-\frac{29272001983873581577806936 7}{313683671243842099200000} + \frac{1712534}{694575}\pi^2 + \frac{128176}{11025}\zeta(3)\right)\mathrm{eulerlog}_1(R) \\
&+ \left(\frac{171522558894084996800503 04}{3254774420865060843 75} - \frac{2063384576}{694575}\pi^2 - \frac{154271744}{11025}\zeta(3)\right)\mathrm{eulerlog}_2(R) \\
&+ \left(-\frac{374229736116490933638717 33}{812775859037542400 00} + \frac{1232010}{343}\pi^2 + \frac{1364688}{49}\zeta(3)\right)\mathrm{eulerlog}_3(R) \\
&- \frac{159516220216108221870899 2}{929935548818588812 5}\mathrm{eulerlog}_4(R) + \frac{282725878023723294921875}{8291735527534018 56}\mathrm{eulerlog}_5(R) \\
&- \frac{121922675995010906 88}{202481230826875}\mathrm{eulerlog}_6(R) - \frac{6033540559350119080 3}{1792954994688000}\mathrm{eulerlog}_7(R) + \frac{690493302243328}{57747104415}\mathrm{eulerlog}_8(R) \\
&- \frac{205891132094649}{168551219200}\mathrm{eulerlog}_9(R) + \frac{792734736884113}{14316991088400}\mathrm{eulerlog}_1^2(R) - \frac{5014132830153349 12}{22370298575625}\mathrm{eulerlog}_2^2(R) \\
&+ \frac{1223478119816547 3}{196392196000}\mathrm{eulerlog}_3^2(R) + \frac{1768982113681408}{127830277575}\mathrm{eulerlog}_4^2(R) - \frac{3087470703125}{159011424}\mathrm{eulerlog}_5^2(R) \\
&+ \frac{72640032768}{17892875}\mathrm{eulerlog}_6^2(R) - \frac{6850136}{3472875}\mathrm{eulerlog}_1^3(R) + \frac{8253538304}{3472875}\mathrm{eulerlog}_2^3(R) - \frac{985608}{343}\mathrm{eulerlog}_3^3(R)\Bigg]\frac{1}{R^{12}}.
\end{aligned}$$



Outline

- Motivation and history of high-order post-Newtonian calculations (analytical and numerical)
- A brief introduction to experimental mathematics and integer relation algorithms
- A sketch of the methods we used to obtain analytic forms of high-order PN coefficients of ΔU from a high-accuracy numerical computation
- First results of a study of the structure of the energy flux at infinity for circular, equatorial orbits in Kerr
- Conclusions

Motivation and history I

- There has recently been considerable interest in very high-order calculations of linear-in-mass-ratio post-Newtonian coefficients using self-force techniques. In principle, one can carry out these computations to arbitrarily high order, though in practice one is limited by the amount of time spent on computation and the combinatorial complexity of the high-order coefficients.

- In particular, there are now analytic calculations of all known gauge invariant quantities have been taken to ~20PN in Schwarzschild (by Fujita [fluxes] and Kavanagh et al. [everything else]).

- Calculations in Kerr are more difficult (and the expansion is more involved), and only exist for the fluxes so far: Fujita has calculated the fluxes analytically to 11PN for circular, equatorial orbits, and Fujita and Sago have calculated them to 4PN and 6th order in the eccentricity for inclined eccentric orbits.

- Numerical calculations in Kerr can go to higher orders more easily, and Abhay Shah has calculated the fluxes to 20PN for circular, equatorial orbits.

Motivation and history II

- There are no analytic calculations of the other invariants in Kerr, so it is possible that numerical calculations (e.g., an extension of the recent work by Maarten van de Meent and Abhay Shah) will provide the first to provide the currently unknown PN coefficients (even at quite lower orders) for these quantities.

- Additionally, the convergence of the PN series becomes slower as one increases the spin (and the ISCO moves closer to the horizon, so the small body can reach even larger speeds)—Fujita estimates that one would need to go to 28PN to obtain a relative error of 10^{-5} in the energy flux at the ISCO for circular, equatorial orbits for $q = 0.9$, as would likely be necessary in order to use PN waveforms to detect EMRIs.

Motivation and history III

- The recent interest in using numerical methods to calculate PN coefficients for EMRIs starts primarily with the work by Shah, Friedman, and Whiting (SFW) on obtaining PN coefficients for Detweiler's redshift observable ΔU for circular orbits in Schwarzschild, where they found the existence of half-integer terms, starting at 5.5PN. Additionally, they were able to infer analytic forms for some simple coefficients (rationals and rationals times π) from a high-accuracy numerical calculation.

- When SFW first appeared on the arXiv, I realized that there was a way to obtain analytic forms of more complicated coefficients by using an integer relation algorithm (PSLQ), and was able to obtain an analytic form for one coefficient that same day using PSLQ, which was then verified by Bini and Damour's purely analytic calculation which appeared the next day.

- Abhay Shah has also applied similar techniques (including application of PSLQ) to the precession and quadrupolar tidal invariants for circular orbits in Schwarzschild, in addition to the aforementioned fluxes in Kerr.

The present work

- Such naïve application of PSLQ to the PN coefficients of the full quantity allows one to obtain analytic forms for some coefficients of moderate complexity (just a few transcendentals) with not too many digits (tens to a few hundred).

- However, if one wants to obtain analytic forms for more complicated coefficients (e.g., 30+ terms) from numerical data of reasonable precision, one has to use a more refined application of PSLQ:

- 1. We consider the individual (retarded) ℓ,m modes, which have a nice structure that is obscured in the sum over all modes.

- 2. We study this structure and find a simplification of the PN expansion of the modes that allows us to predict some (or even all of) certain higher-order coefficients, including some complete leading logarithmic terms in the full quantity.

- 3. We find the general form of the PN coefficients of the renormalized ℓ-modes for large ℓ (and the PN orders we consider), which allows us to perform the sum over all modes analytically, keeping us from having to find high-order regularization coefficients to obtain the requisite numerical accuracy in the infinite ℓ-sum to recognize its analytic expression using PSLQ, which would require calculating to prohibitively high ℓ (~1000).

The present work

An example of the complexity we're considering: The 10PN coefficient of ΔU written in terms of $\mathrm{eulerlog}_m(R) := \gamma + \log(2mR^{-1/2})$, to simplify it, and then the non-log(R) part written out in full

$$
\begin{aligned}
&\Big[-\frac{2389467863446532647991752802\,03}{4522423405833558225000} - \frac{107020844192365086048968\,3}{5865671598538752 0000}\pi^2 + \frac{832229033014028790267991}{1662461581197312000}\pi^4 + \frac{54067065388369}{12884901888}\pi^6 \\
&- \frac{128695611256}{5457375}\zeta(3) + \frac{32768}{5}\zeta(5) - \frac{20416}{35}\log(2/R) - \frac{157982464536376957}{674943865596000}\,\mathrm{eulerlog}_1(R) + \Big(-\frac{1483437716511288604288}{14480466347221875} \\
&+ \frac{46895104}{33075}\pi^2 + \frac{3506176}{525}\zeta(3)\Big)\,\mathrm{eulerlog}_2(R) - \frac{52813127885844357}{10492954472000}\,\mathrm{eulerlog}_3(R) + \frac{8040008069311889408}{82745521984125}\,\mathrm{eulerlog}_4(R) \\
&- \frac{263296063591796875}{8742130068672}\,\mathrm{eulerlog}_5(R) - \frac{13640920722432}{1146520375}\,\mathrm{eulerlog}_6(R) + \frac{6491563697269}{1181466000}\,\mathrm{eulerlog}_7(R) \\
&- \frac{1099511627776}{1688511825}\,\mathrm{eulerlog}_8(R) + \frac{54944178599}{7491884400}\,\mathrm{eulerlog}_1^2(R) + \frac{69907855522816}{3781960875}\,\mathrm{eulerlog}_2^2(R) + \frac{79338802833}{61661600}\,\mathrm{eulerlog}_3^2(R) \\
&- \frac{705049919488}{108056025}\,\mathrm{eulerlog}_4^2(R) + \frac{7548828125}{4077216}\,\mathrm{eulerlog}_5^2(R) - \frac{187580416}{165375}\,\mathrm{eulerlog}_2^3(R)\Big]\frac{1}{R^{11}}
\end{aligned}
$$

$$
\begin{aligned}
\alpha_{10} = &-\frac{238946786344653264799175280203}{4522423405833558225000} - \frac{1070208441923650860489683}{5865671598538752 0000}\pi^2 + \frac{832229033014028790267991}{1662461581197312000}\pi^4 \\
&+ \frac{54067065388369}{12884901888}\pi^6 + \frac{46895104}{33075}\pi^2\gamma + \frac{93790208}{33075}\pi^2\log(2) - \frac{128695611256}{5457375}\zeta(3) + \frac{3506176}{525}\zeta(3)\gamma + \frac{7012352}{525}\zeta(3)\log(2) \\
&+ \frac{32768}{5}\zeta(5) - \frac{8982774011612318193194}{188246062513884375}\gamma + \frac{742653529820948}{49165491375}\gamma^2 - \frac{187580416}{165375}\gamma^3 - \frac{375160832}{55125}\gamma^2\log(2) \\
&+ \frac{2019720449766952}{49165491375}\gamma\log(2) - \frac{750321664}{55125}\gamma\log^2(2) + \frac{79338802833}{30830800}\gamma\log(3) + \frac{7548828125}{2038608}\gamma\log(5) \\
&+ \frac{5597334601144521152662}{188246062513884375}\log(2) + \frac{902678188274452}{49165491375}\log^2(2) - \frac{1500643328}{165375}\log^3(2) + \frac{79338802833}{30830800}\log(2)\log(3) \\
&+ \frac{7548828125}{2038608}\log(2)\log(5) - \frac{177654834337542021}{10492954472000}\log(3) + \frac{79338802833}{61661600}\log^2(3) - \frac{263296063591796875}{8742130068672}\log(5) \\
&+ \frac{7548828125}{4077216}\log^2(5) + \frac{6491563697269}{1181466000}\log(7)
\end{aligned}
$$

The present work (cont.)

- Using the method we outlined above (and will detail shortly) and a calculation of the ℓ,m modes of ΔU to more than 5000 digits (of which we used at most 1240 digits) for 21 radii $R = 10^k M$, $k \in \{50,\ldots,70\}$ (of which we used at most 15 radii), we were able to obtain analytic forms for the PN coefficients to 12.5PN, plus the 13.5PN coefficient (including obtaining all the previously known coefficients "from scratch").

- We then calculated the full ΔU to "merely" ~600 digits at somewhat smaller radii (10^{18} to $9 \times 10^{33} M$) and used this, along with the predictions of higher-order terms from the simplification, to obtain the coefficients to 21.5PN in mixed numerical-analytic form, including all the logarithmic terms at 13PN.

- Also, while we used the standard method of fitting to obtain the PN coefficients of the full ΔU in the second step, to obtain the PN coefficients of the individual modes in the first step we used a different method that uses PSLQ more centrally, using linear combinations of the values of the modes at different radii to obtain the value of a given PN coefficient sufficiently accurately to be able to identify it with PSLQ.

- Before giving more details of this method, we shall first give a brief introduction to experimental mathematics and integer relation algorithms as well as a simple example of applying PSLQ to obtain a coefficient of the full ΔU.

An introduction to experimental mathematics and PSLQ

- The PSLQ integer relation algorithm is a standard (modern) experimental mathematics technique, and is implemented in Mathematica (as of Version 8) as FindIntegerNullVector[].

- PSLQ, discovered by the sculptor-mathematician Heleman Ferguson and computational mathematician David Bailey in 1992, takes in a vector of real numbers and uses Partial Sums-of-squares and the LQ decomposition to return a nonzero integer vector orthogonal to the input and whose (L^2) norm is at most a known factor times those vectors' minimum norm, or a minimum value for the norm of such a relation, in polynomial time (in the number of elements in the vector).

- One can thus apply PSLQ to obtain the rational coefficients of a linear combination of transcendentals from a sufficiently accurate decimal expansion, in addition to many other applications.

An introduction to experimental mathematics and PSLQ

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Helaman Ferguson and his sculpture Invisible Handshake

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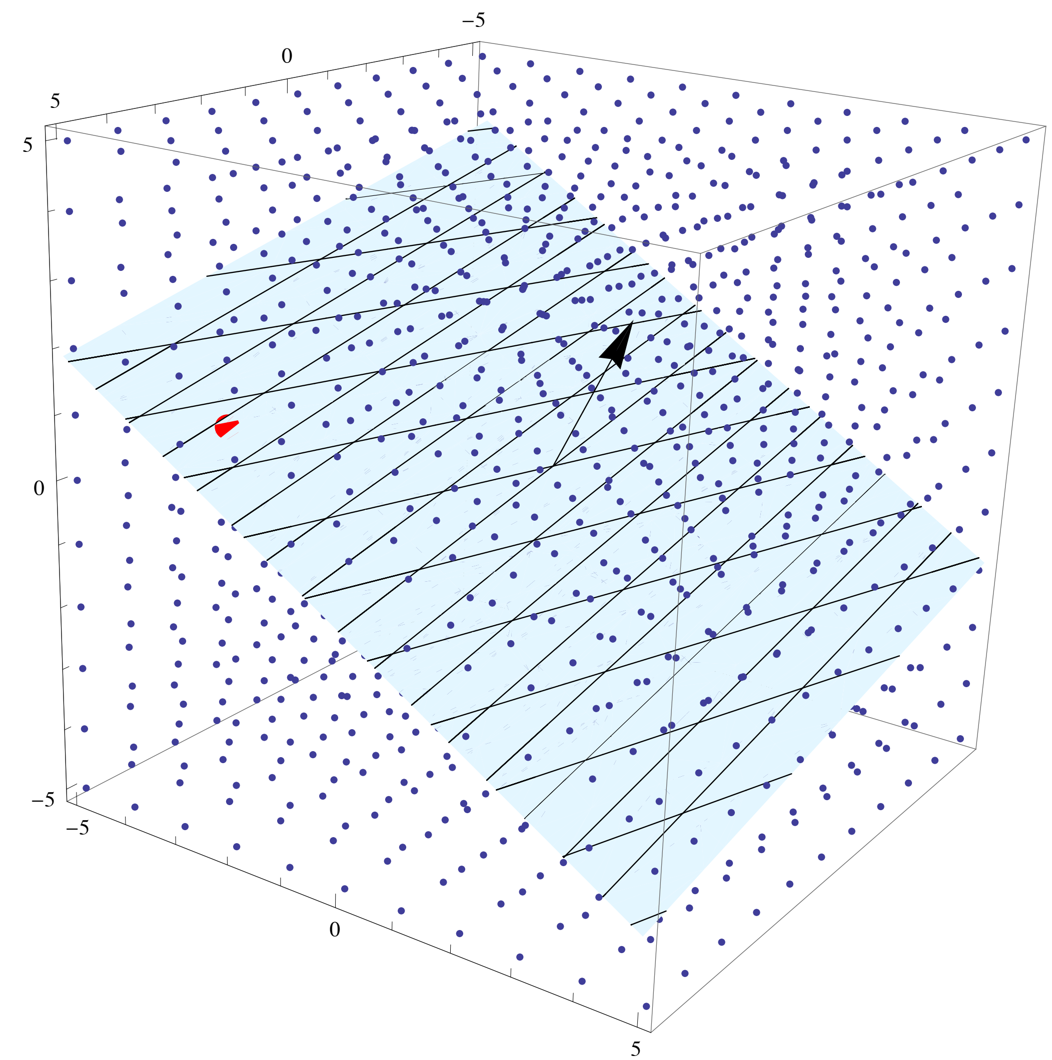
Graphical illustration of the problem PSLQ solves: Find a small norm vector in $\vec{v}^{\perp} \cap \mathbb{Z}^n$.



Notable successes of experimental mathematics (using PSLQ)

- The Bailey-Borwein-Plouffe formula for π, which allows one to calculate the nth digit of its binary hexadecimal expansion without calculating any of the previous digits, was first discovered using PSLQ:

$$\pi = \sum_{k=0}^{\infty} \frac{1}{16^k} \left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6} \right)$$

- Other notable discoveries with PSLQ include the 240-degree polynomial satisfied by B_4, the fourth bifurcation point of the logistic map. This was obtained by physicist David Broadhurst, who has also used PSLQ to evaluate various Feynman integrals in terms of multiple zeta values.

- There is also a computation by Bernie Nickel (2011) of high-order expansion coefficients of the ground state energy of H_2^+ using PSLQ that is similar to what we do here. (The structures of the expansions are even somewhat similar, and they can be computed using similar techniques, though the structure of the H_2^+ expansion is simpler than that of an individual mode of ΔU.)

An example of applying PSLQ to ΔU

- Here we show how to obtain the coefficient of $\log(R)/R^8$ in the PN expansion of ΔU, as I first did using the numerical expressions from SFW.

- While my later study of the structure of the modes would have allowed me to only have to determine the purely rational coefficient here, I shall here show the first way I obtained this, using PSLQ to obtain all the coefficients.

- Specifically, the transcendentals in the coefficient are given by the simplification or (pretty much equivalently in this case) by noting that this order has the first appearance of a $\log^2(R)$ term, which thus only comes from the 2,2 mode, and therefore comes from the square of $\mathrm{eulerlog}_2(R)$, so one can obtain the coefficients of the transcendentals in the $\log(R)$ term at this order from the (purely rational) coefficient of the $\log^2(R)$ term.

An example of applying PSLQ to ΔU

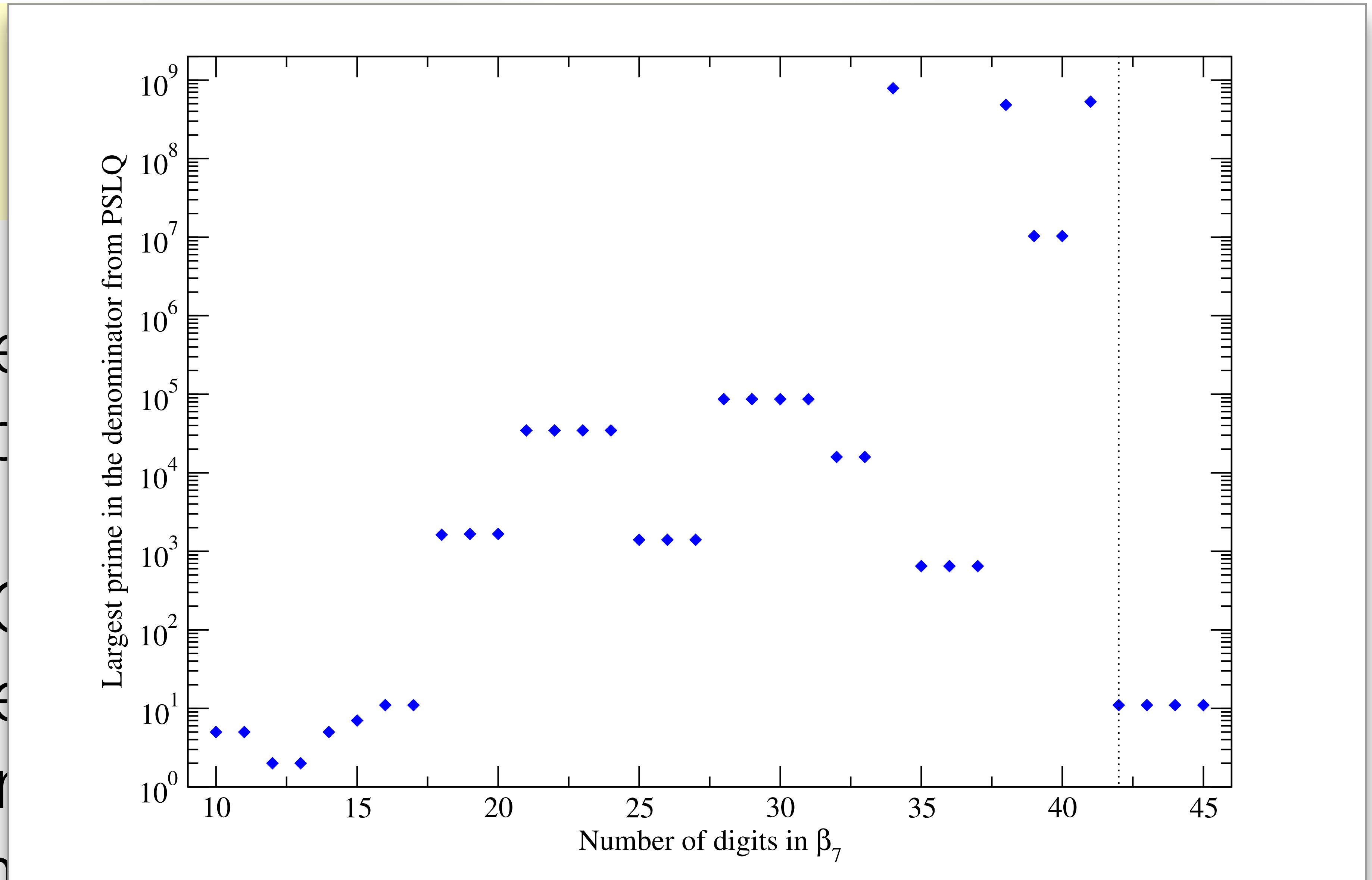
$$\beta_7 = 536.405212471024286871789539475038911270206\ldots$$
$$= \frac{5163722519}{5457375} - \frac{109568}{525}\gamma - \frac{219136}{525}\log(2),$$

One can obtain the given analytic expression from the given 42 digits by applying FindIntegerNullVector to the vector

$$\{\beta_7, 1, \gamma, \log(2)\}$$

Of course, one checks this expression by requiring that it doesn't change if one uses more digits. Additionally, one can make a stringent test by requiring that the denominators of the rationals do not contain abnormally large primes.

An example of applying PSLQ to ΔU



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of the rationals do not contain abnormally large primes.

An example of applying PSLQ to ΔU

Smooth numbers, which only contain small prime factors (e.g., only primes smaller than the logarithm of the number) are quite sparsely distributed as the size of the number increases, which is why this test is so stringent.

For instance, at 12 and 12.5PN, the denominators of the purely rational terms each have 32 digits, but the largest prime in each is only 19. The probability of this being the case for a random 32 digit integer is less than 10^{-21}.

Further consistency checks of the results of PSLQ

- One often scales the coefficient one is considering by a large smooth number (e.g., the denominator of the purely rational term at one PN order lower), as this often allows PSLQ to be able to determine the analytic form with fewer digits. One can then check that the result is independent of small changes to the powers of primes in the smooth number.

- One can also check that the results are independent of whether one obtains the coefficients of ΔU itself, or first scales it by, e.g., some power of U (where the appropriate power can also help in determining the analytic form with fewer digits).

- Finally, if one knows many digits, one can check that the result is independent of whether one uses additional transcendentals in the vector to which one applies PSLQ.

Calculating the infinite ℓ-sum

- To obtain the infinite sum, we note that the general ℓ-dependence of a given PN coefficient of a renormalized ℓ-mode (for large enough ℓ that it is purely rational) has a reasonably simple general form as a sum of rational functions (though the number of rational functions increases with PN order), and solve for the coefficients of these rational functions.

$$\mathcal{T}_k^n(\ell) := \frac{1}{(\ell+k+1/2)^n} + \frac{(-1)^n}{(\ell-k+1/2)^n},$$
$$\mathcal{U}_k^n(\ell) := \frac{1}{(\ell+k)^n} + \frac{(-1)^n}{(\ell-k+1)^n},$$
$$\mathcal{V}^n(\ell) := \frac{1}{(\ell+1/2)^n}.$$

$\mathcal{T}_1^1$,

$\mathcal{T}_{1-2}^1$ & $\mathcal{U}_1^1$,

$\mathcal{T}_{1-3}^1$ & $\mathcal{T}_1^2$ & $\mathcal{U}_{1-2}^1$ & $\mathcal{V}^2$,

$\mathcal{T}_{1-4}^1$ & $\mathcal{T}_1^2$ & $\mathcal{T}_1^3$ & $\mathcal{U}_{1-3}^1$ & $\mathcal{V}^2$,

Form of the general ℓ-dependence through 4PN

$\mathcal{T}_{1-12}^1$ & $\mathcal{T}_{1-9}^2$ & $\mathcal{T}_{1-6}^3$ & $\mathcal{T}_{1-3}^4$ & $\mathcal{T}_1^5$ & $\mathcal{T}_1^6$ & $\mathcal{T}_1^7$ & $\mathcal{T}_1^8$
& $\mathcal{U}_{1-11}^1$ & $\mathcal{U}_{1-8}^2$ & $\mathcal{U}_{1-5}^3$ & $\mathcal{U}_{1-2}^4$ & $\mathcal{U}_1^5$ & $\mathcal{U}_1^6$ & $\mathcal{U}_1^7$
& $\mathcal{V}^2$ & $\mathcal{V}^4$ & $\mathcal{V}^6$ & $\mathcal{V}^8$.

…and at 12PN

- We then check that the putative expression we obtain indeed reproduces the coefficients for the ℓs that were not used in the solve, and then perform the ℓ-sum analytically *à la* Bini and Damour (evaluating it in terms of the Riemann zeta function at even integers, using partial fractions and noting that much of the series telescopes—this can be done automatically by Mathematica).

Sketch of our method for obtaining the PN coefficients of ΔU to 12.5PN analytically and 21.5PN in mixed analytic-numerical form

Calculate the retarded (ℓ, m) modes to very high accuracy (> 5000 digits; we used ~1240 digits here) for large radii [$(10^{50}$–$10^{70})M$]

Calculate the full ΔU to “merely high” accuracy (~600 digits) for somewhat smaller radii [$(10^{18}$–$10^{34})M$]

Both these calculations use a radiation gauge code and the MST formalism

This fit also provides a sensitive check of the accuracy of the analytic forms we obtain

Obtain the analytic PN coefficients of the low-order (ℓ, m) modes using PSLQ (we go to ℓ = 10 for 12.5PN and could automate the process for higher ℓ)

Here we use linear combinations of the values of ΔU at different radii to obtain enough digits to correctly identify the analytic form of a given PN coefficient, so we can subtract it off and move on to the next coefficient.

Fit to obtain the PN coefficients of the full ΔU, after subtracting off the known analytic terms

Notice patterns in the PN coefficients and obtain a simplification that allows one to predict certain higher-order coefficients

Obtain analytic forms for as many PN coefficients as possible, with help from the simplification, and use these to improve the accuracy of the fit

Obtain the analytic PN coefficients of the general renormalized ℓ-modes using PSLQ and a linear solve and sum them analytically

Result I: The analytic PN coefficients of ΔU through 12.5PN + the 13.5PN coefficient, and some leading logarithmic terms to all orders

Result II: The PN coefficients of ΔU through 21.5PN in mixed numerical-analytic form, including analytic forms for all but the non-log(R) term at 13PN

All these PSLQ determinations are required to pass stringent tests (e.g., not having large prime factors in the denominators of the rationals) that give us high confidence that the expressions we obtain are the true ones

A cautionary example

- Of course, even if one finds an analytic form that reproduces the decimal expansion to many digits, this doesn't prove that the analytic form is indeed correct.

- For instance,

$$\int_0^\infty \cos(2x) \prod_{n=1}^{\infty} \cos(x/n)\, \mathrm{d}x$$

$$= 0.392699081698724154807830422909937860524646\underline{5434187231595926}\ldots$$

  only differs from π/8 starting at the 43rd digit.

- Bailey and Borwein give even more extreme examples in some of their articles in the notices of the AMS (2005 & 2011), notably one where the discrepancy would *never* be seen in any realistic numerical computation, since it only occurs after more than a googolplex ($10^{10^{100}}$) digits!

- However, in our case, the structures we find (and expect to see) in the expansion give us good confidence that the analytic expressions we find are correct, which is borne out by agreement with a concurrent completely analytic computation by Kavanagh, Ottewill, and Wardell (2015).

Simplifying the (retarded) ℓ,m modes of ΔU and the energy flux at infinity

- First, a reminder of the basic simplification of the modes of the energy flux at infinity that I found last year. This involves the function

$$S_{\ell m} := (2mv)^{\bar{\nu}_{\ell m}(v)} e^{\pi m v^3} \frac{\Gamma[1 + \bar{\nu}_{\ell m}(v) - 2imv^3]}{\Gamma[1 + 2\bar{\nu}_{\ell m}(v)]}$$

Fractional part of the MST renormalized angular momentum ν

which one can write as

$$S_{\ell m} = \exp\left[\bar{\nu}_{\ell m}(v)\, \mathrm{eulerlog}_m(v) + \pi m v^3 + \sigma_{S_{\ell m}}(v)\right]$$

$$\sigma_{S_{\ell m}}(v) := \sum_{n=2}^{\infty} \frac{\zeta(n)}{n} \left\{\left[-\bar{\nu}_{\ell m}(v) + 2imv^3\right]^n - [-2\bar{\nu}_{\ell m}(v)]^n\right\}$$

using

$$\Gamma(1+z) = \exp\left[-\gamma z + \sum_{n=2}^{\infty} \frac{\zeta(n)}{n} (-z)^n\right]$$

Simplifying the (retarded) ℓ,m modes of ΔU and the energy flux at infinity (cont.)

- The $S_{\ell m}$ factorization turns the ℓ,m mode of the energy flux at infinity for Schwarzschild into a simple integer PN series with rational coefficients through $O(v^{7+4\ell})$.

- We found a similar simplification of the modes of ΔU, though its full form seems more complicated than that of $S_{\ell m}$, and we have only found it as a series expansion, except for one exponential that's the same as in $S_{\ell m}$.
(Unlike for $S_{\ell m}$, it does not seem possible to be able to read off the simplification directly from the MST formalism, though one can see where the exponential part comes from.)

$$\begin{aligned}
\Upsilon^{C1}_{\ell m} &= \Bigg[e^{2\bar{\nu}_{\ell m}\,\mathrm{eulerlog}_m(R)}\Bigg\{\frac{1}{2\bar{\nu}_{\ell m}}\frac{1}{R^2} - \frac{5}{12}\frac{\bar{\nu}_{\ell m}\pi^2}{R^2} + \frac{7}{3}\frac{\bar{\nu}^2_{\ell m}\zeta(3)}{R^2} + (2m)^2\frac{\zeta(3)}{R^5} - \frac{m^2\bar{\nu}_{\ell m}}{15}\frac{\pi^4}{R^5} - (2m)^4\frac{\zeta(5)}{R^8} - \frac{\bar{\nu}_{\ell m}}{4m}\frac{\pi}{R^{0.5}} \\
&\quad + \frac{\bar{\nu}^3_{\ell m}}{24m}\frac{\pi^3}{R^{0.5}} - \frac{m\bar{\nu}_{\ell m}}{3}\frac{\pi^3}{R^{3.5}} - 2m\bar{\nu}^2_{\ell m}\frac{\pi\zeta(3)}{R^{3.5}} + \frac{4m^3\bar{\nu}_{\ell m}}{45}\frac{\pi^5}{R^{6.5}}\Bigg\} - \frac{1}{2\bar{\nu}_{\ell m}}\frac{1}{R^2}\Bigg]\sum_{k=0}^{\infty}\frac{A^{(k)}_{\ell m}}{R^{k+\ell+1+\varepsilon_{\ell m}}} \\
&= \Bigg[e^{2\bar{\nu}_{\ell m}\,\mathrm{eulerlog}_m(R)}\Bigg\{\frac{1}{2\bar{\nu}_{\ell m}}\frac{1}{R^2} - \bar{\nu}_{\ell m}\left[\frac{5}{2}\zeta(2) - \frac{7}{3}\bar{\nu}_{\ell m}\zeta(3)\right]\frac{1}{R^2} + (2m)^2\left[\zeta(3) - \frac{3}{2}\bar{\nu}_{\ell m}\zeta(4)\right]\frac{1}{R^5} - (2m)^4\frac{\zeta(5)}{R^8} \\
&\quad - \frac{\bar{\nu}_{\ell m}}{4m}\left[1 - \bar{\nu}^2_{\ell m}\zeta(2)\right]\frac{\pi}{R^{0.5}} - 2m\bar{\nu}_{\ell m}\left[\zeta(2) + \bar{\nu}_{\ell m}\zeta(3)\right]\frac{\pi}{R^{3.5}} + (2m)^3\bar{\nu}_{\ell m}\zeta(4)\frac{\pi}{R^{6.5}}\Bigg\} - \frac{1}{2\bar{\nu}_{\ell m}}\frac{1}{R^2}\Bigg]\sum_{k=0}^{\infty}\frac{A^{(k)}_{\ell m}}{R^{k+\ell+1+\varepsilon_{\ell m}}} \\
&=: C^{[1]}_{\ell m}\sum_{k=0}^{\infty}\frac{A^{(k)}_{\ell m}}{R^k}.
\end{aligned}$$

same as in $|S_{\ell m}|^2$

Simplifying the (retarded) ℓ,m modes of ΔU and the energy flux at infinity (cont.)

- Additionally, the simplification of ΔU does not act by division, but rather by subtraction. Here is the simplified version of the 2,2 mode of ΔU/U to 12.5PN + 13.5PN (and a bit of 14PN), as an example:

$$
\begin{aligned}
\Upsilon_{22} - \Upsilon_{22}^{C1} = {} & \sum_{k=1}^{13} \frac{\alpha_{22}^{(k)}}{R^k} + \left[-\frac{2^6}{5^1}\frac{1}{R^9} - \frac{2^5 3^2}{5^1}\frac{1}{R^{10}} - \frac{2^3 4507^1}{3^1 5^1 7^1}\frac{1}{R^{11}} - \frac{2^2 191^1 124343^1}{3^2 5^3 7^2}\frac{1}{R^{12}} - \frac{1493^1 185557^1}{2^1 3^1 5^3 7^2}\frac{1}{R^{13}} \right] \log(2/R) \\
& + \frac{2^{16}}{3^1 5^2}\frac{\log(2/R)\,\mathrm{eulerlog}_2(R)}{R^{13}} + \left[\frac{2^9 107^1}{3^1 5^2 7^1}\frac{1}{R^{12}} + \frac{2^8 3^1 107^1}{5^2 7^1}\frac{1}{R^{13}} \right] \log^2(2/R) + \left[-\frac{2^{10} 107^1}{3^2 5^2 7^1}\frac{1}{R^{12}} \right. \\
& \left. - \frac{2^9 11^1 13^2}{3^2 5^2 7^1}\frac{1}{R^{13}} \right] \pi^2 + \left[-\frac{2^{11}}{5^1}\frac{1}{R^{12}} - \frac{2^{10} 3^2}{5^1}\frac{1}{R^{13}} \right] \zeta(3) - \frac{2^{13}}{3^1 5^2}\frac{\pi}{R^{11.5}} - \frac{2^{12} 19^1}{3^2 5^2 7^1}\frac{\pi}{R^{12.5}} - \frac{2^{10} 11^1 673^1}{3^4 5^2 7^1}\frac{\pi}{R^{13.5}} \\
& + \left\{ \frac{1}{R^{14}} \text{ and higher terms that we do not yet know all of} \right\} + \left[-\frac{2^9 240013637^1}{3^3 5^4 7^2 11^1} + \frac{2^{17} 107^1}{3^2 5^3 7^1}\mathrm{eulerlog}_2(R) \right. \\
& \left. + \frac{2^{16} 107^1}{3^1 5^3 7^1}\log(2/R) - \frac{2^{17}}{3^2 5^2}\pi^2 \right] \frac{\pi}{R^{14.5}} - \frac{2^{13} 107^2}{3^3 5^3 7^2}\frac{\log^3(2/R)}{R^{15}} + O\left(\frac{1}{R^{15.5}}\right),
\end{aligned}
$$

Here $\alpha_{22}^{(k)} \in \mathbb{Q}$.

Simplifying the (retarded) ℓ,m modes of ΔU and the energy flux at infinity (cont.)

- Add… by divis… plified vers… 5PN (and…

Υ_{22} …

Her…

Here is the unsimplified expression for the 2,2 mode of ΔU/U, only to 10.5PN, for comparison

$$
\begin{aligned}
\Upsilon_{22} = {} & \frac{3^1}{2^2}\frac{1}{R} + \frac{19^1}{2^3 7^1}\frac{1}{R^2} + \frac{11^1 673^1}{2^5 3^2 7^1}\frac{1}{R^3} + \frac{5813077^1}{2^6 3^1 5^2 7^2 11^1}\frac{1}{R^4} + \biggl[\frac{19^1 727^1 78938159^1}{2^9 3^3 5^2 7^3 11^1 13^1} - \frac{2^7}{5^1}\,\mathrm{eulerlog}_2(R)\biggr]\frac{1}{R^5} \\
& + \biggl[\frac{29^1 71^1 733^1 3301^1 3323^1}{2^{10} 3^5 5^2 7^3 11^1 13^1} + \frac{2^6 151^1}{3^1 5^1 7^1}\,\mathrm{eulerlog}_2(R)\biggr]\frac{1}{R^6} - \frac{2^7 107^1}{3^1 5^2 7^1}\frac{\pi}{R^{6.5}} + \biggl[-\frac{6217^1 51217^1 95947^1 108961^1}{2^{12} 3^4 5^4 7^4 11^2 13^1 17^1} \\
& + \frac{2^4 6899^1}{3^3 5^1 7^2}\,\mathrm{eulerlog}_2(R)\biggr]\frac{1}{R^7} + \frac{2^6 107^1 151^1}{3^2 5^2 7^2}\frac{\pi}{R^{7.5}} + \biggl[\frac{181399^1 1081890218404289908 7^1}{2^{13} 3^6 5^4 7^5 11^2 13^2 17^1 19^1} - \frac{2^8 107^1}{3^2 5^1 7^1}\pi^2 - \frac{2^{11}}{5^1}\zeta(3) \\
& - \frac{2^3 1529121911^1}{3^4 5^3 7^2 11^1}\,\mathrm{eulerlog}_2(R) + \frac{2^{10} 107^1}{3^1 5^2 7^1}\,\mathrm{eulerlog}_2^2(R)\biggr]\frac{1}{R^8} + \frac{2^4 107^1 6899^1}{3^4 5^2 7^3}\frac{\pi}{R^{8.5}} \\
& + \biggl[-\frac{257^1 701^1 25142051358546602833^1}{2^{17} 3^7 5^3 7^5 11^2 13^2 17^1 19^1} + \frac{2^7 107^1 151^1}{3^3 5^1 7^2}\pi^2 + \frac{2^{10} 151^1}{3^1 5^1 7^1}\zeta(3) - \frac{2^6}{5^1}\log(2/R) \\
& + \frac{53^1 467003681161^1}{3^6 5^3 7^3 11^1 13^1}\,\mathrm{eulerlog}_2(R) - \frac{2^9 107^1 151^1}{3^2 5^2 7^2}\,\mathrm{eulerlog}_2^2(R)\biggr]\frac{1}{R^9} + \biggl[-\frac{2^3 34871408153^1}{3^5 5^3 7^3 11^1} - \frac{2^{11} 107^1}{3^2 5^2 7^1}\pi^2 \\
& + \frac{2^{11} 107^2}{3^2 5^3 7^2}\,\mathrm{eulerlog}_2(R)\biggr]\frac{\pi}{R^{9.5}} + \biggl[-\frac{21937^1 29041277^1 1386244347381727739 17^1}{2^{18} 3^7 5^5 7^6 11^3 17^2 19^1 23^1} + \frac{2^5 107^1 6899^1}{3^5 5^1 7^3}\pi^2 + \frac{2^8 6899^1}{3^3 5^1 7^2}\zeta(3) \\
& - \frac{2^5 3^2}{5^1}\log(2/R) + \frac{157^1 37456616009^1}{2^1 3^6 5^2 7^4 13^1}\,\mathrm{eulerlog}_2(R) - \frac{2^7 107^1 6899^1}{3^4 5^2 7^3}\,\mathrm{eulerlog}_2^2(R)\biggr]\frac{1}{R^{10}} + \biggl[\frac{239^1 2819^1 4306391899^1}{3^7 5^4 7^4 11^1 13^1} \\
& + \frac{2^{10} 107^1 151^1}{3^3 5^2 7^2}\pi^2 - \frac{2^{10} 107^2 151^1}{3^3 5^3 7^3}\,\mathrm{eulerlog}_2(R)\biggr]\frac{\pi}{R^{10.5}} + \biggl[\frac{283^1 4280593261770893438785729019605702 07^1}{2^{20} 3^8 5^6 7^7 11^3 13^3 17^2 19^2 23^1} \\
& - \frac{2^4 34871408153^1}{3^6 5^2 7^3 11^1}\pi^2 - \frac{2^{12} 107^1}{3^2 5^3 7^1}\pi^4 - \frac{2^7 29^1 58564963^1}{3^4 5^3 7^2 11^1}\zeta(3) + \frac{2^{15}}{5^1}\zeta(5) - \frac{2^3 4507^1}{3^1 5^1 7^1}\log(2/R) \\
& + \biggl(-\frac{23^1 37^1 179^1 1871^1 7907^1 58007867^1}{2^4 3^6 5^5 7^4 11^2 13^1 17^1} + \frac{2^{12} 107^2}{3^3 5^2 7^2}\pi^2 + \frac{2^{15} 107^1}{3^1 5^2 7^1}\zeta(3)\biggr)\mathrm{eulerlog}_2(R) + \frac{2^6 34871408153^1}{3^5 5^3 7^3 11^1}\,\mathrm{eulerlog}_2^2(R) \\
& - \frac{2^{14} 107^2}{3^3 5^3 7^2}\,\mathrm{eulerlog}_2^3(R)\biggr]\frac{1}{R^{11}} + \biggl[\frac{1001321^1 3135169339^1}{2^1 3^7 5^4 7^5 13^1} + \frac{2^8 107^1 6899^1}{3^5 5^2 7^3}\pi^2 - \frac{2^8 107^2 6899^1}{3^5 5^3 7^4}\,\mathrm{eulerlog}_2(R)\biggr]\frac{\pi}{R^{11.5}}
\end{aligned}
$$

Simplifying the (retarded) ℓ,m modes of ΔU and the energy flux at infinity (cont.)

The general (rather naïve) idea of these simplifications is that much of the complexity—transcendentals and powers of log(v)—of higher-order terms in the PN expansion is “inessential,” coming from propagation effects, while the “truly new” information at each order is much simpler (for the energy flux at infinity, possibly just purely rational).

Indeed, Goldberger and Ross (2010) and Goldberger, Ross, and Rothstein (2014) find leading logarithms in both the energy flux and binding energy using the beta function for the multipole moments in the effective field theory picture, though they just compute the beta function to leading order, and thus only obtain the very simplest predictions of the simplifications we consider here.

Simplifying the full ΔU

- One can use the simplifications of the low-order modes, including a similar simplification that removes most of the powers of log(2/R), to simplify the full ΔU:

The even powers of π here come from the infinite ℓ-sum

$$
\begin{aligned}
\frac{\Delta U}{u} = &-1 - 2u - 5u^2 + \left[-\frac{121}{3} + \frac{41}{32}\pi^2\right] v^3 + \left[-\frac{1157}{15} + \frac{677}{512}\pi^2\right] u^4 + \left[\frac{1606877}{3150} - \frac{60343}{768}\pi^2\right] u^5 \\
&+ \left[\frac{17083661}{4050} - \frac{1246056911}{1769472}\pi^2 + \frac{2800873}{262144}\pi^4\right] u^6 + \left[\frac{12624956532163}{382016250} - \frac{8826302018257}{2477260800}\pi^2 - \frac{23851025}{16777216}\pi^4\right] u^7 \\
&+ \left[-\frac{7516581717416867}{34763478750} - \frac{741674600438227}{55490641920}\pi^2 + \frac{22759807747673}{6442450944}\pi^4\right] u^8 \\
&+ \left[-\frac{1048036213737050821493 3}{2044301131372500} - \frac{2585088013590862390 7}{494421619507200}\pi^2 + \frac{32962327798317273}{549755813888}\pi^4 - \frac{27101981341}{100663296}\pi^6\right] u^9 \\
&+ \left[-\frac{2389467863446532647991752802 03}{452242340583355822500 0} - \frac{1892790098556378449660704 1}{1583731331605463040000}\pi^2\right. \\
&\left.+ \frac{1664643109929546979475 63}{332492316239462400}\pi^4 + \frac{54067065388369}{12884901888}\pi^6\right] u^{10} - \frac{8192}{75}\pi u^{10.5} \\
&+ \left[\frac{3814229145040080910470246242071097}{1379871188591686265475 0000} - \frac{3130615191892001882037620681308 1}{2305912818817554186240000}\pi^2\right. \\
&\left.+ \frac{485483700323866025378261 33}{893739346051674931 20}\pi^4 - \frac{5276940898567193189}{25975962206208}\pi^6\right] u^{11} - \frac{2192896}{6741}\pi u^{11.5} \\
&+ \left[\frac{17653826452609603902567425186317655993151 1}{120807253404998011369005988500 00} - \frac{23059527171486685697289627056191334439 1}{558842583466071892143636480000}\pi^2\right. \\
&- \frac{902366950567138330511081959572149}{188757749886113745469440 00}\pi^4 - \frac{1675177917102535631869 33}{265993852991569 92}\pi^6 + \frac{44336492264184971}{2473901162496}\pi^8 \\
&\left.+ \frac{65536}{75}\log(2u)\,\mathrm{eulerlog}_2(u)\right] u^{12} - \frac{31486592}{43335}\pi u^{12.5} + \sum_{\ell=2}^{10}\sum_{m=1}^{\ell} C^{[1]}_{\ell m}\Upsilon^{\mathcal{S}1}_{\ell m} + \sum_{\ell=2}^{4}\sum_{m=1}^{\ell} C^{[2]}_{\ell m}\Upsilon^{\mathcal{S}2}_{\ell m} + O(u^{13}),
\end{aligned}
$$

rational coefficients

Predicting “leading logarithmic” terms to all orders

- In addition to simplifying the full ΔU, the simplification also predicts some “leading logarithmic” coefficients to all orders, due to the exponential term. (The same is also true for the energy flux at infinity.)

- Specifically, the simplification predicts the coefficient of the first five appearances of a given power of $\log(R)$ in both the integer and half-integer PN terms.

- In addition to the complete terms predicted by the simplification, it also gives the coefficients of various transcendentals in other terms, which is very helpful in obtaining the remainder of the term using PSLQ.

Predicting “leading logarithmic” terms to all orders

The expressions used to make the predictions and some example results

$$
\begin{aligned}
&C^{\mathcal{S}1}_{22}\left[-\frac{128}{5}+\frac{5632}{105}\frac{1}{R}+\frac{452096}{6615}\frac{1}{R^2}-\frac{1654278784}{779625}\frac{1}{R^3}+\frac{436456134656}{178783605}\frac{1}{R^4}\right]\\
&+C^{\mathcal{S}1}_{21}\left[-\frac{32}{45}-\frac{80}{63}\frac{1}{R}-\frac{14234}{3969}\frac{1}{R^2}-\frac{61065898}{2338875}\frac{1}{R^3}\right]+C^{\mathcal{S}1}_{33}\left[-\frac{243}{7}+\frac{1215}{7}\frac{1}{R}-\frac{43254}{385}\frac{1}{R^2}-\frac{26690187609}{4904900}\frac{1}{R^3}\right]\\
&+C^{\mathcal{S}1}_{32}\left[-\frac{128}{63}+\frac{7424}{2835}\frac{1}{R}+\frac{2432}{1403325}\frac{1}{R^2}\right]+C^{\mathcal{S}1}_{31}\left[-\frac{1}{315}+\frac{1}{135}\frac{1}{R}-\frac{206}{10395}\frac{1}{R^2}-\frac{116962723}{1986484500}\frac{1}{R^3}\right]\\
&+C^{\mathcal{S}1}_{44}\left[-\frac{32768}{567}+\frac{14024704}{31185}\frac{1}{R}-\frac{29572726784}{31216185}\frac{1}{R^2}\right]+C^{\mathcal{S}1}_{43}\left[-\frac{729}{175}+\frac{6561}{385}\frac{1}{R}\right]\\
&+C^{\mathcal{S}1}_{42}\left[-\frac{128}{3969}+\frac{34816}{218295}\frac{1}{R}-\frac{398729216}{1092566475}\frac{1}{R^2}\right]+C^{\mathcal{S}1}_{41}\left[-\frac{1}{11025}+\frac{103}{363825}\frac{1}{R}\right]\\
&+C^{\mathcal{S}1}_{55}\left[-\frac{1953125}{19008}+\frac{798828125}{741312}\frac{1}{R}\right]-\frac{524288}{66825}C^{\mathcal{S}1}_{54}+C^{\mathcal{S}1}_{53}\left[-\frac{2187}{17600}+\frac{19683}{20800}\frac{1}{R}\right]-\frac{512}{200475}C^{\mathcal{S}1}_{52}\\
&+C^{\mathcal{S}1}_{51}\left[-\frac{1}{4989600}+\frac{241}{194594400}\frac{1}{R}\right]-\frac{3359232}{17875}C^{\mathcal{S}1}_{66}-\frac{16777216}{47779875}C^{\mathcal{S}1}_{64}-\frac{512}{28667925}C^{\mathcal{S}1}_{62}-\frac{64}{5}C^{\mathcal{S}2}_{22}\\
&=\{\text{leading logarithmic terms through 12.5PN}\}\\
&+\left[\left(\frac{28553817216903148928}{3019990307709375}-\frac{40142209024}{17364375}\gamma-\frac{80284418048}{17364375}\log(2)\right)\log^3(R)+\frac{5017776128}{17364375}\log^4(R)\right]\frac{1}{R^{14}}\\
&+\frac{266856694218789586}{39313483295625}\frac{\pi\log^2(R)}{R^{14.5}}+\left[\left(\frac{127860848443786766 16752}{335890033113009375}-\frac{230609748224}{72930375}\gamma+\frac{613208455936}{364651875}\log(2)\right.\right.\\
&\left.\left.-\frac{19219356}{2401}\log(3)\right)\log^3(R)+\frac{28826218528}{72930375}\log^4(R)\right]\frac{1}{R^{15}}+\left[\left(-\frac{116106125514835430463890813}{1773499374836689500000}-\frac{80284418048}{17364375}\pi^2\right.\right.\\
&\left.\left.+\frac{8590432731136}{607753125}\gamma+\frac{17180865462272}{607753125}\log(2)\right)\log^2(R)-\frac{4295216365568}{1823259375}\log^3(R)\right]\frac{\pi}{R^{15.5}}-\frac{7401983886289495 48}{353821349660625}\frac{\log^4(R)}{R^{16}}\ \text{etc.}
\end{aligned}
$$

$$C^{\mathcal{S}1}_{\ell m}:=e^{2\bar\nu_{\ell m}\,\mathrm{eulerlog}_m(R)}\left[\frac{1}{2\bar\nu_{\ell m}}\frac{1}{R^2}-\frac{\bar\nu_{\ell m}}{4m}\frac{\pi}{R^{0.5}}-\frac{m\bar\nu_{\ell m}}{3}\frac{\pi^3}{R^{3.5}}\right]\frac{1}{R^{\ell+1+\varepsilon_{\ell m}}}-\frac{1}{2\bar\nu_{\ell m}}\frac{1}{R^{\ell+3+\varepsilon_{\ell m}}}\qquad C^{\mathcal{S}2}_{\ell m}:=\left[e^{2\bar\nu_{\ell m}\log(2/R)}-1\right]\frac{1}{2\bar\nu_{\ell m}R^{5+2\ell+\epsilon_{\ell m}}}$$

Why can't we predict more of these coefficients?
A lesson from the "complete simplification" of the energy flux at infinity

- One can reduce the modes of the point-particle-in-a-circular-orbit-around-Schwarzschild energy flux at infinity to a form that has only rationals in the infinite series by a study of the expressions in the MST formalism, viz.,

These give powers of log(2/R)

$$\underbrace{(2z)^{2\bar\nu_{\ell m}} e^{-\pi\epsilon}\frac{|\Gamma(1+\bar\nu_{\ell m}+\mathrm{i}\epsilon)|^2}{\Gamma^2(1+2\bar\nu_{\ell m})}}_{|S_{\ell m}|^2}\left|\sum_{k=0}^{\infty} c_k^{\ell m} v^k + (2v^2)^{2\bar\nu_{\ell m}}\frac{\Gamma(1-2\bar\nu_{\ell m})}{\Gamma(1+2\bar\nu_{\ell m})}\left[\frac{\Gamma(1+\bar\nu_{\ell m}-\mathrm{i}\epsilon)}{\Gamma(1-\bar\nu_{\ell m}-\mathrm{i}\epsilon)}\right]^2\sum_{k=0}^{\infty} d_k^{\ell m} v^k\right|^2$$

$$\times\left|1-e^{-\mathrm{i}\pi\bar\nu_{\ell m}}\frac{\sin\pi(\bar\nu_{\ell m}+\mathrm{i}\epsilon)}{\sin\pi(\bar\nu_{\ell m}-\mathrm{i}\epsilon)}(2\epsilon)^{2\bar\nu_{\ell m}}\left[\frac{\Gamma(1-2\bar\nu_{\ell m})}{\Gamma(1+2\bar\nu_{\ell m})}\frac{\Gamma(1+\bar\nu_{\ell m}-\mathrm{i}\epsilon)}{\Gamma(1-\bar\nu_{\ell m}-\mathrm{i}\epsilon)}\right]^2\frac{\Gamma(1+\bar\nu_{\ell m}+\mathrm{i}\epsilon)}{\Gamma(1-\bar\nu_{\ell m}+\mathrm{i}\epsilon)}\sum_{k=0}^{\infty} e_k^{\ell m} v^k\right|^{-2}$$

Here the c_k and d_k series have both even and odd powers.
The d_k and e_k terms only enter at higher orders, and parts of them correspond to V'_{lm}

- However, this may not be the simplest form, since ν and the c_k, d_k, and e_k series have large primes in the denominator which are not seen in the PN series of any invariant.

Explicit expressions for some of these predictions, including first results for the energy flux at infinity for a circular, equatorial orbit in Kerr

- One can derive explicit expressions for the predictions of the simplification.
- Here we show the predictions for the coefficients of the first few appearances of a given power of log(v) in both the integer and half-integer terms both for ΔU for a circular orbit in Schwarzschild, and the energy flux at infinity for a circular equatorial orbit in Kerr. We write the expansion in terms of v for both, for easy comparison.

ΔU

$$\left\{\frac{64}{5}\left(\frac{856}{105}\right)^{n-1}\left[v^4-\left(\frac{856}{105}\right)^2\pi v^{13}\right]+\left[-\frac{2816}{105}\left(\frac{856}{105}\right)^{n-1}+\frac{16}{45}\left(\frac{214}{105}\right)^{n-1}+\frac{243}{14}\left(\frac{78}{7}\right)^{n-1}+\frac{1}{630}\left(\frac{26}{21}\right)^{n-1}\right]v^6\right.$$
$$\left.+\left[\frac{704}{105}\left(\frac{856}{105}\right)^{n+1}-\frac{8}{45}\left(\frac{214}{105}\right)^{n+1}-\frac{243}{84}\left(\frac{78}{7}\right)^{n+1}-\frac{1}{1260}\left(\frac{26}{21}\right)^{n+1}\right]\pi v^{15}\right\}\frac{(-2)^n v^{6n}\log^n(v)}{n!}$$

Energy flux at infinity for Kerr

$$\left[\left(\frac{856}{105}\right)^n\left\{1+\left[4\pi-\left(\frac{8}{3}+\frac{1}{12\cdot 4^n}+\frac{140}{107}n\right)q\right]v^3\right\}\right.$$
$$+\left\{\frac{1}{36}\left(\frac{214}{105}\right)^n-\frac{107}{21}\left(\frac{856}{105}\right)^n+\frac{1}{8064}\left(\frac{26}{21}\right)^n+\frac{1215}{896}\left(\frac{78}{7}\right)^n\right\}v^2$$
$$+\left\{\frac{4784}{1323}\left(\frac{856}{105}\right)^n-\frac{17}{504}\left(\frac{214}{105}\right)^n-\frac{1}{1512}\left(\frac{26}{21}\right)^n+\frac{5}{63}\left(\frac{104}{21}\right)^n-\frac{1215}{112}\left(\frac{78}{7}\right)^n+\frac{5}{3969}\left(\frac{12568}{3465}\right)^n\right.$$
$$\left.+\frac{1280}{567}\left(\frac{50272}{3465}\right)^n+\frac{1}{16}\left(\frac{214}{105}\right)^n(1+32\cdot 4^n)q^2\right\}v^4$$
$$+\left\{\left[\frac{1}{18}\left(\frac{214}{105}\right)^n-\frac{428}{21}\left(\frac{856}{105}\right)^n+\frac{3645}{448}\left(\frac{78}{7}\right)^n+\frac{1}{4032}\left(\frac{26}{21}\right)^n\right]\pi\right.$$
$$+\left[\left(\frac{52}{27}+\frac{20}{3}n\right)\left(\frac{856}{105}\right)^n+\left(\frac{215}{9072}-\frac{35}{3852}n\right)\left(\frac{214}{105}\right)^n-\frac{40}{189}\left(\frac{104}{21}\right)^n-\left(\frac{1215}{224}+\frac{5103}{3328}n\right)\left(\frac{78}{7}\right)^n\right.$$
$$\left.\left.\left.-\left(\frac{25}{18144}+\frac{7}{449280}n\right)\left(\frac{26}{21}\right)^n\right]q\right\}v^5\right]\frac{(-2)^n v^{6n}\log^n(v)}{n!}$$

Here q is the dimensionless spin

Explicit expressions for some of these predictions, including first results for the energy flux at infinity for a circular, equatorial orbit in Kerr

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These predictions for the energy flux at infinity come from the $S_{\ell m}$ simplification, where the only change necessary for Kerr is to use the Kerr version of ν. However, there are transcendental functions of the spin that enter at higher orders and are not accounted for by $S_{\ell m}$.

Energy flux at infinity for Kerr

$$
\begin{aligned}
&\Bigg[\left(\frac{856}{105}\right)^n\left\{1+\left[4\pi-\left(\frac{8}{3}+\frac{1}{12\cdot 4^n}+\frac{140}{107}n\right)q\right]v^3\right\}\\
&+\left\{\frac{1}{36}\left(\frac{214}{105}\right)^n-\frac{107}{21}\left(\frac{856}{105}\right)^n+\frac{1}{8064}\left(\frac{26}{21}\right)^n+\frac{1215}{896}\left(\frac{78}{7}\right)^n\right\}v^2\\
&+\left\{\frac{4784}{1323}\left(\frac{856}{105}\right)^n-\frac{17}{504}\left(\frac{214}{105}\right)^n-\frac{1}{1512}\left(\frac{26}{21}\right)^n+\frac{5}{63}\left(\frac{104}{21}\right)^n-\frac{1215}{112}\left(\frac{78}{7}\right)^n+\frac{5}{3969}\left(\frac{12568}{3465}\right)^n\right.\\
&\quad\left.+\frac{1280}{567}\left(\frac{50272}{3465}\right)^n+\frac{1}{16}\left(\frac{214}{105}\right)^n(1+32\cdot 4^n)\,q^2\right\}v^4\\
&+\left\{\left[\frac{1}{18}\left(\frac{214}{105}\right)^n-\frac{428}{21}\left(\frac{856}{105}\right)^n+\frac{3645}{448}\left(\frac{78}{7}\right)^n+\frac{1}{4032}\left(\frac{26}{21}\right)^n\right]\pi\right.\\
&\quad+\left[\left(\frac{52}{27}+\frac{20}{3}n\right)\left(\frac{856}{105}\right)^n+\left(\frac{215}{9072}-\frac{35}{3852}n\right)\left(\frac{214}{105}\right)^n-\frac{40}{189}\left(\frac{104}{21}\right)^n-\left(\frac{1215}{224}+\frac{5103}{3328}n\right)\left(\frac{78}{7}\right)^n\right.\\
&\quad\left.\left.-\left(\frac{25}{18144}+\frac{7}{449280}n\right)\left(\frac{26}{21}\right)^n\right]q\right\}v^5\Bigg]\frac{(-2)^n v^{6n}\log^n(v)}{n!}
\end{aligned}
$$

Here q is the dimensionless spin

Explicit expressions for some of these predictions, including first results for the energy flux at infinity for a circular, equatorial orbit in Kerr

- One can derive explicit expressions for the predictions of the simplification.
- Here ... ces of a given ... for a circul ... equatorial orbit ... arison.

Example of a newly-obtained spin-dependent piece of a coefficient of the energy flux at infinity for a circular, equatorial orbit in Kerr

$$\left\{\left[-\frac{108649121065357026606921642567 89}{1635692044476644942850562 50}-\frac{8415139871298263}{137838408750}\pi^2+\frac{787775914908495197}{9648688612500}\gamma-\frac{2871797852601011}{119119612500}\log(2)+\frac{705518534565}{3764768}\log(3)\right]q+\frac{2220827389239310949101 93}{13077237083456700 00}\pi q^2+\frac{151795795077131601174 3041}{1961585562518505000}q^3+\frac{357978390331}{486202500}\pi q^4-\frac{56001342517411}{1458607500}q^5\right\}\log^4(v)v^{35}$$

ΔU

$$\left\{\frac{64}{5} \cdots\right]v^6$$

$$+\left[\frac{\cdots}{105}\left(\frac{\cdots}{105}\right) \cdots -\frac{\cdots}{45}\left(\frac{\cdots}{105}\right) \cdots -\frac{\cdots}{84}\left(\frac{\cdots}{7}\right) \cdots -\frac{\cdots}{1260}\left(\frac{\cdots}{21}\right)\right]\pi v^{15}\Bigg\}\frac{\cdots}{n!}$$

Energy flux at infinity for Kerr

$$\left[\left(\frac{856}{105}\right)^n\left\{1+\left[4\pi-\left(\frac{8}{3}+\frac{1}{12\cdot 4^n}+\frac{140}{107}n\right)q\right]v^3\right\}\right.$$
$$+\left\{\frac{1}{36}\left(\frac{214}{105}\right)^n-\frac{107}{21}\left(\frac{856}{105}\right)^n+\frac{1}{8064}\left(\frac{26}{21}\right)^n+\frac{1215}{896}\left(\frac{78}{7}\right)^n\right\}v^2$$
$$+\left\{\frac{4784}{1323}\left(\frac{856}{105}\right)^n-\frac{17}{504}\left(\frac{214}{105}\right)^n-\frac{1}{1512}\left(\frac{26}{21}\right)^n+\frac{5}{63}\left(\frac{104}{21}\right)^n-\frac{1215}{112}\left(\frac{78}{7}\right)^n+\frac{5}{3969}\left(\frac{12568}{3465}\right)^n\right.$$
$$\left.+\frac{1280}{567}\left(\frac{50272}{3465}\right)^n+\frac{1}{16}\left(\frac{214}{105}\right)^n(1+32\cdot 4^n)q^2\right\}v^4$$
$$+\left\{\left[\frac{1}{18}\left(\frac{214}{105}\right)^n-\frac{428}{21}\left(\frac{856}{105}\right)^n+\frac{3645}{448}\left(\frac{78}{7}\right)^n+\frac{1}{4032}\left(\frac{26}{21}\right)^n\right]\pi\right.$$
$$+\left[\left(\frac{52}{27}+\frac{20}{3}n\right)\left(\frac{856}{105}\right)^n+\left(\frac{215}{9072}-\frac{35}{3852}n\right)\left(\frac{214}{105}\right)^n-\frac{40}{189}\left(\frac{104}{21}\right)^n-\left(\frac{1215}{224}+\frac{5103}{3328}n\right)\left(\frac{78}{7}\right)^n\right.$$
$$\left.\left.\left.-\left(\frac{25}{18144}+\frac{7}{449280}n\right)\left(\frac{26}{21}\right)^n\right]q\right\}v^5\right]\frac{(-2)^n v^{6n}\log^n(v)}{n!}$$

Here q is the dimensionless spin

Why such high order PN expansions?

- Why do we bother making such high-order PN expansions? In particular, there is a fit for ΔU and other invariants for circular orbits in Schwarzschild that reproduces the numerical values with quite high accuracy up to the light ring (Akcay et al., 2012).

- We mostly do this as a warm-up for Kerr, where such fits do not yet exist, so high-order PN expansions will be quite useful. Additionally, since the PN expansions of conservative invariants are only known to very low order in Kerr, it would be useful to be able to compute analytic PN coefficients for them, e.g., to aid comparable-mass PN calculations.

- Additionally, these perturbative series can often be used to obtain nonperturbative pieces (the principle of resurgence). One can also frequently make interesting studies of the structure of the expansion and properties of the coefficients (as is often done for quantum field theory series); it is already interesting to compare the structure of the series for ΔU with that of the energy flux at infinity.

Conclusions and outlook

- It is possible to use experimental mathematics techniques, notably the application of an integer relation algorithm, to obtain analytic forms of quite complicated high-order PN coefficients with high confidence.
- To do this efficiently, one looks at the individual modes and studies their structure, finding a simplification that allows one to predict various higher-order terms, letting one obtain the complete coefficient with PSLQ with fewer digits.
- Additionally, one can perform the infinite ℓ-sum analytically, rather than numerically, by obtaining the general form of the ℓ-dependence of a given PN coefficient.

Conclusions and outlook II

- We have currently applied this method to ΔU for a circular orbit in Schwarzschild, as an illustration, obtaining the PN coefficients to 12.5PN + 13.5PN completely analytically, and the coefficients to 21PN in mixed numerical-analytic form. Our confidence in these analytic forms has been validated by the exact agreement with the concurrent purely analytic calculation by Kavanagh, Ottewill, and Wardell.

- The methods we have developed should be applicable to more complicated cases without too many changes—we plan on considering circular equatorial orbits in Kerr next and have started studying the structure of the energy flux at infinity, since this was a good guide in the Schwarzschild case. Even at these early stages of this study, we have already been able to obtain a few new coefficients of logarithmic terms in the energy flux at infinity from Abhay's pre-existing numerical data.

Extra Slides

Example of our method of using linear combinations of values at different radii to increase the accuracy of a given PN coefficient

- Assume that we are at a point in the expansion where it looks like:

No log(R) term for simplicity of describing the remainder—this will never occur in practice

$$S_N(R) = \frac{\alpha_{N,0}}{R^N} + \frac{\alpha_{N+1,0} + \alpha_{N+1,1}\log(R)}{R^{N+1}} + \frac{\alpha_{N+2,0}}{R^{N+2}} + O(R^{-N-3})$$

- Then, if we know $S_N(R)$ at R = 10^k, 10^{k+p}, and 10^{k+q}, we can use the following expression to remove the $O(R^{-N-1})$ pieces and obtain $\alpha_{N,0}$ to ~2k digits:

$$\alpha_{N,0} = 10^{kN}\frac{(q-p)S_N(10^k) - q10^{(N+1)p}S_N(10^{k+p}) + p10^{(N+1)q}S_N(10^{k+q})}{q-p-q10^p+p10^q} + \underbrace{\frac{q-p-q10^{-p}+p10^{-q}}{q-p-q10^p+p10^q}\frac{\alpha_{N+2,0}}{10^{2k}}}_{\mathcal{R}}$$

Convergence

- We find that the additional PN terms we have obtained improve the accuracy of the PN series for ΔU, even in the strong-field regime inside the ISCO. However, the convergence becomes less rapid with increasing field strength, as expected.

- Exponential resummation helps some with the convergence, though not as much as it does for the energy flux at infinity, where it works very well when applied mode-by-mode.

- Of course, we primarily calculated these terms as a first application of our method, not simply to increase the accuracy of the PN series for ΔU. These high-order calculations also give useful insight into the structure of the PN expansion, showing similar structures in both conservative and dissipative quantities, due to tail effects from wave propagation on curved spacetime.

Convergence

- We find tha … rove the accuracy o … gime inside the ISCO. How … ncreasing field strength, a …
- Exponentia … , though not as much a … rks very well when appli …
- Of course, … plication of our method, n … es for ΔU. These high … e structure of the PN exp … rvative and dissipative … tion on curved spacetime. …